

A Reduction-Based Creative Telescoping Algorithm for Multiple Integrals

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A symbolic integration problem

$$\text{Let } I(t) = \int_{\gamma} \frac{dx dy dz}{1 - (1-xy)z - txyz(1-x)(1-y)(1-z)} \quad (\text{g.f. of Apéry numbers})$$

The objective is to compute a lin. ord. diff. eq. (LODE) for I :

$$t^2(t^2 - 34t + 1) \frac{\partial^3 I}{\partial t^3} + 3t(2t^2 - 51t + 1) \frac{\partial^2 I}{\partial t^2} + (7t^2 - 112t + 1) \frac{\partial I}{\partial t} + (t - 5)I = 0.$$

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With this LODE it is possible to

1. compute a series expansion,
2. evaluate the integral numerically,
3. prove identities involving $I(t)$.

Algebra of differential operators

Let D be the algebra $\mathbb{K}(t)[\underline{x}] \langle \partial_t, \underline{\partial}_x \rangle$ where each pair of variables commutes except $\partial_i x_i = x_i \partial_i + 1$, $\partial_t t = t \partial_t + 1$

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- It acts on C^∞ functions: $x_i \cdot f = x_i f$ and $\partial_i \cdot f = \frac{\partial f}{\partial x_i}$ (resp. t, ∂_t).
- It has a monomial basis as $\mathbb{K}(t)$ -vector space: $(\underline{x}^\alpha \underline{\partial}_x^\beta \partial_t^\gamma)_{\alpha, \beta, \gamma}$.
- It has a notion of degree:

$$\deg \left(\sum_{\alpha, \beta, \gamma} c_{\alpha, \beta, \gamma} \underline{x}^\alpha \underline{\partial}_x^\beta \partial_t^\gamma \right) = \max\{|\alpha| + |\beta| + \gamma \mid c_{\alpha, \beta, \gamma} \neq 0\}.$$

- Any left (resp. right) ideal I of D has a finite Gröbner basis.

Holonomy

Holonomic function

A function f is holonomic if for each ∂_i and for ∂_t it satisfies a LODE with polynomial coefficients.

Annihilator of f

The set $\text{Ann}(f) \stackrel{\text{def}}{=} \{L \in D \mid L \cdot f = 0\}$ is a left ideal

Link between functions and operators

f is characterized by its annihilator: $D \cdot f \simeq D / \text{Ann}(f)$.

Creative telescoping

Assumption: the integral of a derivative is zero

For any $b \in D$ and $i \in \llbracket 1, n \rrbracket$, $\int_{\gamma} \partial_i b \cdot f \underline{dx} = 0$.

Creative telescoping

Find $\ell \in \mathbb{N}$, $a_1, \dots, a_{\ell} \in \mathbb{K}(t)$ and $b_1, \dots, b_n \in D$ such that

$$\left(a_{\ell}(t) \partial_t^{\ell} + \dots + a_1(t) \partial_t + a_0(t) \right) \cdot f = \sum_{i=1}^n \partial_i b_i \cdot f$$

after integration we get

$$\left(a_{\ell}(t) \partial_t^{\ell} + \dots + a_1(t) \partial_t + a_0 \right) \cdot \int_{\gamma} f \underline{dx} = 0$$

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Compute the telescoping ideal $\mathcal{I}(f) = \left(\text{Ann}(f) + \sum_{i=1}^n \partial_i D \right) \cap \mathbb{K}(t) \langle \partial_t \rangle$

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Theorem

f holonomic $\implies \mathcal{I}(f)$ non-zero.

Takayama's algorithm for integration¹

Difficulty: $\text{Ann}(f) + \sum_{i=1}^n \partial_i D$ is not a left D -ideal.

But it is an infinite rank left $\mathbb{K}(t)\langle \partial_t, \underline{\partial_x} \rangle$ -module.

¹Takayama 1990, (improved in Chyzak-Salvy98 but not presented today)

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Takayama's algorithm

Input: $\text{Ann}(f)$

Output: non-trivial subset of $(\text{Ann}(f) + \sum_{i=1}^n \partial_i D) \cap \mathbb{K}(t)\langle \partial_t \rangle$

- Take generators of $\text{Ann}(f) + \sum_{i=1}^n \partial_i D$ up to a certain degree.
- Eliminate \underline{x} and $\underline{\partial}_x$ via Gröbner basis computation
- If we find an operator without \underline{x} and $\underline{\partial}_x$ return it otherwise increase the degree bound and start again.

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Creative telescoping using reductions

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- $a - [a] \in \text{Ann}(f) + \sum_{i=1}^n \partial_i D$ (reduction)
- $[a] = 0$ iff $a \in \text{Ann}(f) + \sum_{i=1}^n \partial_i D$ (canonical form)

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Creative telescoping algorithm:¹

- 1 $p_0 \leftarrow [a]; \ell \leftarrow 0$
- 2 **while** there is no $\mathbb{K}(t)$ -linear relation $\sum_{i=0}^{\ell} \lambda_i p_i = 0$
- 3 $p_{\ell+1} \leftarrow [\partial_t p_{\ell}]; \ell \leftarrow \ell + 1$
- 4 **return** $\sum_{i=0}^{\ell} \lambda_i \partial_t^i$

¹Bostan, Chen, Chyzak, Li, Xin

A first “naïve” reduction

Algorithm $[\cdot]_{GD}$:

```
1 repeat  
2    $a \leftarrow RRem(a, \sum_{i=1}^n \partial_i D)$   
3    $a \leftarrow LRem(a, Ann(f))$   
4 until no term in  $a$  can be further reduced  
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The Griffiths-Dwork reduction is a special case of the above algorithm.

Theorem (Dwork 62, Griffiths 69)

If $f = e^q$ with $q \in \mathbb{K}(t)[\underline{x}]$ homogeneous and $V(q)$ is smooth then $[\cdot]_{GD}$ is a canonical form.

A more effective reduction

Precomputation (reminiscent of Takayama's algorithm)

- 1 $B_\sigma \leftarrow \emptyset$
- 2 **for each** gen. g of $\text{Ann}(f) + \sum_{i=1}^n \partial_i D$ as $\mathbb{K}(t)\langle \partial_t, \underline{\partial}_x \rangle$ -module
s.t. $\deg_{\underline{x}}(g) \leq \sigma$
- 3 $B_\sigma \leftarrow B_\sigma \cup \{[g]_{GD}\}$
- 4 $B_\sigma \leftarrow GB(B_\sigma)$ over $\mathbb{K}(t)\langle \partial_t \rangle$
- 5 **return** B_σ

Algorithm $[\cdot]_{\mathcal{T}}^\sigma$:

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Theorem

For any $a \in \text{Ann}(f) + \sum_{i=1}^n \partial_i D$, $[a]_{\mathcal{T}}^\sigma = 0$ for σ large enough.

Future work

- Algorithm to find a good σ (done)
- Variant that uses evaluation/interpolation (WIP)
- Implementation in Julia (WIP)