[ddesolver] A Maple package for Discrete Differential Equations

Journées Nationales de Calcul Formel, 4-8 March 2024

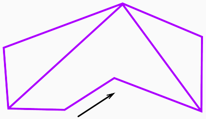
Hadrien Notarantonio (Inria Saclay – Sorbonne Université)

Implementation of a joint work with:

Alin Bostan (Inria Saclay)

Mohab Safey El Din (Sorbonne Université)

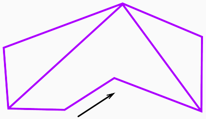
First nontrivial example: planar maps enumeration



$$F(t,u) = 1 + tu\left(uF(t,u)^2 + \frac{uF(t,u) - F(t,1)}{u-1}\right)$$

[Tutte '68]

First nontrivial example: planar maps enumeration



$$F(t,u) = 1 + tu\left(uF(t,u)^2 + \frac{uF(t,u) - F(t,1)}{u-1}\right)$$ [Tutte '68]

$a_n := \#$ {planar maps with n edges}

$\downarrow$ refinement

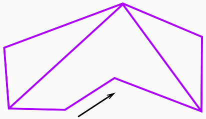
$a_{n,d} := \#${planar maps with n edges, d of them on the external face}

$\sum_{n=0}^{\infty} a_n t^n$ generating function

$\downarrow$ refinement

$F(t,u) := \sum_{n=0}^{\infty} \sum_{d=0}^{2n} a_{n,d} u^d t^n$ complete generating function

First nontrivial example: planar maps enumeration



$$F(t,u) = 1 + tu\left(uF(t,u)^2 + \frac{uF(t,u) - F(t,1)}{u-1}\right)$$ [Tutte '68]

$a_n := \#\ \{\text{planar maps with } n \text{ edges}\}$

↓ refinement

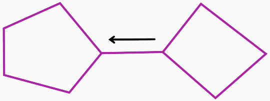
$a_{n,d} := \#\{\text{planar maps with } n \text{ edges}, d \text{ of them on the external face}\}$

$\sum_{n=0}^{\infty} a_n t^n$ generating function

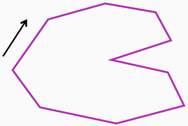
↓ refinement

$F(t,u) := \sum_{n=0}^{\infty} \sum_{d=0}^{2n} a_{n,d} u^d t^n$ complete generating function

1

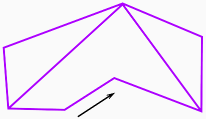


$tu^2 F(t,u)^2$



$tu \frac{uF(t,u) - F(t,1)}{u-1}$

First nontrivial example: planar maps enumeration



$$F(t,u) = 1 + tu\left(uF(t,u)^2 + \frac{uF(t,u) - F(t,1)}{u-1}\right)$$ [Tutte '68]

$a_n := \#\ \{\text{planar maps with } n \text{ edges}\}$

↓ refinement

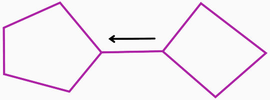
$a_{n,d} := \#\{\text{planar maps with } n \text{ edges, } d \text{ of them on the external face}\}$

$\sum_{n=0}^{\infty} a_n t^n$ generating function

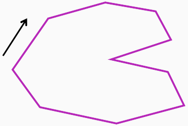
↓ refinement

$F(t,u) := \sum_{n=0}^{\infty} \sum_{d=0}^{2n} a_{n,d} u^d t^n$ complete generating function

1



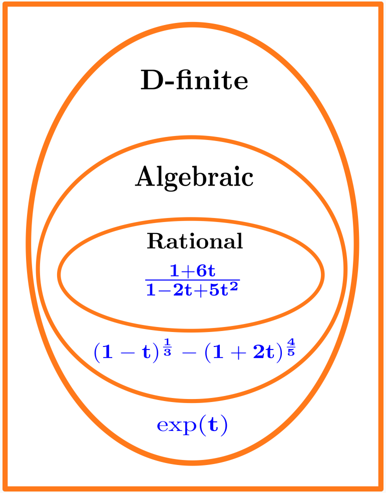
$tu^2 F(t,u)^2$



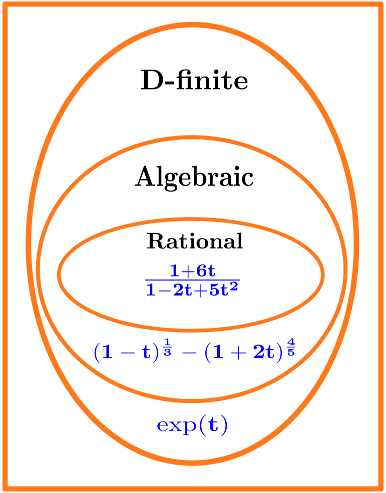
$tu\,\frac{uF(t,u) - F(t,1)}{u-1}$

$$F(t,1) = \sum_{n=0}^{\infty} a_n t^n$$

Solving functional equations



Solving functional equations



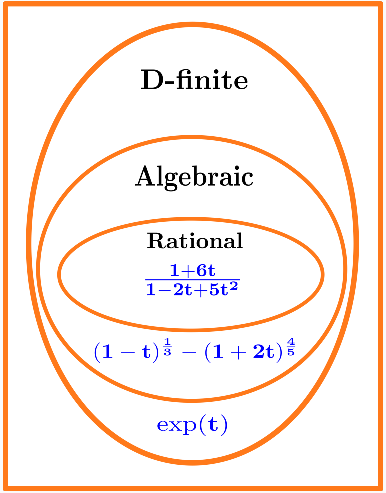
In this talk

Solving $=$ Classifying the initial series $F(t,1)$

$+$ Computing a witness of this classification

(e.g. $R \in \mathbb{Q}[z,t]$ s.t. $R(F(t,1),t) = 0$)

Solving functional equations



In this talk

Solving $=$ Classifying the initial series $F(t,1)$

$+$ Computing a witness of this classification

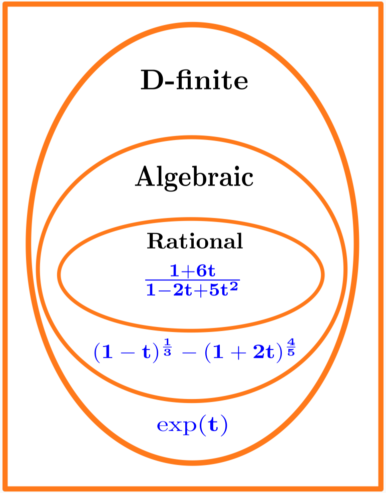
(e.g. $R \in \mathbb{Q}[z,t]$ s.t. $R(F(t,1),t) = 0$)

Going back to our planar maps...

$F(t,1) = 1 + 2t + 9t^2 + 54t^3 + 378t^4 + \cdots \in \mathbb{Q}[[t]]$

annihilated by $R = 27t^2z^2 + (1-18t)z + 16t - 1 \in \mathbb{Q}[z,t]$

Solving functional equations



In this talk

Solving $=$ Classifying the initial series $F(t,1)$

$+$ Computing a witness of this classification

(e.g. $R \in \mathbb{Q}[z,t]$ s.t. $R(F(t,1),t)=0$)

Going back to our planar maps...

$F(t,1) = 1 + 2t + 9t^2 + 54t^3 + 378t^4 + \cdots \quad \in \mathbb{Q}[[t]]$

annihilated by $R = 27t^2z^2 + (1-18t)z + 16t - 1 \in \mathbb{Q}[z,t]$

From R:

- **(Recurrence)** $a_0 = 1$ and $(n+3)a_{n+1} - 6(2n+1)a_n = 0$,
- **(Closed-form)** $a_n = 2\frac{3^n(2n)!}{n(n+2)!}$,
- **(Asymptotics)** $a_n \sim 2\frac{12^n}{\sqrt{\pi n^5}}$, when $n \to +\infty$.

Discrete Differential Equations

Definition

Given $f \in \mathbb{Q}[u]$, $k \geq 1$, and $Q \in \mathbb{Q}[y_0, \ldots, y_k, t, u]$,

$$F = f + t \cdot Q(F, \Delta F, \ldots, \Delta^k F, t, u) \qquad \textbf{(DDE)}$$

is a **Discrete Differential Equation**, where $\Delta : F \in \mathbb{Q}[u][[t]] \mapsto \frac{F(t,u) - F(t,1)}{u-1} \in \mathbb{Q}[u][[t]]$, and where for $\ell \geq 1$ we define $\Delta^{\ell+1} = \Delta^{\ell} \circ \Delta$.

Discrete Differential Equations

Definition

Given $f \in \mathbb{Q}[u]$, $k \geq 1$, and $Q \in \mathbb{Q}[y_0, \ldots, y_k, t, u]$,

$$F = f + t \cdot Q(F, \Delta F, \ldots, \Delta^k F, t, u) \qquad \textbf{(DDE)}$$

is a **Discrete Differential Equation**, where $\Delta : F \in \mathbb{Q}[u][[t]] \mapsto \frac{F(t,u) - F(t,1)}{u-1} \in \mathbb{Q}[u][[t]]$, and where for $\ell \geq 1$ we define $\Delta^{\ell+1} = \Delta^{\ell} \circ \Delta$.

Are 3-constellations of this shape? YES!

$$F(t,u) = 1 + tu\Bigg(F(t,u)^3 + (2F(t,u) + F(t,1))\frac{F(t,u) - F(t,1)}{u-1} + \frac{F(t,u) - F(t,1) - (u-1)\partial_u F(t,1)}{(u-1)^2}\Bigg)$$

Discrete Differential Equations

Definition

Given $f \in \mathbb{Q}[u]$, $k \geq 1$, and $Q \in \mathbb{Q}[y_0, \dots, y_k, t, u]$,

$$F = f + t \cdot Q(F, \Delta F, \dots, \Delta^k F, t, u) \qquad \textbf{(DDE)}$$

is a **Discrete Differential Equation**, where $\Delta : F \in \mathbb{Q}[u][[t]] \mapsto \frac{F(t,u) - F(t,1)}{u-1} \in \mathbb{Q}[u][[t]]$, and where for $\ell \geq 1$ we define $\Delta^{\ell+1} = \Delta^{\ell} \circ \Delta$.

Are 3-constellations of this shape? YES!

$$F(t,u) = 1 + tu\Bigg(F(t,u)^3 + (2F(t,u) + F(t,1))\frac{F(t,u) - F(t,1)}{u-1} + \frac{F(t,u) - F(t,1) - (u-1)\partial_u F(t,1)}{(u-1)^2}\Bigg)$$

Theorem
[Bousquet-Mélou, Jehanne '06]

The unique solution in $\mathbb{Q}[u][[t]]$ *of* **(DDE)** *is* **algebraic** *over* $\mathbb{Q}(t,u)$.

Bousquet-Mélou and Jehanne's trick

Input: $F(t,u) = 1 + tu\left(F(t,u)^3 + (2F(t,u) + F(t,1))\frac{F(t,u)-F(t,1)}{u-1} + \frac{F(t,u)-F(t,1)-(u-1)\partial_u F(t,1)}{(u-1)^2}\right),$

Output: $81t^2F(t,1)^3 - 9t(9t-2)F(t,1)^2 + (27t^2-66t+1)F(t,1) - 3t^2 + 47t - 1 = 0.$

Bousquet-Mélou and Jehanne's trick

Input: $F(t,u) = 1 + tu\left(F(t,u)^3 + (2F(t,u) + F(t,1))\frac{F(t,u)-F(t,1)}{u-1} + \frac{F(t,u)-F(t,1)-(u-1)\partial_u F(t,1)}{(u-1)^2}\right),$

Output: $81t^2F(t,1)^3 - 9t(9t-2)F(t,1)^2 + (27t^2 - 66t + 1)F(t,1) - 3t^2 + 47t - 1 = 0.$

- **Compute** $P \in \mathbb{Q}(t)[x, u, z_0, z_1]$ such that $P(F(t,u), u, F(t,1), \partial_u F(t,1)) = 0,$

Bousquet-Mélou and Jehanne's trick

Input: $F(t,u) = 1 + tu\left(F(t,u)^3 + (2F(t,u) + F(t,1))\frac{F(t,u)-F(t,1)}{u-1} + \frac{F(t,u)-F(t,1)-(u-1)\partial_u F(t,1)}{(u-1)^2}\right)$,

Output: $81t^2F(t,1)^3 - 9t(9t-2)F(t,1)^2 + (27t^2-66t+1)F(t,1) - 3t^2 + 47t - 1 = 0.$

- **Compute** $P \in \mathbb{Q}(t)[x,u,z_0,z_1]$ such that $P(F(t,u),u,F(t,1),\partial_u F(t,1)) = 0$,
- **Consider**

$$\partial_u F(t,u) \cdot \partial_x P(F(t,u),u,F(t,1),\partial_u F(t,1)) + \partial_u P(F(t,u),u,F(t,1),\partial_u F(t,1)) = 0,$$

Bousquet-Mélou and Jehanne's trick

Input: $F(t,u) = 1 + tu\left(F(t,u)^3 + (2F(t,u) + F(t,1))\frac{F(t,u)-F(t,1)}{u-1} + \frac{F(t,u)-F(t,1)-(u-1)\partial_u F(t,1)}{(u-1)^2}\right)$,

Output: $81t^2F(t,1)^3 - 9t(9t-2)F(t,1)^2 + (27t^2 - 66t + 1)F(t,1) - 3t^2 + 47t - 1 = 0$.

- **Compute** $P \in \mathbb{Q}(t)[x, u, z_0, z_1]$ such that $P(F(t,u), u, F(t,1), \partial_u F(t,1)) = 0$,
- **Consider**

$$\partial_u F(t,u) \cdot \partial_x P(F(t,u), u, F(t,1), \partial_u F(t,1)) + \partial_u P(F(t,u), u, F(t,1), \partial_u F(t,1)) = 0,$$

- **Show** that there exist distinct $U_1, U_2 \in \bigcup_{d \geq 1} \overline{\mathbb{Q}}[[t^{\frac{1}{d}}]]$ s.t. $\partial_x P(F(t,U_i), U_i, F(t,1), \partial_u F(t,1)) = 0$,

Bousquet-Mélou and Jehanne's trick

Input: $F(t,u) = 1 + tu\left(F(t,u)^3 + (2F(t,u) + F(t,1))\frac{F(t,u)-F(t,1)}{u-1} + \frac{F(t,u)-F(t,1)-(u-1)\partial_u F(t,1)}{(u-1)^2}\right),$

Output: $81t^2F(t,1)^3 - 9t(9t-2)F(t,1)^2 + (27t^2 - 66t + 1)F(t,1) - 3t^2 + 47t - 1 = 0.$

- **Compute** $P \in \mathbb{Q}(t)[x, u, z_0, z_1]$ such that $P(F(t,u), u, F(t,1), \partial_u F(t,1)) = 0,$
- **Consider**

$$\partial_u F(t,u) \cdot \partial_x P(F(t,u), u, F(t,1), \partial_u F(t,1)) + \partial_u P(F(t,u), u, F(t,1), \partial_u F(t,1)) = 0,$$

- **Show** that there exist distinct $U_1, U_2 \in \bigcup_{d \geq 1} \overline{\mathbb{Q}}[[t^{\frac{1}{d}}]]$ s.t. $\partial_x P(F(t,U_i), U_i, F(t,1), \partial_u F(t,1)) = 0,$

Let's do a break here... 
This problem is **highly** geometric!

Geometric modelling of the problem (with Bostan and Safey El Din)

There exist 2 solutions $(x, \mathbf{u}) \in \overline{\mathbb{Q}(t)}^2$ with **distinct** $\mathbf{u}$-coordinates to

$$\begin{cases} \mathbf{P}(\mathbf{x}, \mathbf{u}, \mathbf{F}(\mathbf{t}, \mathbf{1}), \partial_{\mathbf{u}}\mathbf{F}(\mathbf{t}, \mathbf{1})) = \mathbf{0}, \\ \partial_{\mathbf{x}}\mathbf{P}(\mathbf{x}, \mathbf{u}, \mathbf{F}(\mathbf{t}, \mathbf{1}), \partial_{\mathbf{u}}\mathbf{F}(\mathbf{t}, \mathbf{1})) = \mathbf{0}, \quad \mathbf{u} \neq \mathbf{1}, \\ \partial_{\mathbf{u}}\mathbf{P}(\mathbf{x}, \mathbf{u}, \mathbf{F}(\mathbf{t}, \mathbf{1}), \partial_{\mathbf{u}}\mathbf{F}(\mathbf{t}, \mathbf{1})) = \mathbf{0}. \end{cases}$$

There exist 2 solutions $(x, \mathbf{u}) \in \overline{\mathbb{Q}(t)}^2$ with **distinct** $\mathbf{u}$-coordinates to

$$\begin{cases} \mathbf{P}(\mathbf{x}, \mathbf{u}, \mathbf{F}(\mathbf{t}, \mathbf{1}), \partial_{\mathbf{u}}\mathbf{F}(\mathbf{t}, \mathbf{1})) = \mathbf{0}, \\ \partial_{\mathbf{x}}\mathbf{P}(\mathbf{x}, \mathbf{u}, \mathbf{F}(\mathbf{t}, \mathbf{1}), \partial_{\mathbf{u}}\mathbf{F}(\mathbf{t}, \mathbf{1})) = \mathbf{0}, \quad \mathbf{u} \neq \mathbf{1}, \\ \partial_{\mathbf{u}}\mathbf{P}(\mathbf{x}, \mathbf{u}, \mathbf{F}(\mathbf{t}, \mathbf{1}), \partial_{\mathbf{u}}\mathbf{F}(\mathbf{t}, \mathbf{1})) = \mathbf{0}. \end{cases}$$

$$\pi_x : (x, \mathbf{u}, z_0, z_1) \in \overline{\mathbb{Q}(t)}^4 \mapsto (\mathbf{u}, z_0, z_1) \in \overline{\mathbb{Q}(t)}^3,$$

$$\mathbf{W} := \pi_x(V(\mathbf{P}, \partial_{\mathbf{x}}\mathbf{P}, \partial_{\mathbf{u}}\mathbf{P}) \setminus V(\mathbf{u} - 1))$$

$$\pi_u : (\mathbf{u}, z_0, z_1) \in \overline{\mathbb{Q}(t)}^3 \mapsto (z_0, z_1) \in \overline{\mathbb{Q}(t)}^2,$$

Geometric modelling of the problem (with Bostan and Safey El Din)

There exist 2 solutions $(x, \mathbf{u}) \in \overline{\mathbb{Q}(t)}^2$ with **distinct** $\mathbf{u}$-coordinates to

$$\begin{cases} \mathbf{P}(\mathbf{x}, \mathbf{u}, \mathbf{F}(\mathbf{t}, \mathbf{1}), \partial_\mathbf{u}\mathbf{F}(\mathbf{t}, \mathbf{1})) = \mathbf{0}, \\ \partial_\mathbf{x}\mathbf{P}(\mathbf{x}, \mathbf{u}, \mathbf{F}(\mathbf{t}, \mathbf{1}), \partial_\mathbf{u}\mathbf{F}(\mathbf{t}, \mathbf{1})) = \mathbf{0}, \quad \mathbf{u} \neq \mathbf{1}, \\ \partial_\mathbf{u}\mathbf{P}(\mathbf{x}, \mathbf{u}, \mathbf{F}(\mathbf{t}, \mathbf{1}), \partial_\mathbf{u}\mathbf{F}(\mathbf{t}, \mathbf{1})) = \mathbf{0}. \end{cases}$$

$$\pi_x : (x, \mathbf{u}, z_0, z_1) \in \overline{\mathbb{Q}(t)}^4 \mapsto (\mathbf{u}, z_0, z_1) \in \overline{\mathbb{Q}(t)}^3,$$

$$\mathbf{W} := \pi_x(V(\mathbf{P}, \partial_\mathbf{x}\mathbf{P}, \partial_\mathbf{u}\mathbf{P}) \setminus V(\mathbf{u} - 1))$$

$$\pi_u : (\mathbf{u}, z_0, z_1) \in \overline{\mathbb{Q}(t)}^3 \mapsto (z_0, z_1) \in \overline{\mathbb{Q}(t)}^2,$$

Characterize with polynomial constraints

$$\mathcal{F}_2 := \{\alpha_{\underline{z}} \in \overline{\mathbb{Q}(t)}^2 \mid \# \, \pi_u^{-1}(\alpha_{\underline{z}}) \cap \mathbf{W} \geq 2\}$$

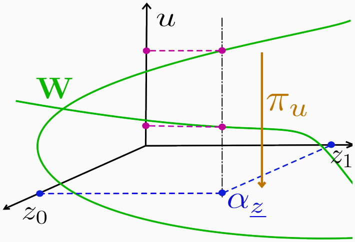
There exist 2 solutions $(x, \mathbf{u}) \in \overline{\mathbb{Q}(t)}^2$ with **distinct** $\mathbf{u}$-coordinates to

$$\begin{cases} \mathbf{P}(\mathbf{x}, \mathbf{u}, \mathbf{F}(\mathbf{t}, \mathbf{1}), \partial_{\mathbf{u}}\mathbf{F}(\mathbf{t}, \mathbf{1})) = \mathbf{0}, & \\ \partial_{\mathbf{x}}\mathbf{P}(\mathbf{x}, \mathbf{u}, \mathbf{F}(\mathbf{t}, \mathbf{1}), \partial_{\mathbf{u}}\mathbf{F}(\mathbf{t}, \mathbf{1})) = \mathbf{0}, & \mathbf{u} \neq \mathbf{1}, \\ \partial_{\mathbf{u}}\mathbf{P}(\mathbf{x}, \mathbf{u}, \mathbf{F}(\mathbf{t}, \mathbf{1}), \partial_{\mathbf{u}}\mathbf{F}(\mathbf{t}, \mathbf{1})) = \mathbf{0}. & \end{cases}$$

$$\pi_x : (x, \mathbf{u}, z_0, z_1) \in \overline{\mathbb{Q}(t)}^4 \mapsto (\mathbf{u}, z_0, z_1) \in \overline{\mathbb{Q}(t)}^3,$$

$$\mathbf{W} := \pi_x(V(\mathbf{P}, \partial_{\mathbf{x}}\mathbf{P}, \partial_{\mathbf{u}}\mathbf{P}) \setminus V(\mathbf{u} - 1))$$

$$\pi_u : (\mathbf{u}, z_0, z_1) \in \overline{\mathbb{Q}(t)}^3 \mapsto (z_0, z_1) \in \overline{\mathbb{Q}(t)}^2,$$



$$\# \pi_u^{-1}(\alpha_{\underline{z}}) \cap \mathbf{W} = 2$$

Characterize with polynomial constraints

$$\mathcal{F}_2 := \{\alpha_{\underline{z}} \in \overline{\mathbb{Q}(t)}^2 \mid \# \pi_u^{-1}(\alpha_{\underline{z}}) \cap \mathbf{W} \geq 2\}$$

Solving a toy example... (with Bostan and Safey El Din)

Input: $F(t,u) = 1 + t\left(uF(t,u) + \frac{F(t,u)-F(t,0)-u\partial_u F(t,0)}{u^2}\right)$, **k = 2**

Output: $t^3 F(t,0)^3 - F(t,0) + 1 = 0.$

Solving a toy example... (with Bostan and Safey El Din)

Input: $F(t,u) = 1 + t\left(uF(t,u) + \frac{F(t,u) - F(t,0) - u\partial_u F(t,0)}{u^2}\right)$, **k = 2**

Output: $t^3 F(t,0)^3 - F(t,0) + 1 = 0.$

- **Compute** $P \in \mathbb{Q}(t)[x, u, z_0, z_1]$ such that $P(F(t,u), u, F(t,0), \partial_u F(t,0)) = 0$,

Solving a toy example... (with Bostan and Safey El Din)

Input: $F(t,u) = 1 + t\left(uF(t,u) + \frac{F(t,u) - F(t,0) - u\partial_u F(t,0)}{u^2}\right)$, **k = 2**

Output: $t^3 F(t,0)^3 - F(t,0) + 1 = 0.$

- **Compute** $P \in \mathbb{Q}(t)[x, u, z_0, z_1]$ such that $P(F(t,u), u, F(t,0), \partial_u F(t,0)) = 0$,
- **Compute** G_u Gröbner basis of $\langle P, \partial_1 P, \partial_2 P, m \cdot u - 1\rangle \cap \mathbb{Q}(t)[u, z_0, z_1]$ for $\{u\} \succ_{plex} \{z_0, z_1\}$:

Solving a toy example... (with Bostan and Safey El Din)

Input: $F(t,u) = 1 + t\left(uF(t,u) + \frac{F(t,u) - F(t,0) - u\partial_u F(t,0)}{u^2}\right),$ **k = 2**

Output: $t^3 F(t,0)^3 - F(t,0) + 1 = 0.$

- **Compute** $P \in \mathbb{Q}(t)[x, u, z_0, z_1]$ such that $P(F(t,u), u, F(t,0), \partial_u F(t,0)) = 0,$
- **Compute** G_u Gröbner basis of $\langle P, \partial_1 P, \partial_2 P, m \cdot u - 1\rangle \cap \mathbb{Q}(t)[u, z_0, z_1]$ for $\{u\} \succ_{plex} \{z_0, z_1\}$:

$$\mathbf{B_0} : \quad \gamma_0$$

$$\mathbf{B_1} : \begin{cases} \beta_1 \cdot u + \gamma_1 \\ \vdots \\ \beta_r \cdot u + \gamma_r \end{cases}, \gamma_i, \beta_j \in \mathbb{Q}(t)[z_0, z_1]$$

$$\mathbf{B_2} : \quad \mathbf{g_2} := u^2 + \beta_{r+1} \cdot u + \gamma_{r+1}$$

"At $\alpha \in \pi_u(V(G_u)) \subset \overline{\mathbb{Q}(t)}^2$, there exist two distinct solutions in u"

Input: $F(t,u) = 1 + t\left(uF(t,u) + \frac{F(t,u) - F(t,0) - u\partial_u F(t,0)}{u^2}\right)$, **k = 2**

Output: $t^3 F(t,0)^3 - F(t,0) + 1 = 0$.

- **Compute** $P \in \mathbb{Q}(t)[x,u,z_0,z_1]$ such that $P(F(t,u),u,F(t,0),\partial_u F(t,0)) = 0$,
- **Compute** G_u Gröbner basis of $\langle P, \partial_1 P, \partial_2 P, m\cdot u - 1\rangle \cap \mathbb{Q}(t)[u,z_0,z_1]$ for $\{u\} \succ_{plex} \{z_0,z_1\}$:

$\mathbf{B_0}$: γ_0

$\mathbf{B_1}$: $\left\{\begin{array}{c} \beta_1 \cdot u + \gamma_1 \\ \vdots \\ \beta_r \cdot u + \gamma_r \end{array}\right.$ $, \gamma_i, \beta_j \in \mathbb{Q}(t)[z_0,z_1]$

$\mathbf{B_2}$: $\mathbf{g_2} := u^2 + \beta_{r+1}\cdot u + \gamma_{r+1}$

"At $\alpha \in \pi_u(V(G_u)) \subset \overline{\mathbb{Q}(t)}^2$, there exist two distinct solutions in u"

At $\alpha \in V(G_u \cap \mathbb{K}[t,z_0,z_1])$ fixed, there exist two solutions in u
$\implies \beta_i, \gamma_j = 0$ (**equations**)

Input: $F(t,u) = 1 + t\left(uF(t,u) + \frac{F(t,u) - F(t,0) - u\partial_u F(t,0)}{u^2}\right)$, $\quad$ $\mathbf{k=2}$

Output: $t^3 F(t,0)^3 - F(t,0) + 1 = 0.$

- **Compute** $P \in \mathbb{Q}(t)[x, u, z_0, z_1]$ such that $P(F(t,u), u, F(t,0), \partial_u F(t,0)) = 0$,
- **Compute** G_u Gröbner basis of $\langle P, \partial_1 P, \partial_2 P, m \cdot u - 1\rangle \cap \mathbb{Q}(t)[u, z_0, z_1]$ for $\{u\} \succ_{plex} \{z_0, z_1\}$:

$\mathbf{B_0}$: γ_0

$\mathbf{B_1}$: $\begin{cases} \beta_1 \cdot u + \gamma_1 \\ \vdots \\ \beta_r \cdot u + \gamma_r \end{cases}$, $\gamma_i, \beta_j \in \mathbb{Q}(t)[z_0, z_1]$

$\mathbf{B_2}$: $\mathbf{g_2} := u^2 + \beta_{r+1} \cdot u + \gamma_{r+1}$

"At $\alpha \in \pi_u(V(G_u)) \subset \overline{\mathbb{Q}(t)}^2$, there exist two distinct solutions in u"

At $\alpha \in V(G_u \cap \mathbb{K}[t, z_0, z_1])$ fixed, there exist two solutions in u

$\implies \beta_i, \gamma_j = 0$ (**equations**)

[Extension theorem]

$\alpha \in \pi_u(V(G_u)) \implies \mathrm{LeadingCoeff}_u(\mathbf{g_2}) \neq 0$

Distinct solutions in $u \implies \mathrm{disc}_u(\mathbf{g_2}) \neq 0$ (**inequations**)

ddesolver: package presentation

ddesolver: package presentation

- **Maple package** dedicated to solving discrete differential equations,

ddesolver: package presentation

- **Maple package** dedicated to solving discrete differential equations,
- **Available** in a **Git repository**,

ddesolver: package presentation

- **Maple package** dedicated to solving discrete differential equations,
- **Available** in a **Git repository**,
- Relies on **evaluation–interpolation** and **fast multi-modular arithmetic**,

ddesolver: package presentation

- **Maple package** dedicated to solving discrete differential equations,
- **Available** in a **Git repository**,
- Relies on **evaluation–interpolation** and **fast multi-modular arithmetic**,
- **4 implemented algorithms** in a single function "annihilating_polynomial",

ddesolver: package presentation

- **Maple package** dedicated to solving discrete differential equations,
- **Available** in a **Git repository**,
- Relies on **evaluation–interpolation** and **fast multi-modular arithmetic**,
- **4 implemented algorithms** in a single function "annihilating_polynomial",
- **One of the algorithms** uses the Maple package **gfun** for guessing annihilating polynomials,

ddesolver: package presentation

- **Maple package** dedicated to solving discrete differential equations,
- **Available** in a **Git repository**,
- Relies on **evaluation–interpolation** and **fast multi-modular arithmetic**,
- **4 implemented algorithms** in a single function "annihilating_polynomial",
- **One of the algorithms** uses the Maple package **gfun** for guessing annihilating polynomials,
- **Can be coupled with libraries** for efficient Gröbner bases computations like the C library **msolve**.

Maple worksheet: default procedure of ddesolver

annihilating_polynomial

Input:

- $P \in \mathbb{Q}[x, z_0, \dots, z_{k-1}, t, u]$ the polynomial associated to the "numerator DDE",
- k the order of the DDE,
- $[x, z_0, \dots, z_{k-1}, t, u]$ the variables in P (this order matters!).

Output:

- $R \in \mathbb{Q}[t, z_0]$ annihilating the univariate series associated to z_0 in the DDE.

Maple worksheet: default procedure of ddesolver

annihilating_polynomial

Input:

- $P \in \mathbb{Q}[x, z_0, \dots, z_{k-1}, t, u]$ the polynomial associated to the "numerator DDE",
- k the order of the DDE,
- $[x, z_0, \dots, z_{k-1}, t, u]$ the variables in P (this order matters!).

Output:

- $R \in \mathbb{Q}[t, z_0]$ annihilating the univariate series associated to z_0 in the DDE.

- *with*(*ddesolver*);

  [annihilating_polynomial]

- $P := (u(u-1)^2x^3+2u(u-1)x^2-u(uz_0-z_0-1)x-u(uz_0^2+uz_1-z_0^2+z_0-z_1))t-(u-1)^2x+(u-1)^2 :$

Maple worksheet: default procedure of ddesolver

annihilating_polynomial

Input:

- $P \in \mathbb{Q}[x, z_0, \dots, z_{k-1}, t, u]$ the polynomial associated to the "numerator DDE",
- k the order of the DDE,
- $[x, z_0, \dots, z_{k-1}, t, u]$ the variables in P (this order matters!).

Output:

- $R \in \mathbb{Q}[t, z_0]$ annihilating the univariate series associated to z_0 in the DDE.

- $with(ddesolver);$

$$[\text{annihilating\_polynomial}]$$

- $P := (u(u-1)^2x^3+2u(u-1)x^2-u(uz_0-z_0-1)x-u(uz_0^2+uz_1-z_0^2+z_0-z_1))t-(u-1)^2x+(u-1)^2 :$
- annihilating_polynomial$(P, 2, [x, z_0, z_1, t, u]);$

$$(16tz_0^2-8tz_0+t-16)\cdot(81t^2z_0^3-81t^2z_0^2+27t^2z_0+18tz_0^2-3t^2-66tz_0+47t+z_0-1)$$

Maple worksheet: options of annihilating_polynomial

annihilating_polynomial

Input:

- $P \in \mathbb{Q}[x, z_0, \dots, z_{k-1}, t, u]$: polynomial associated to the "numerator DDE",
- k: order of the DDE,
- $[x, z_0, \dots, z_{k-1}, t, u]$ the variables in P (this order matters!),
- algorithm: to choose between {"duplication", "elimination", "geometry", "hybrid"},
- variable: variable among t and z_0 on which to perform evaluation–interpolation.

Output:

- $R \in \mathbb{Q}[t, z_0]$ annihilating the univariate series associated to z_0 in the DDE.

Maple worksheet: options of annihilating_polynomial

annihilating_polynomial

Input:

- $P \in \mathbb{Q}[x, z_0, \ldots, z_{k-1}, t, u]$: polynomial associated to the "numerator DDE",
- k: order of the DDE,
- $[x, z_0, \ldots, z_{k-1}, t, u]$ the variables in P (this order matters!),
- algorithm: to choose between {"duplication", "elimination", "geometry", "hybrid"},
- variable: variable among t and z_0 on which to perform evaluation–interpolation.

Output:

- $R \in \mathbb{Q}[t, z_0]$ annihilating the univariate series associated to z_0 in the DDE.

- annihilating_polynomial$(P, 2, [x, z_0, z_1, t, u], \text{"}\textit{elimination}\text{''}, t)$;

$$(16tz_0^2 - 8tz_0 + t - 16) \cdot (81t^2z_0^3 - 81t^2z_0^2 + 27t^2z_0 + 18tz_0^2 - 3t^2 - 66tz_0 + 47t + z_0 - 1)$$

- annihilating_polynomial$(P, 2, [x, z_0, z_1, t, u], \text{"}\textit{hybrid}\text{''}, t)$;

$$81t^2z_0^3 - 81t^2z_0^2 + 27t^2z_0 + 18tz_0^2 - 3t^2 - 66tz_0 + 47t + z_0 - 1$$

Timings with ddesolver

Data	[3]		[6]		[7]		[8]	
k	2		3		3		4	
variable	z_0	t	z_0	t	z_0	t	z_0	t
“duplication”	41m*	10m*	13h	27h	∞	∞	∞	∞
“elimination”	1h20m*	4m*	**2m20s**	35m	1m	7m30s	2d19h	2d
“geometry”	54m*	**2m***	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
“hybrid”	∞		1h42m		**34s**		**2h41m**	
$(\deg_t(R), \deg_{z_0}(R))$	$(132, 6)$		$(5, 16)$		$(2, 4)$		$(3, 9)$	

- [3]: Enumeration of non-separable near-triangulations in which all intern vertices have degree at least 5,
- [6]: Enumeration of 3-Tamari lattices,
- [7]: Enumeration of 3-greedy Tamari intervals,
- [8]: Enumeration of 5-constellations.

- ∞: computations did not finish within 5 days,
- $\times$: algorithm not implemented for $k > 2$,
- $\cdot^*$: added the constraint $(1+t^3)(1-t^3)t \neq 0$.

Intel® Xeon® Gold CPU 6246R v4 @ 3.40GHz, 1.5TB of RAM with 1 thread, Gröbner bases computations run with msolve.

Summary

- **ddesolver**: first Maple package dedicated to solving DDEs,
- Can be downloaded from the **git repository**,
- **Play with it** and feel free to report any bug!
- Having a hard computation even with this package? **Email me!**

https://github.com/HNotarantonio/ddesolver

Summary

- **ddesolver**: first Maple package dedicated to solving DDEs,
- Can be downloaded from the **git repository**,
- **Play with it** and feel free to report any bug!
- Having a hard computation even with this package? **Email me!**

https://github.com/HNotarantonio/ddesolver

Want to know more?

- *Fast Algorithms for Discrete Differential Equations* (*with Alin Bostan and Mohab Safey El Din*)
- *Effective algebraicity for solutions of systems of functional equations with one catalytic variable* (*with Sergey Yurkevich*)
- *Systems of Discrete Differential Equations, Constructive Algebraicity of the Solutions* (*with Sergey Yurkevich*)