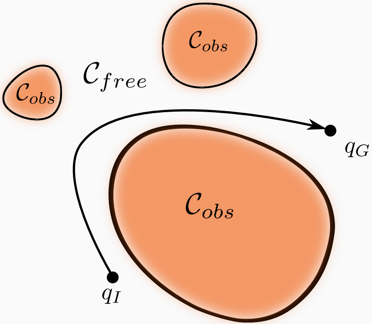
KU LEUVEN

Faster algorithms for connectivity queries in unbounded real algebraic sets

5th March 2024

JNCF '24



Rémi Prébet

Joint works with M. Safey El Din, É. Schost
N. Islam, A. Poteaux
J. Capco, P. Wenger

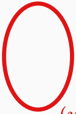
Slides:
rprebet.github.io/#talks

Computational real algebraic geometry

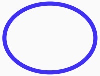
Semi-algebraic sets

Set of **real** solutions of systems of **polynomial equations** and **inequalities**

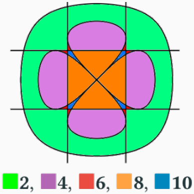
$$\begin{cases} 4y + x^3 - 4x^2 - 2x - 8 = 0 \\ -2 \leq x \leq 0 \end{cases}$$



$$\frac{x^2}{4} + y^2 - 1 = 0$$

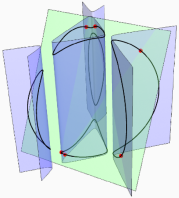


$$(x-1)^2 + \frac{(y-1)^2}{9} - 1 = 0$$



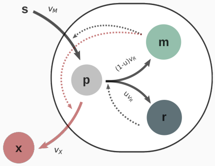
Physics

[Le, Safey El Din; '22]



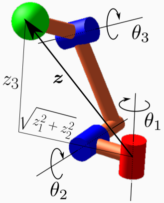
Computational geometry

[Le, Manevich, Plaumann; '21]



Biology

[Yabo, Safey El Din, Caillau, Gouzé; '23]



Robotics

[Chablat, P., Safey El Din, Salunkhe, Wenger; '22]

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Stability [Tarski-Seidenberg]

The family of s.a. sets is stable by projection

Finiteness

Finite number of connected components

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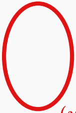

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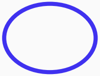

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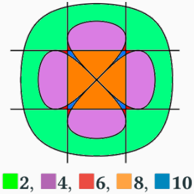

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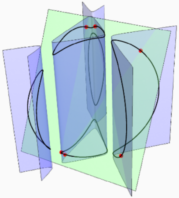

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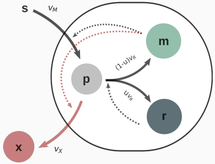

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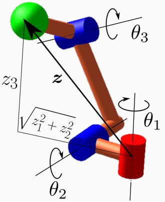

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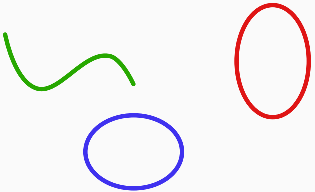

Fundamental problems in computational real algebraic geometry

(P) compute a projection: one block quantifier elimination

(S) compute at least one point in each connected component

(C) decide if two points lie in the same connected component

(N) count the number of connected components

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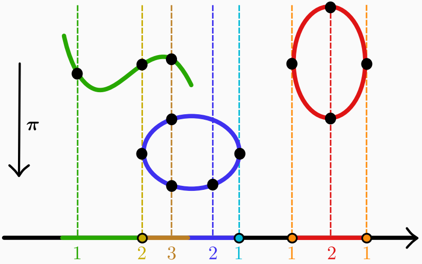

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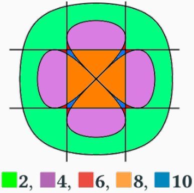

Kuramoto oscillators

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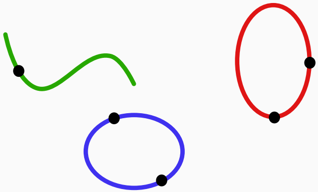

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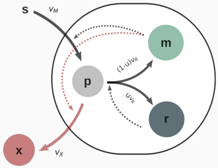

Dynamical systems

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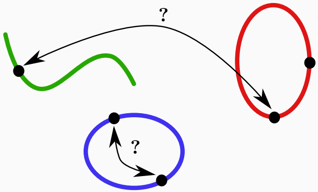

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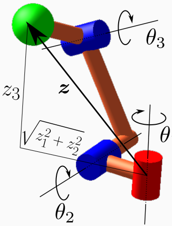

Cuspidality decision

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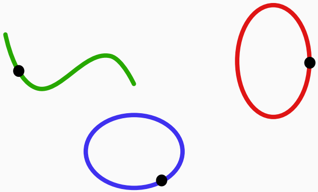

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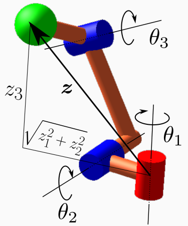

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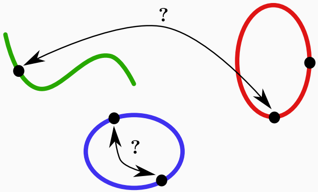

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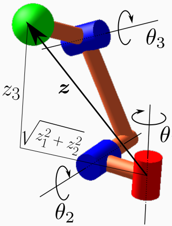


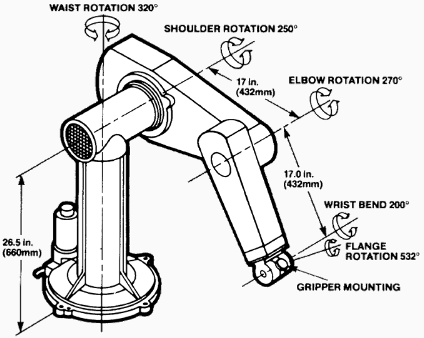
Cuspidality decision

A challenging application in robotics

$\mathrm{Jac}_{v_2,\ldots,v_5}(\mathcal{K})$ for a PUMA-type robot with a non-zero offset in the wrist

$$\begin{bmatrix} (v_3+v_2)(1-v_2v_3) & 0 & A(\boldsymbol{v}) & d_3A(\boldsymbol{v}) & a_2(v_3^2+1)(v_2^2-1)-a_3A(\boldsymbol{v}) & 2d_3(v_3+v_2)(v_2v_3-1) \\ 0 & v_3^2+1 & 0 & 2a_2v_3 & 0 & (a_3-a_2)v_3^2+a_2+2a_3 \\ 0 & 1 & 0 & 0 & 0 & 2a_3 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ v_4 & 1-v_4^2 & 0 & d_4(1-v_4^2) & -2d_4v_4 & 0 \\ (v_4^2-1)v_5 & 4v_4v_5 & (1-v_5^2)(v_4^2+1) & (1-v_5^2)(v_4^2-1)d_5+4d_4v_4v_5 & 2d_5v_4(1-v_5^2)+2d_4v_5(1-v_4^2) & -2d_5v_5(v_4^2+1) \end{bmatrix}$$

where $A(\boldsymbol{v}) = (v_3^2-1)(v_2^2-1)-4v_2v_3$



A PUMA 560 [Unimation, 1984]

Fix generic parameters $(a_2, a_3, d_3, d_4, d_5) \in (\mathbb{Q}_{>0})^5$
v_2, v_3, v_4, v_5: half-angle tangents of rotations

Robotic problem

Count the number of aspects of this robot.

⇕

Semi-algebraic problem

Compute the number of connected components of $S = \{\boldsymbol{v} \in \mathbb{R}^4 \mid \det(M(\boldsymbol{v})) \neq 0\}$

⇕

Algebraic problem

Compute the number of connected components of $V_{\mathbb{R}} = \{(\boldsymbol{v}, t) \in \mathbb{R}^5 \mid \det(M(\boldsymbol{v})) \cdot t = 1\}$ where t is a new variable.

Computing connectivity properties: Roadmaps

[Canny, 1988] Compute $\mathscr{R} \subset S$ one-dimensional, sharing its connectivity

Roadmap of $(S, \mathcal{P})$

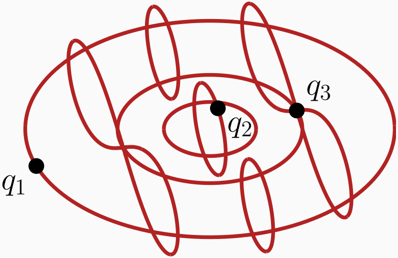
A *semi-algebraic curve* $\mathscr{R} \subset S$, containing query points $(q_1, \ldots, q_N)$ s.t. for all connected components C of S: $C \cap \mathscr{R}$ is *non-empty* and *connected*

Proposition

q_i and q_j are path-connected in $S \iff$ they are in $\mathscr{R}$

Problem reduction

Arbitrary dimension $\underset{\text{Roadmap}}{\Longrightarrow}$ Dimension 1



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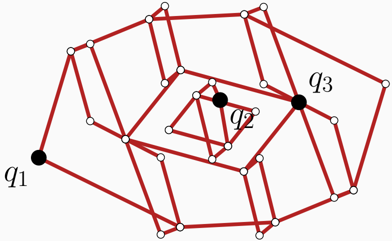
q_i and q_j are path-connected in $S \iff$ they are in $\mathscr{R} \iff$ they are in $\mathscr{G}$

Problem reduction

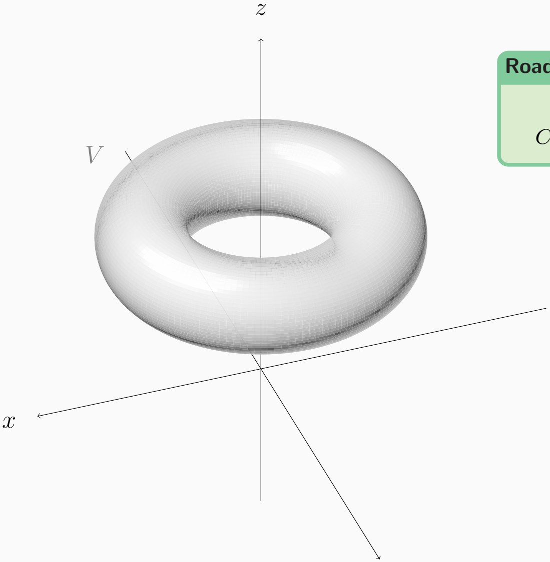
Arbitrary dimension $\underset{\text{Roadmap}}{\Longrightarrow}$ Dimension 1 $\underset{\text{Topology}}{\Longrightarrow}$ Finite graph $\mathscr{G}$



Roadmap algorithms for unbounded algebraic sets

joint work with M. Safey El Din and É. Schost

Canny's strategy



Roadmap property

$\forall C$ connected component,

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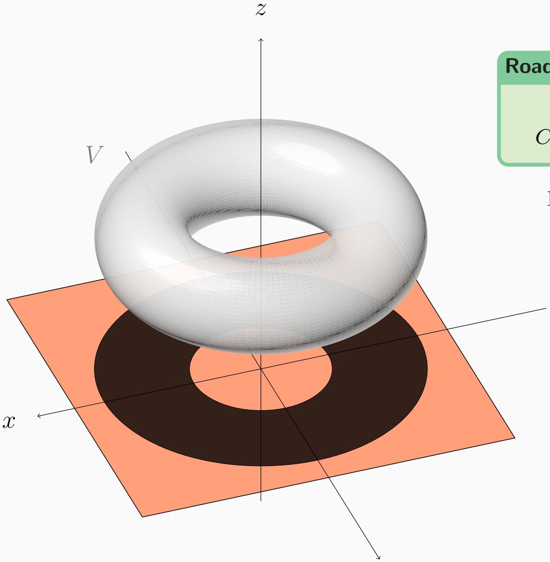

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Projection through:
$\pi_2 : (x_1, \ldots, x_n) \mapsto (x_1, x_2)$

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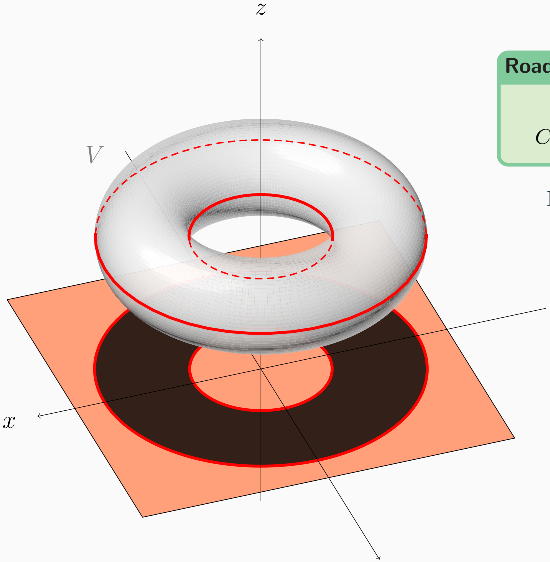

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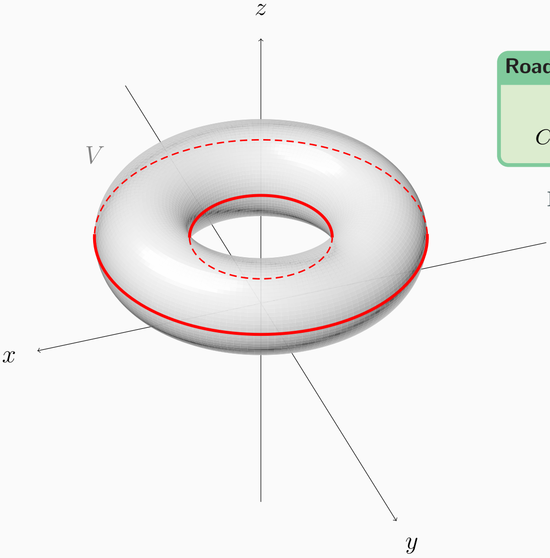
$W(\pi_2, V)$ critical locus of π_2.

Intersects all the
connected components of V



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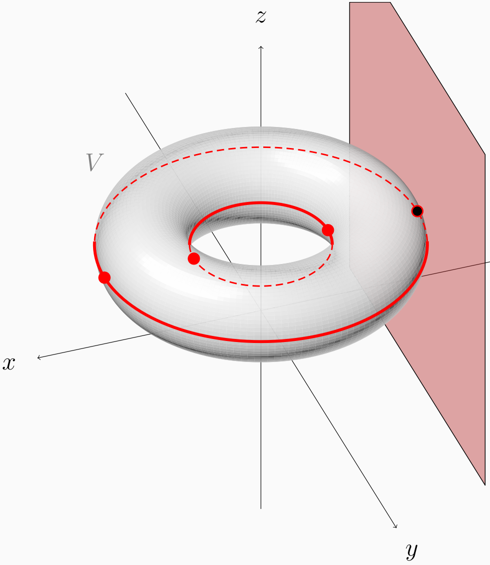

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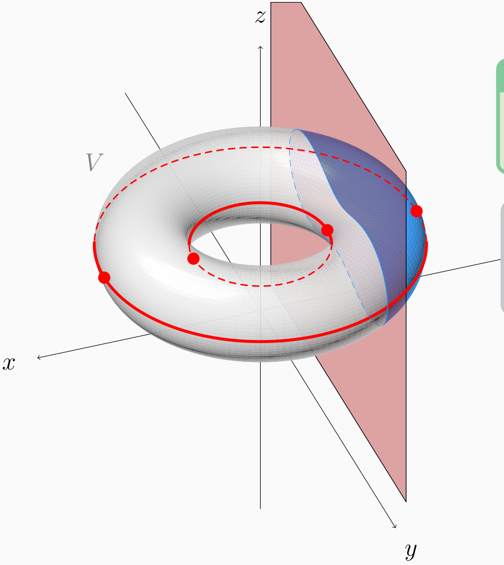
Morse theory

"Scan" $\boldsymbol{W(\pi_2, V)}$ at the critical values of π_1

- We repair the connectivity failures with critical fibers
- We repeat the process at every critical value

Canny's strategy



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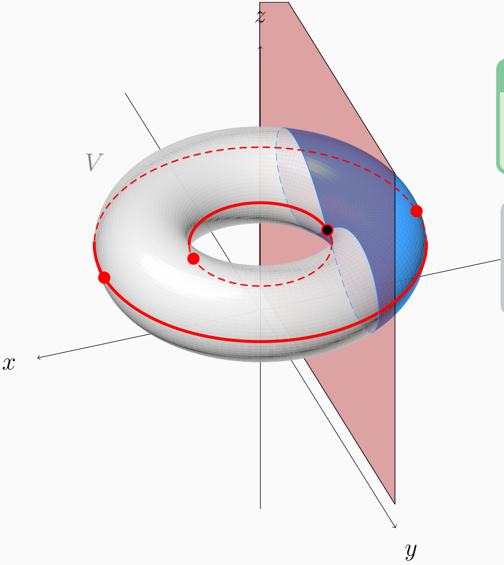

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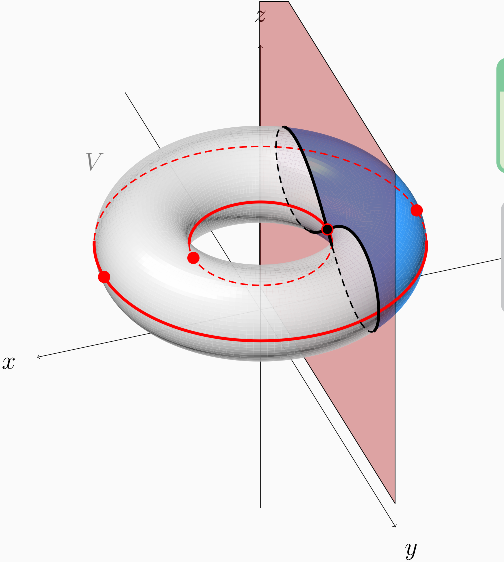

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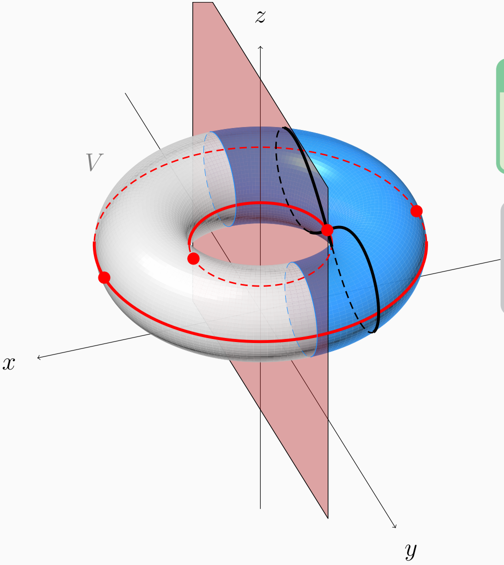

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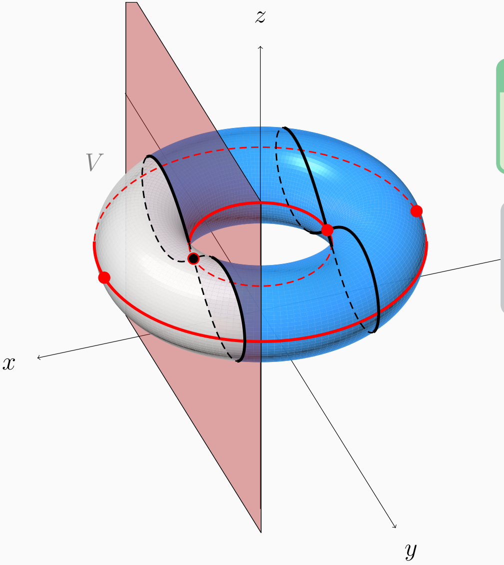

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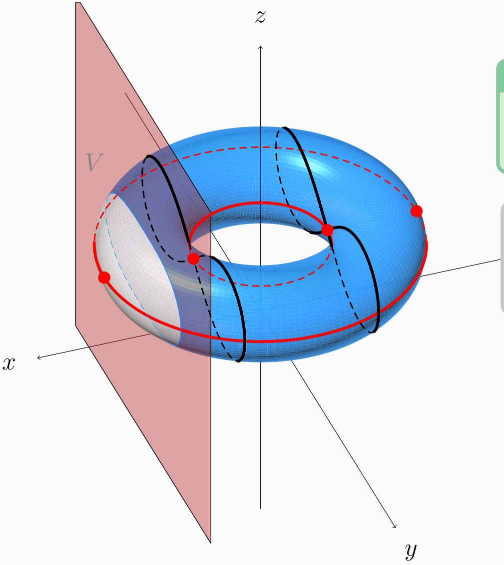

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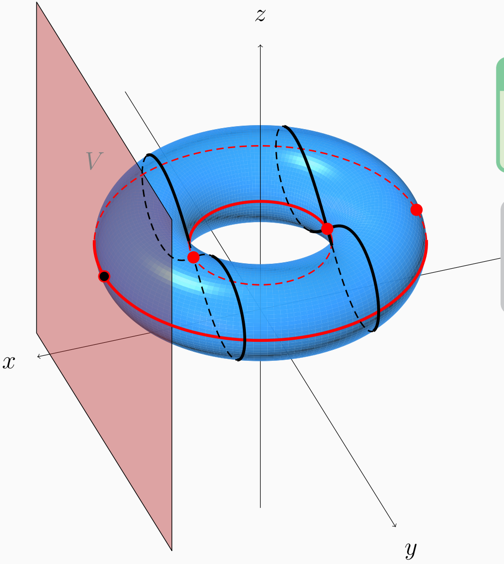

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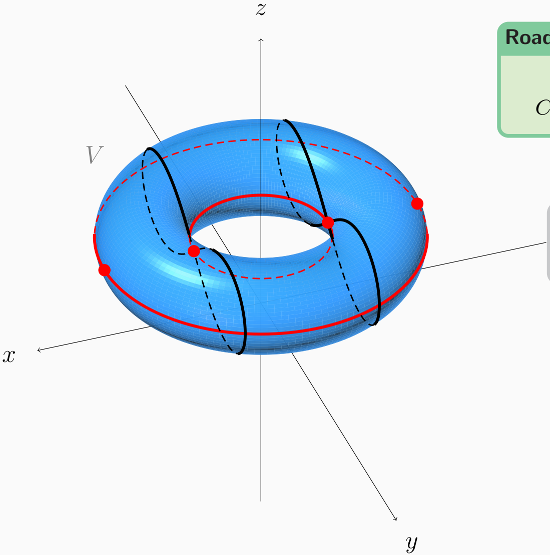

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$W(\pi_2, V)$ polar variety
F critical fibers

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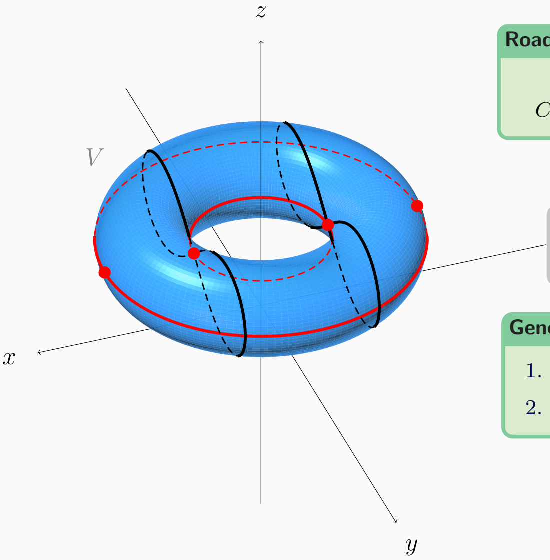

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Genericity assumptions

1. $W(\pi_2, V)$ has dimension 1
2. F has dimension $\dim(V) - 1$

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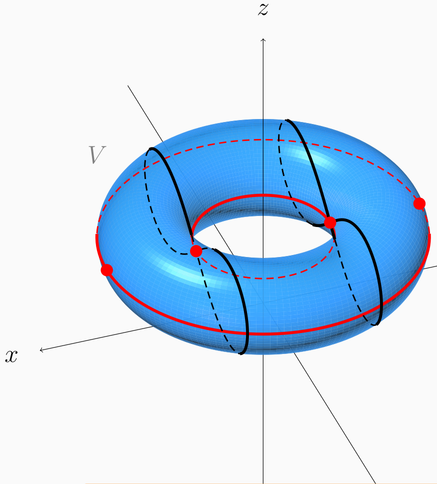

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$\boldsymbol{W(\pi_2, V)}$ **polar variety**
$\boldsymbol{F}$ **critical fibers**

Genericity assumptions

1. $\boldsymbol{W(\pi_2, V)}$ has dimension 1
2. $\boldsymbol{F}$ has dimension $\dim(V) - 1$

Theorem [Canny, 1988]

If V is bounded, $\boldsymbol{W(\pi_2, V)} \bigcup \boldsymbol{F}$ has dimension $\dim(V) - 1$
and satisfies the Roadmap property

On the complexity of computing roadmaps

$S \subset \mathbb{R}^n$ semi alg. set of <u>dimension d</u> and defined by s polynomials of <u>degree $\leqslant D$</u>

Connectivity result [Canny, 1988]

If V is bounded, $W(\pi_2, V) \cup F$ has dimension $d-1$
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Author·s	Complexity	Assumptions
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$S \subset \mathbb{R}^n$ semi alg. set of dimension d and defined by s polynomials of degree $\leqslant D$

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If V is bounded, $W(\pi_i, V) \cup F_i$ has dimension $\max(i-1, d-i+1)$
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[Safey El Din & Schost, 2017]	$(n^2 D)^{\mathbf{6} n \log_2(d) + O(n)}$	Smooth, bounded algebraic sets

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Remove the boundedness assumption is a *costly* step

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Remove the boundedness assumption is a *costly* step

Not polynomial in the output size

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Remove the boundedness assumption is a *costly* step

Not polynomial in the output size

Necessity of a new theorem in the **unbounded** case!

On the extension of Canny's result

Projection on 2 coordinates

$$\pi_2 : \begin{array}{ccc} \mathbb{C}^n & \to & \mathbb{C}^2 \\ (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) & \mapsto & (\boldsymbol{x}_1, \boldsymbol{x}_2) \end{array}$$

- $W(\pi_2, V)$ polar variety
- $F_2 = \pi_1^{-1}(\pi_1(K)) \cap V$ critical fibers
- K = critical points of π_1 on $W(\pi_2, V)$

Connectivity result [Canny, 1988]

If V is bounded, $W(\pi_2, V) \cup F_2$ has dimension $d - 1$
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On the extension of Canny's result

Projection on i coordinates

$$\pi_i : \begin{array}{ccc} \mathbb{C}^n & \to & \mathbb{C}^i \\ (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) & \mapsto & (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_i) \end{array}$$

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If V is bounded, $W(\pi_i, V) \cup F_i$ has dimension $\max(i-1, d-i+1)$
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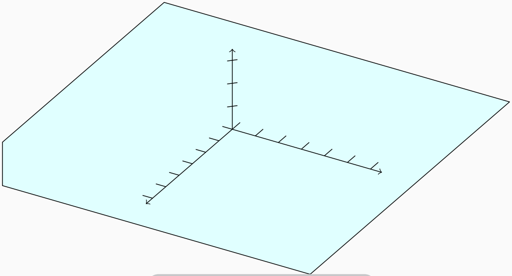


No critical points...

On the extension of Canny's result

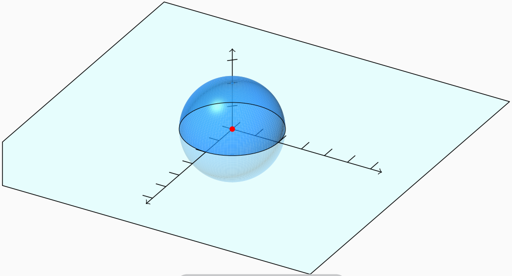
Non-negative <u>proper</u> polynomial map

$$\boldsymbol{\varphi}_i : \begin{array}{ccc} \mathbb{C}^n & \longrightarrow & \mathbb{C}^i \\ \boldsymbol{x} & \mapsto & (\psi_1(\boldsymbol{x}), \ldots, \psi_i(\boldsymbol{x})) \end{array}$$

- $W(\boldsymbol{\varphi}_i, V)$ generalized polar variety
- $F_i = \boldsymbol{\varphi}_{i-1}^{-1}(\boldsymbol{\varphi}_{i-1}(K)) \cap V$ critical fibers.
- K = critical points of $\boldsymbol{\varphi}_1$ on $W(\boldsymbol{\varphi}_i, V)$

Connectivity result [P. & Safey El Din & Schost, 2024] NEW!

~~If V is bounded,~~ $W(\boldsymbol{\varphi}_i, V) \cup F_i$ has dimension $\max(i-1, d-i+1)$ and satisfies the Roadmap property



Critical point!

⇝ Sard's lemma

⇝ Thom's isotopy lemma

⇝ Puiseux series

How to use it?

Assumptions to satisfy in the new result

(R) $\operatorname{sing}(V)$ is finite

(P) φ_1 is a proper map bounded from below

For all $1 \leqslant i \leqslant \dim(V)/2$,

(N) φ_{i-1} has finite fibers on W_i

(W) $\dim W_i = i - 1$ and $\operatorname{sing}(W_i) \subset \operatorname{sing}(V)$

(F) $\dim F_i = n - d + 1$ and $\operatorname{sing}(F_i)$ is finite

How to use it?

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Assumption on the input

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By construction of φ

A successful candidate

Choose *generic* $(\boldsymbol{a}, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_n) \in \mathbb{R}^{n^2}$ and:

$$\boldsymbol{\varphi} = \left(\sum_{i=1}^{n} (x_i - a_i)^2 \,,\, \boldsymbol{b}_2^{\mathsf{T}} \vec{\boldsymbol{x}} \,,\, \ldots \,,\, \boldsymbol{b}_n^{\mathsf{T}} \vec{\boldsymbol{x}} \right) \quad \text{where} \quad a_i \in \mathbb{R}, \quad \boldsymbol{b}_i \in \mathbb{R}^n$$

It satisfies the assumptions! NEW!

How to use it?

Assumptions to satisfy in the new result

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Generalization of Noether position from [Safey El Din & Schost, 2003]

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Jacobian criterion

$\oplus$

Thom's transversality theorem

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It satisfies the assumptions! NEW!

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$\oplus$

Noether position

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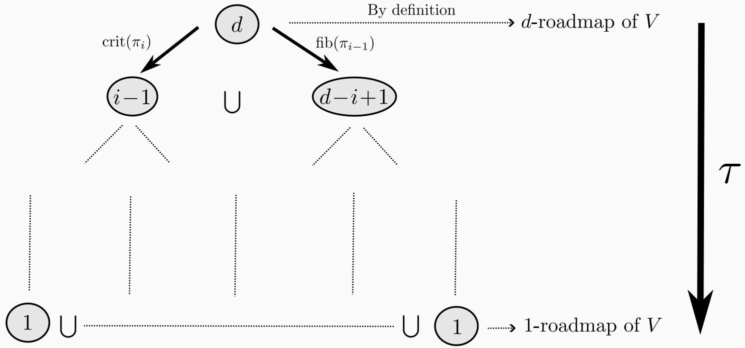
It satisfies the assumptions! NEW!

An algorithm for unbounded algebraic set

Consider an algebraic set $V \subset \mathbb{C}^n$ with dimension d



Depth of recursion tree : τ

$\Rightarrow$ complexity: $(nD)^{O(n\tau)}$

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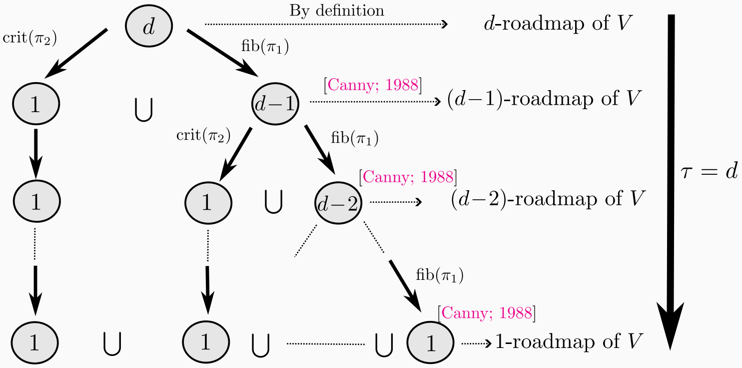

Depth of recursion tree : d

$\Rightarrow$ complexity: $(nD)^{O(nd)}$

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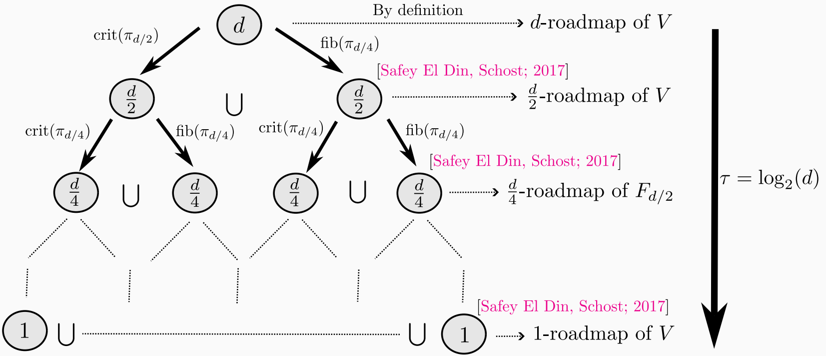

Depth of recursion tree : $\log_2(d)$

$\Rightarrow$ complexity: $(nD)^{O(n \log_2(d))}$

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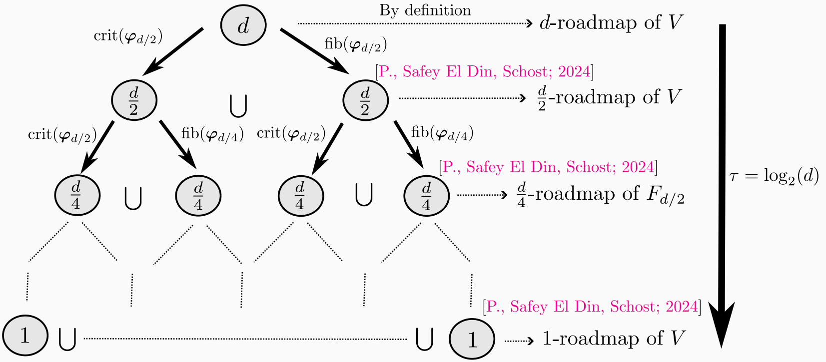

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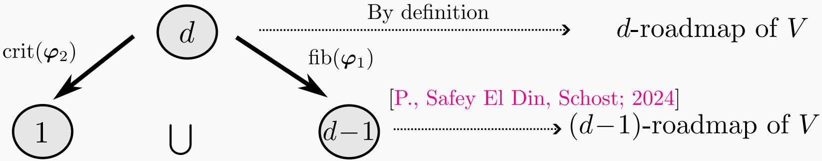

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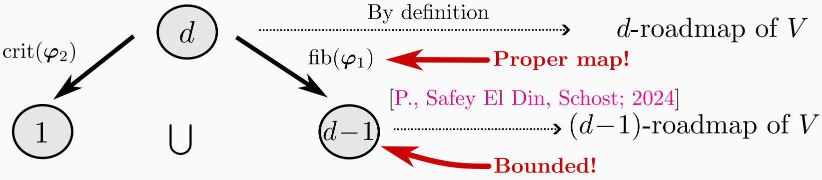

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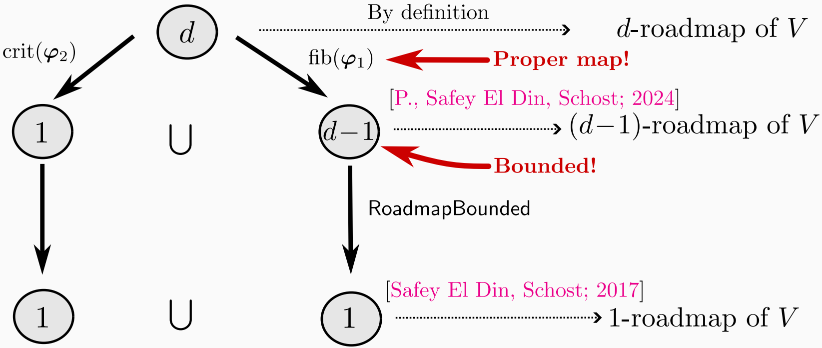

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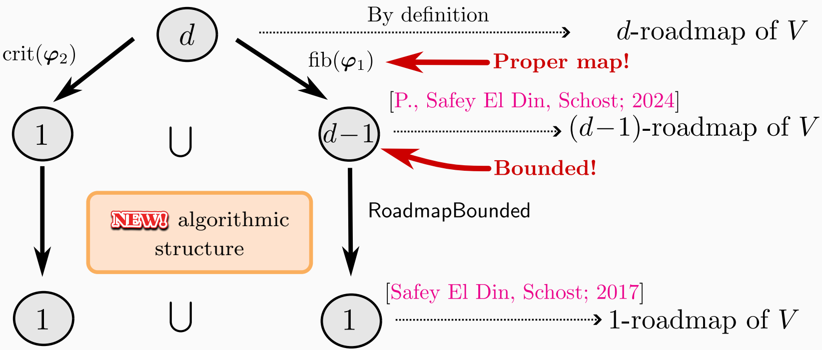
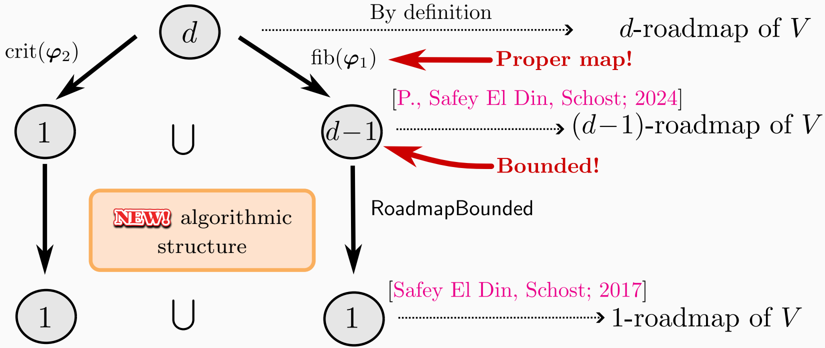


Quantitative estimate

	Output size	Complexity
RoadmapBounded(fib(φ_1)) Compute crit(φ_2) & fib(φ_1)		
Overall		

An algorithm for unbounded algebraic set

Consider an algebraic set $V \subset \mathbb{C}^n$ with dimension d

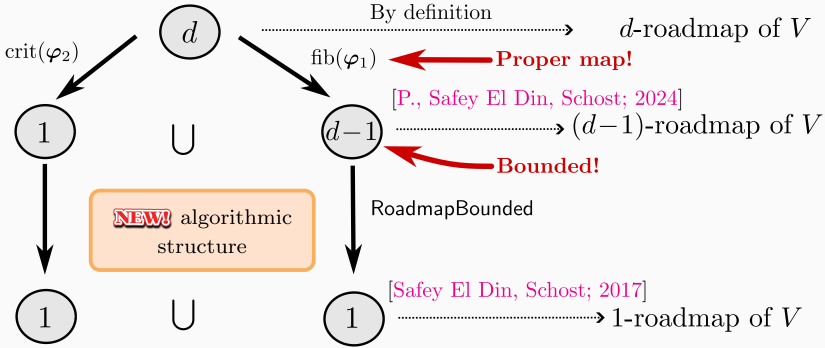


Quantitative estimate

	Output size	Complexity
RoadmapBounded(fib(φ_1)) Compute crit(φ_2) & fib(φ_1)	$(n^2 D)^{4n \log_2 d + O(n)}$	$(n^2 D)^{6n \log_2 d + O(n)}$
Overall		

An algorithm for unbounded algebraic set

Consider an algebraic set $V \subset \mathbb{C}^n$ with dimension d



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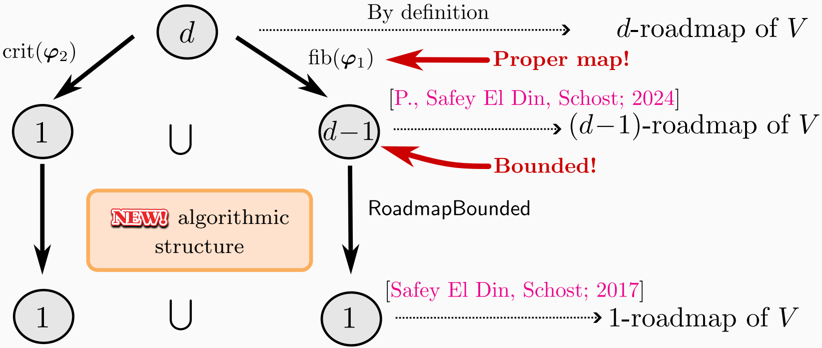

Quantitative estimate

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RoadmapBounded(fib(φ_1))	$(n^2D)^{4n\log_2 d+O(n)}$	$(n^2D)^{6n\log_2 d+O(n)}$
Compute crit(φ_2) & fib(φ_1)	$(nD)^{O(n)}$	$(nD)^{O(n)}$
Overall	$(n^2D)^{4n\log_2 d+O(n)}$	$(n^2D)^{6n\log_2 d+O(n)}$

Summary

Input

Polynomials in $\mathbb{Q}[x_1, \ldots x_n]$ of max degree D defining a smooth algebraic set of dim. d

Connectivity reduction process - before

Arbitrary dimension	$\overset{\text{Roadmap}}{\Longrightarrow}$	Dimension: 1	$\overset{\text{Topology}}{\Longrightarrow}$	Finite graph $\mathscr{G}$
	$\downarrow$		$\downarrow$	

Summary

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Arbitrary dimension	$\xrightarrow{\text{Roadmap}}$	Dimension: 1 Size: $(nD)^{O(n\log(n))}$	$\xrightarrow{\text{Topology}}$	Finite graph $\mathscr{G}$
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	$(nD)^{O(n\log^2(n))}$ [Basu, Roy; 2014]			

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Roadmap ↓ $(nD)^{O(n\log^2(n))}$ [Basu, Roy; 2014]

Topology ↓ $(\text{Size})^{O(1)} = (nD)^{O(n\log(n))}$ [Safey El Din, Schost; 2011]

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Arbitrary dimension $\xrightarrow{\text{Roadmap}}$ Dimension: 1, Size: $(n^2D)^{4n\log_2(d)+O(n)}$ $\Longrightarrow$ Finite graph $\mathscr{G}$

↓ $(n^2D)^{6n\log_2(d)+O(n)}$ [P., Safey El Din, Schost; 2024]

↓

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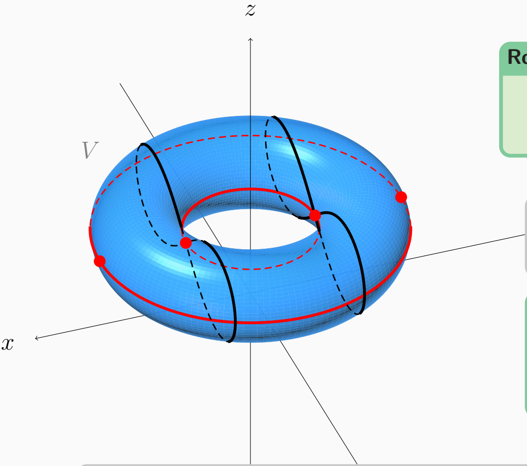
Computing roadmaps in unbounded smooth real algebraic sets I: connectivity results, 2024
with M. Safey El Din and É. Schost

Computing roadmaps in unbounded smooth real algebraic sets II: algorithm and complexity, 2024
with M. Safey El Din and É. Schost

Algorithm for connectivity queries on real algebraic curves, 2023
with Md N. Islam and A. Poteaux

Analysis of the kinematic singularities of a PUMA robot

with J.Capco, M.Safey El Din and P.Wenger



Roadmap property

$\forall C$ connected component,
$C \cap \mathscr{R}$ is non-empty and connected

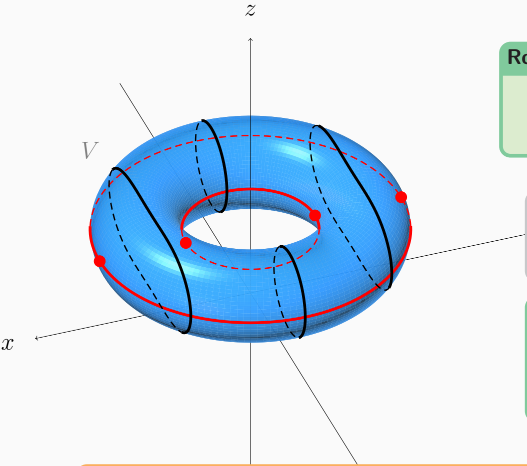
$\boldsymbol{W(\pi_2, V)}$ **polar variety**
$\boldsymbol{F}$ **critical fibers**

Genericity assumptions

1. $\boldsymbol{W(\pi_2, V)}$ has dimension 1
2. $\boldsymbol{F}$ has dimension $\dim(V) - 1$

Theorem [Canny, 1988]

If V is bounded, $\boldsymbol{W(\pi_2, V)} \bigcup \boldsymbol{F}$ has dimension $\dim(V) - 1$
and satisfies the Roadmap property



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$W(\pi_2, V)$ polar variety
F regular fibers

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2. F has dimension $\dim(V) - 1$

Theorem [Mezzaroba & Safey El Din, 2006]

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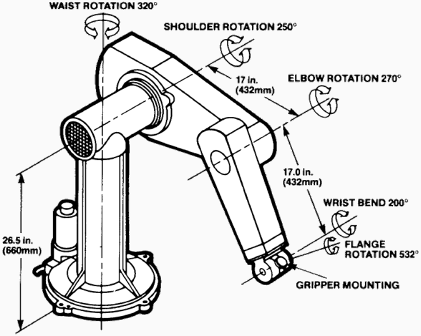
Roadmap computation for robotics

Matrix M associated to a PUMA-type robot with a non-zero offset in the wrist

$$\begin{bmatrix} (v_3+v_2)(1-v_2v_3) & 0 & A(\boldsymbol{v}) & d_3A(\boldsymbol{v}) & a_2(v_3^2+1)(v_2^2-1)-a_3A(\boldsymbol{v}) & 2d_3(v_3+v_2)(v_2v_3-1) \\ 0 & v_3^2+1 & 0 & 2a_2v_3 & 0 & (a_3-a_2)v_3^2+a_2+2a_3 \\ 0 & 1 & 0 & 0 & 0 & 2a_3 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ v_4 & 1-v_4^2 & 0 & d_4(1-v_4^2) & -2d_4v_4 & 0 \\ (v_4^2-1)v_5 & 4v_4v_5 & (1-v_5^2)(v_4^2+1) & (1-v_5^2)(v_4^2-1)d_5+4d_4v_4v_5 & 2d_5v_4(1-v_5^2)+2d_4v_5(1-v_4^2) & -2d_5v_5(v_4^2+1) \end{bmatrix}$$

`https://msolve.lip6.fr`

⇝ Multivariate system solving
⇝ Real roots isolation



A PUMA 560 [Unimation, 1984]

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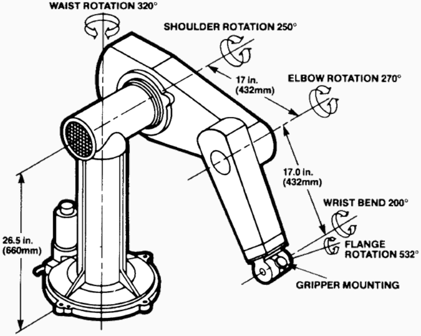
First step

Max. deg without splitting: **1858**

Locus	Degrees	$\mathbb{R}$-roots	Tot. time
Critical points	400 & 934	96 & 182	9.7 min
Critical curves	182 & 220	∞	3h46



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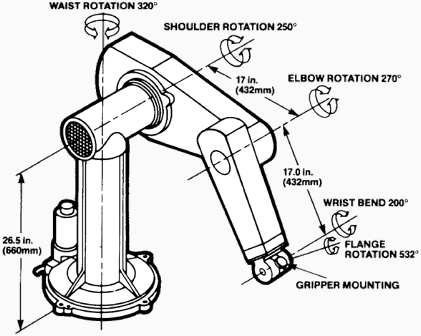

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Recursive step over 95 fibers

Data are for one fiber

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- ⇝ Multivariate system solving
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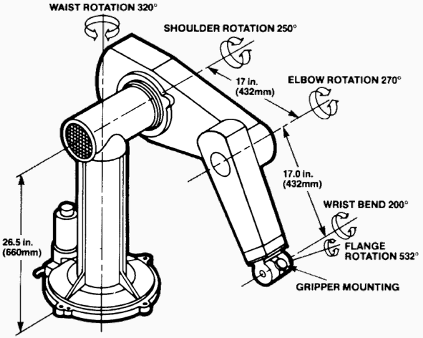

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Roadmap computation NEW!

Output degree: **4847**
Time: **4h10** (msolve)

Perspectives

Algorithms

Roadmap algorithms:

- Adapt the algorithms to structured systems: quadratic case (J.A.K.Elliott, M.Safey El Din, É.Schost)
- Generalize the connectivity result to semi-algebraic sets
- Design optimal roadmap algorithms with complexity exponential in $O(n)$

Connectivity of s.a. curves:

- Adapt to algebraic curves given as union (A.Poteaux)
- Generalize to semi-algebraic curves

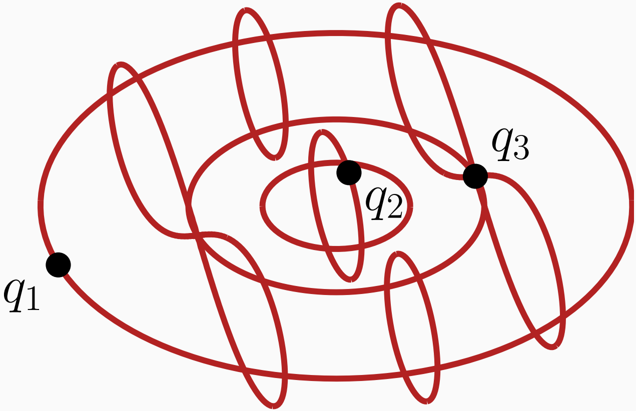
Applications

- Analyze challenging class of robots (D.Salunkhe, P.Wenger)
- Obtain practical version of modern roadmap algorithms

Software

- Curves: subresultant/GCD computations deg ~ 100 (now) $\to \sim 200$ (target)
- Build a Julia library for computational real algebraic geometry (C.Eder, R.Mohr)
- Implement a ready-to-use toolbox for roboticians

Union of curves



Reduce data size