Secure Computations on Shared Polynomials in MPC

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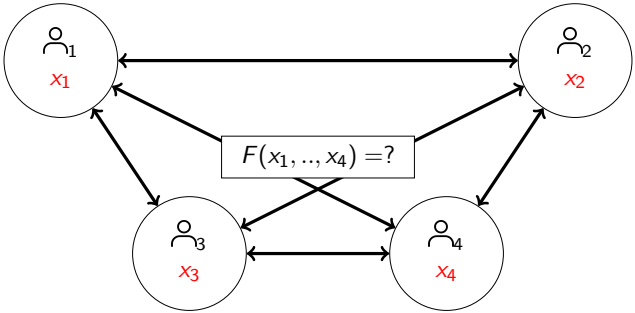
Multiparty Computation



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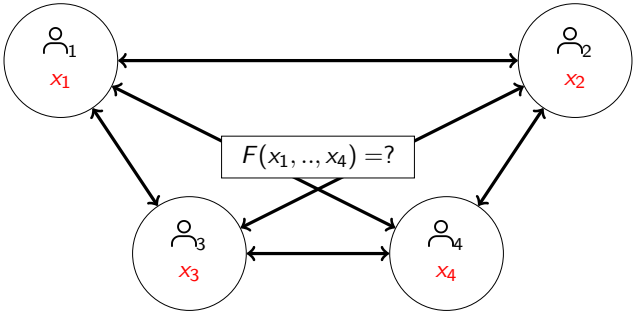
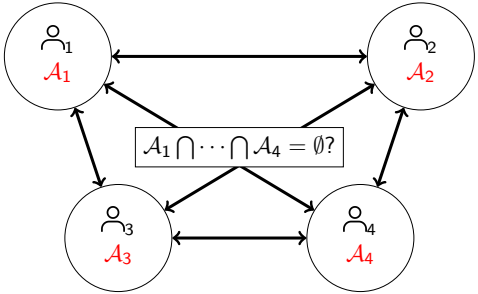


Motivation

Private Disjointess Test

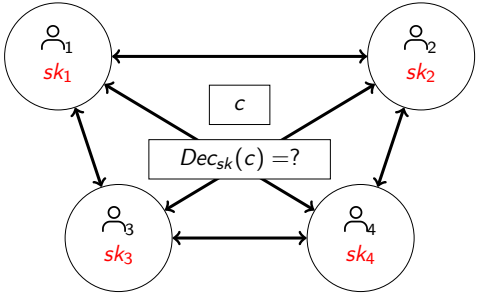


$P_{\mathcal{A}} = \prod_{\mathcal{A}}(X - \alpha)$

$P_{\cap} = \sum r_i P_{\mathcal{A}_i}$

$\Rightarrow$ Polynomial evaluation

Threshold McEliece Scheme

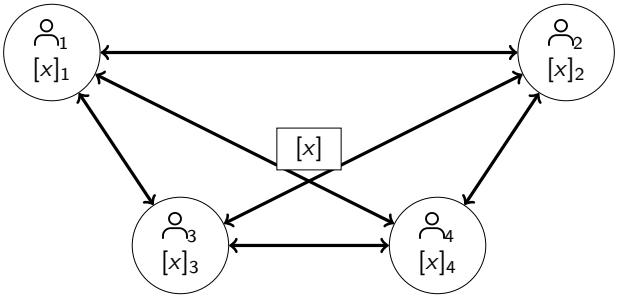


$S_c, g \rightarrow S_c^{-1} \mod g$

g irreducible

$\Rightarrow$ Modular inversion

Secret Sharing in MPC



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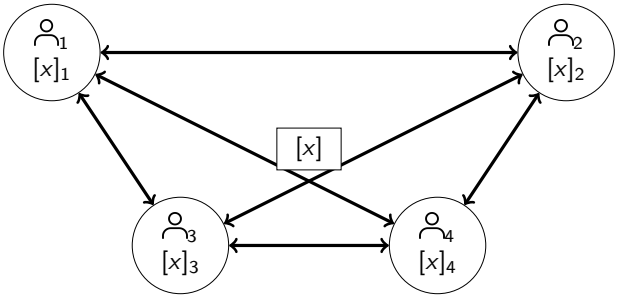

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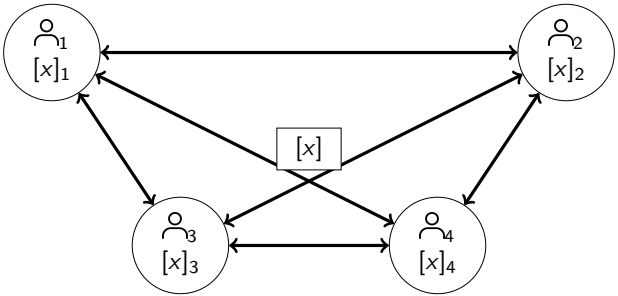

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- Goal: Faster solutions for specific problems

Computing in this model

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- Goal: For a given operation, as few sec. mult. as possible and constant round.

Previous work and our contribution

- [Mohassel, Franklin,2006]: specific operations on shared polynomials

	prod.	div.	fan-in prod.		interpol. + multi eval.		mod. inv.	
	[MF,06]		[MF,06]	Ours	[MF,06]	Ours	[MF,06]	Ours
Sec. mult.	$\mathcal{O}(d)$	$\mathcal{O}(d)$	$\mathcal{O}(n^2 d)$	$\mathcal{O}(\tau n^{1+1/\tau} d)$	$\mathcal{O}(d^2)$	$\mathcal{O}(\tau d^{1+1/\tau})$	$\mathcal{O}(d^3)$	$\mathcal{O}(d^2)$
Round	$\mathcal{O}(1)$	$\mathcal{O}(1)$	$\mathcal{O}(1)$	$\mathcal{O}(\tau)$	$\mathcal{O}(1)$	$\mathcal{O}(\tau)$	$\mathcal{O}(1)$	$\mathcal{O}(1)$

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Only secure multiplications!

$$[f], [g], (\mathsf{a_i})_{\leq 2d+1} \xrightarrow{local} [f(a_i)][g(a_i)] \xrightarrow{\mathcal{O}(d)} [f(a_i)g(a_i)] \xrightarrow{local} [fg]$$

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Fan-in mult. :
$[f_1], \ldots, [f_n] \rightarrow [f_1 \times \cdots \times f_n]$

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Evaluation on **shared** points :

$$[f], [\alpha_1], \ldots, [\alpha_d] \rightarrow [f(\alpha_1)], \ldots, [f(\alpha_d)]$$

Improved fan-in multiplication

$$\underline{\text{Op:}}\ [f_1], \ldots, [f_n]\ (\textbf{non-zero}\ \text{and}\ \deg < d) \longrightarrow\ [f_1 \times \cdots \times f_n]$$

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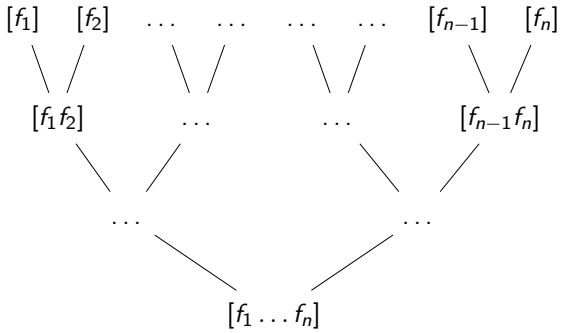
Binary tree



2 poly at a time

$\log n \times \mathcal{O}(nd)$ sec. mult.

$\mathcal{O}(\log n)$ rounds

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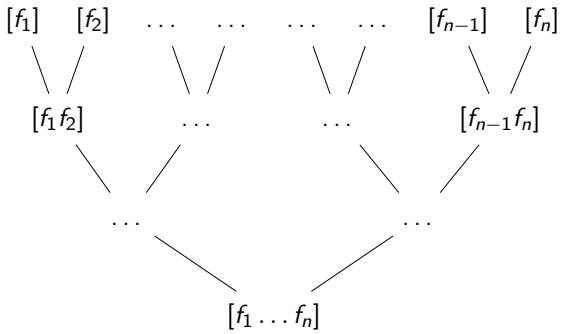

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[Mohassel, Franklin,2006]

$[f_i]$ = *element of* $F_q[X]/P$ ($\deg P \sim nd$)

1. $[p_i] = [r_{i-1}]^{-1}[f_i][r_i]$ *comm.*
   $[r_i] \leftarrow^{\$} (F_q[X]/P)^*$
2. $\prod p_i = \prod f_i \cdot r_n$ *local*
3. $\prod p_i [r_n^{-1}] = [\prod f_i]$ *local*

n poly at a time

$1 \times \mathcal{O}(n^2 d)$ sec. mult.

$\mathcal{O}(1)$ rounds

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Our sol: "Squished" tree

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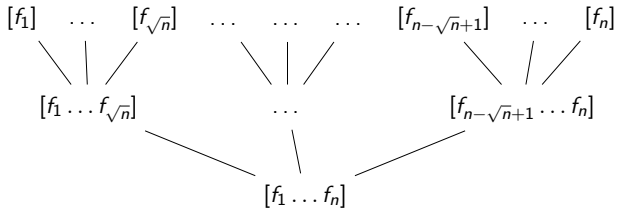

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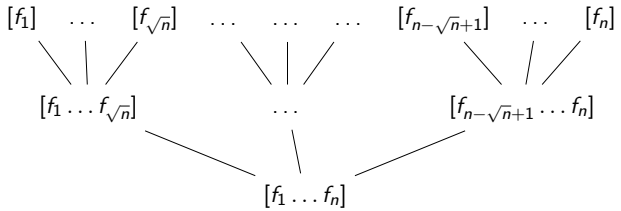

$\sqrt{n}$ poly at a time

$\sqrt{n} \times \mathcal{O}((\sqrt{n})^2 d) = \mathcal{O}(n^{3/2} d)$ sec. mult.

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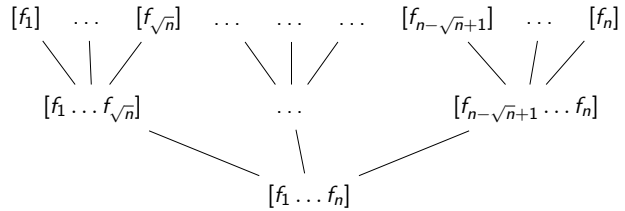

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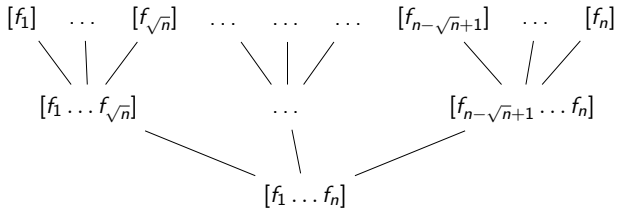

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- With same ideas, interpolation and multi-point evaluation on d shared points in $\mathcal{O}(\tau d^{1+1/\tau})$ sec. mult.

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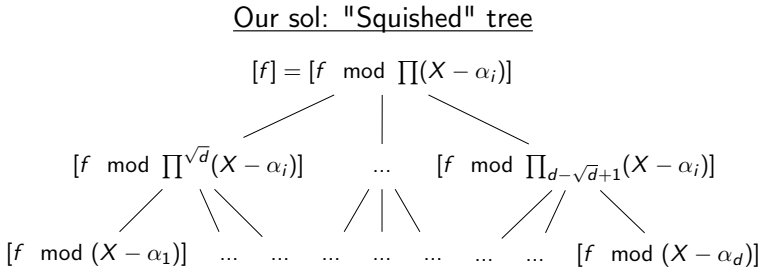

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$$\underline{\text{Op:}}\ [f], [g]\ (g \text{ irred and } \deg f < d, \deg g = d) \longrightarrow\ [f^{-1}]\ \mod [g]$$

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- [Mohassel, Franklin,2006]: Sylvester matrices, costly...

1. $[M] \leftarrow^{\$} (\mathbb{F}_q^{n \times n})^*$
2. $[A] \leftarrow [Syl(f, g)][M]$, reveal A
3. $[M]A^{-1} = [Syl(f, g)^{-1}]$

- $\mathcal{O}(d^3)$ sec. mult.

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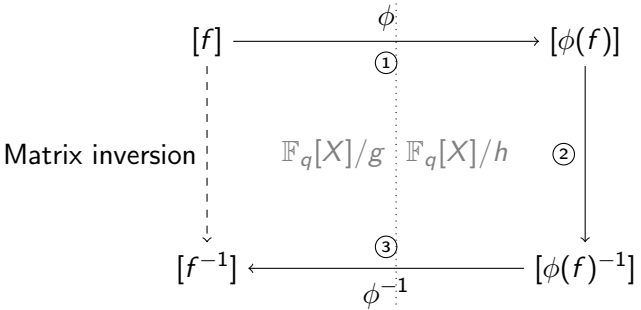

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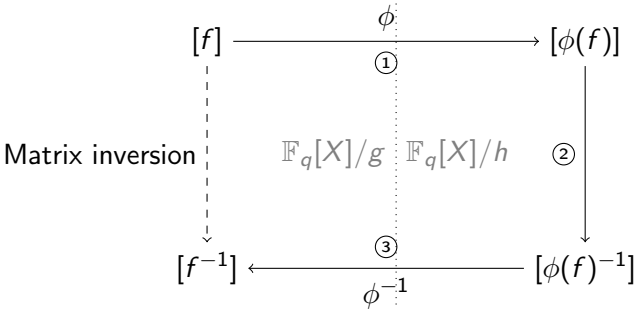

- Step 1: modular composition
  [Brent, Kung,1978]
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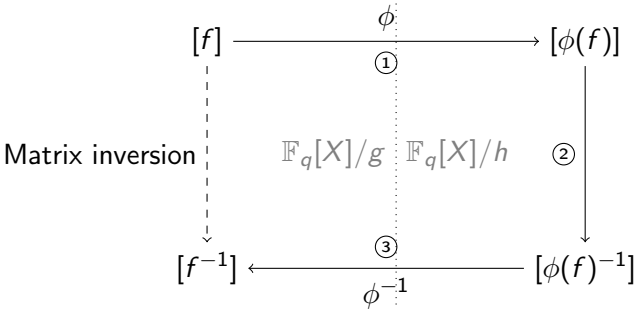

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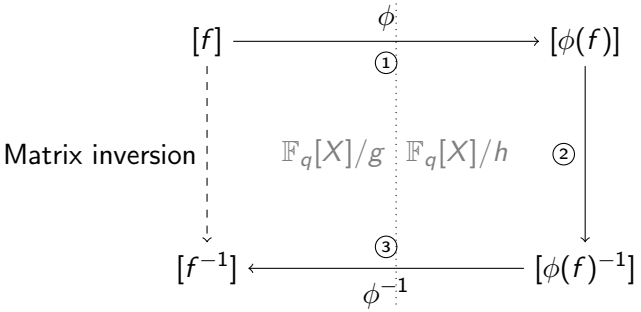

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  $\mathcal{O}(d)$ sec. mult.
- Step 3: Dual representation
  [De Feo, Doliskani, Schost,2014]
  $\mathcal{O}(d^2)$ sec. mult., $[1/g' \mod g]$

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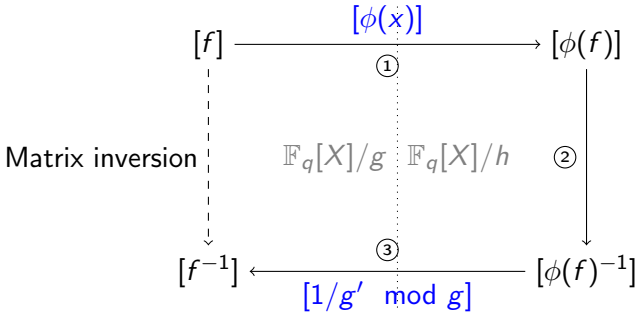

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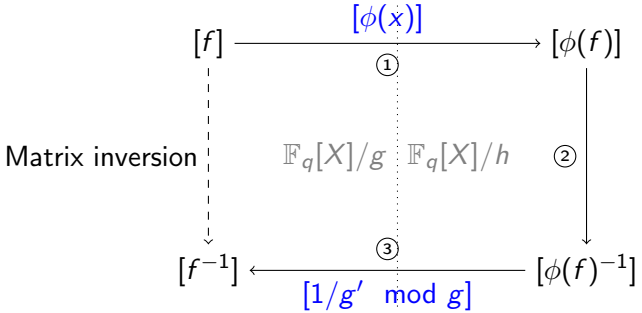

- $\mathcal{O}(d^3) \rightarrow \mathcal{O}(d^2)$ sec. mult. (with auxiliary inputs only on $[g]$)

Conclusion

- evaluation $\Rightarrow$ Private Disjointess Test (m people, n elements per set):

	Comm.	Rounds	crypto hyp.
[Sathya Narayanan et al. ,CANS2009]	$\mathcal{O}(mn^2)$	$\mathcal{O}(\log(mn))$	No
[Chandran et. al.,DCC2021]	$\mathcal{O}(mn\log^2(m))$	$\mathcal{O}(log(m\log(n)))$	Yes
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- Modular inversion $\Rightarrow$ Threshold McEliece Scheme:
  - Improvements from [Takahashi et al.,DCC2023]: shorter private key and some faster steps in decryption

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  - Improvements from [Takahashi et al.,DCC2023]: shorter private key and some faster steps in decryption
  - Future work: improvements on key generation and other steps of decryption
    $\Rightarrow$ generating shared ired. polynomial, rational reconstruction

Conclusion

- evaluation $\Rightarrow$ Private Disjointess Test (m people, n elements per set):

	Comm.	Rounds	crypto hyp.
[Sathya Narayanan et al. ,CANS2009]	$\mathcal{O}(mn^2)$	$\mathcal{O}(\log(mn))$	No
[Chandran et. al.,DCC2021]	$\mathcal{O}(mn\log^2(m))$	$\mathcal{O}(log(m\log(n)))$	Yes
Our protocol	$\mathcal{O}(nm + \tau n^{1+1/\tau})$	$\mathcal{O}(\tau)$	No

- Modular inversion $\Rightarrow$ Threshold McEliece Scheme:
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Thank you for your attention!