On Bounding The Degree Of Irreducible Darboux Polynomials

Maxime Bridoux

Inria Rennes, Hycomes

Joint work with Khalil Ghorbal
JNCF 2024

Darboux Polynomials

To a system of polynomial ODEs:

$$\left\{ \begin{array}{ll} \dot{x_1} & = f_1(x_1, \ldots, x_n) \\ & \ldots \\ \dot{x_n} & = f_n(x_1, \ldots, x_n) \end{array} \right. \qquad \text{s.t.} \qquad f_1, \ldots, f_n \in k[x_1, \ldots, x_n]$$

we associate a differential operator $D = \sum_{i=1}^{n} f_i . \partial_i$

We say that $p \in k[x]$ is a *Darboux polynomial* for D if there is $q \in k[x]$ such that :

$$D(p) = q.p$$

Mains Applications

- generation of first integrals (Prelle-Singer)
- verification of dynamical and hybrid system

Motivation

Let $\mu \in \mathbb{N}^*$, the following dynamics

$$\begin{cases} \dot{x_1} &= \mu x_1 &= f_1(x_1, x_2) \\ \dot{x_2} &= x_2 &= f_2(x_1, x_2) \end{cases}$$

has $x_1 - x_2^{\mu}$ as an irreducible Darboux polynomial

Upper bound on irreducible Darboux polynomials

The upper bound of the total degree of an irreducible Darboux polynomial does not only depend on $\max_i \deg f_i$, but also on the coefficients of the polynomials f_i.

Every Darboux polynomial generation algorithm requires a fixed bound on the total degree as an input (that is eventually incremented)

Multivariate Division

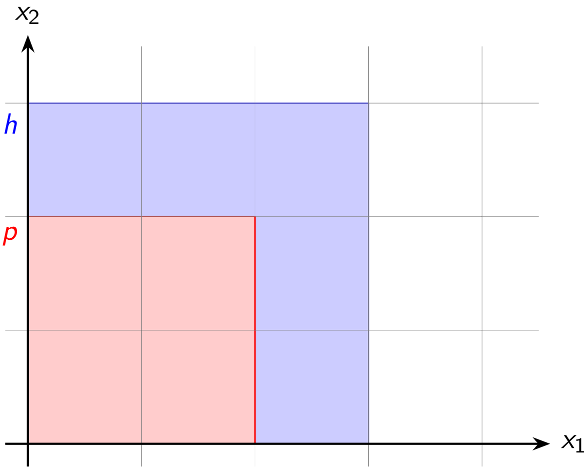
Our idea

Perform the reduction of $D(p)$ w.r.t p so that $D(p) = q.p + r$ (which requires a monomial order), but such that $\mathrm{LT}(p) = x^d$ is symbolic. We then solve $r = 0$ for the multidegree d and the coefficients of p

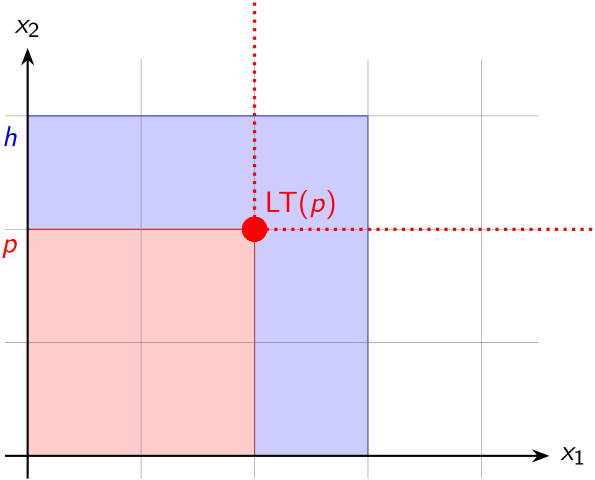
No computer algebra system implements polynomial reduction where $\mathrm{LT}(p)$ is symbolic!

Usually, p is fixed with a certain shape first and $\mathrm{LT}(p)$ follows from the choice of a monomial order (Mann, 1984). On the contrary, we fix a symbolic $\mathrm{LT}(p) = x^d$ and the shape of p then follows from the choice of a monomial order.

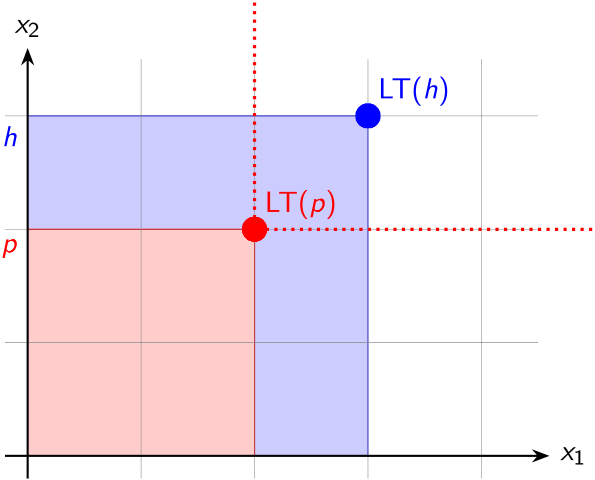
Polynomial reduction of h w.r.t p



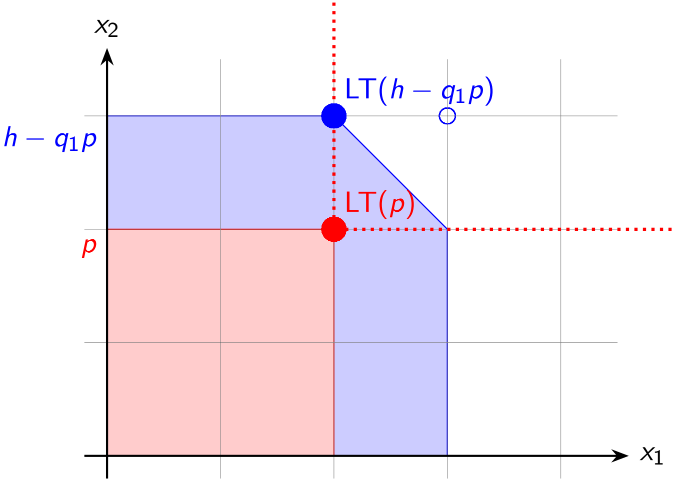
Polynomial reduction of h w.r.t p



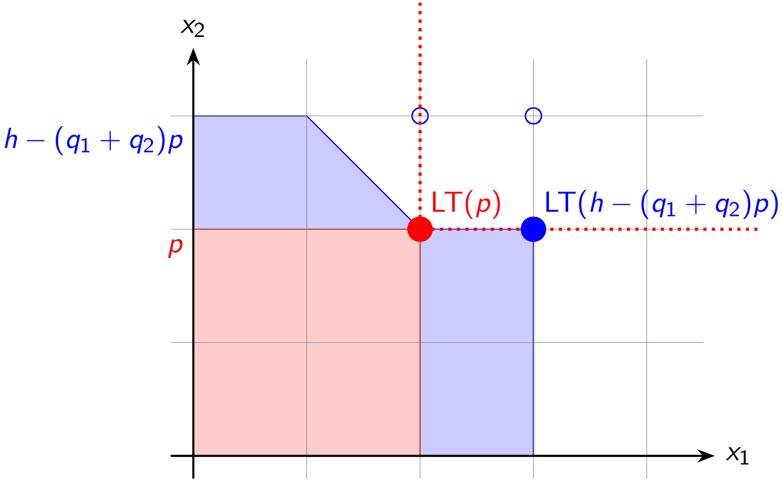
Polynomial reduction of h w.r.t p



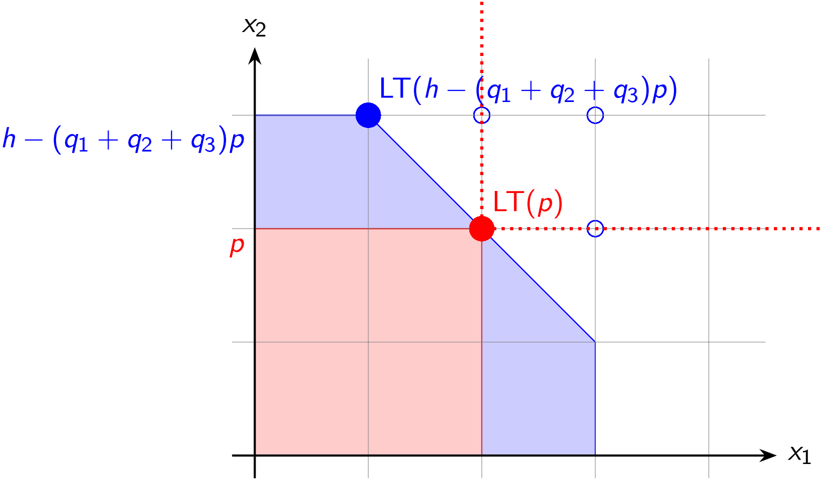
Polynomial reduction of h w.r.t p



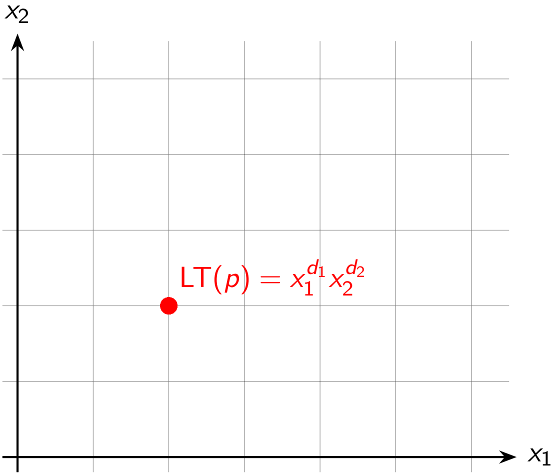
Polynomial reduction of h w.r.t p



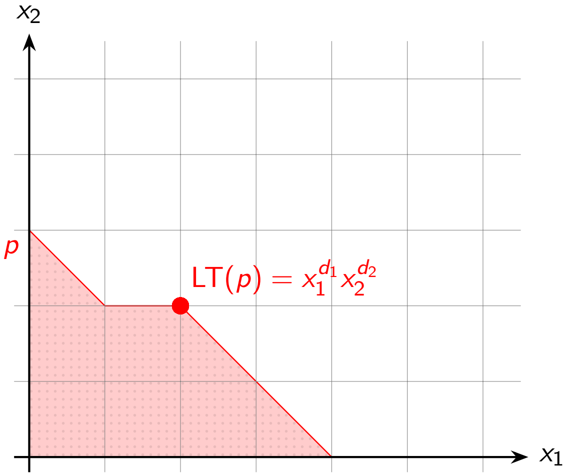
Polynomial reduction of h w.r.t p



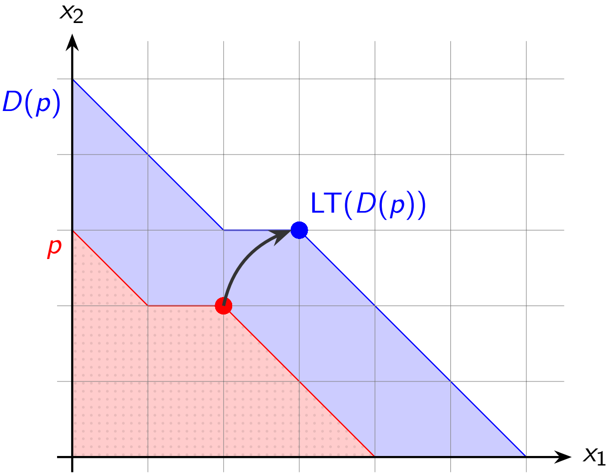
Polynomial reduction with $\mathrm{LT}(p)$ symbolic



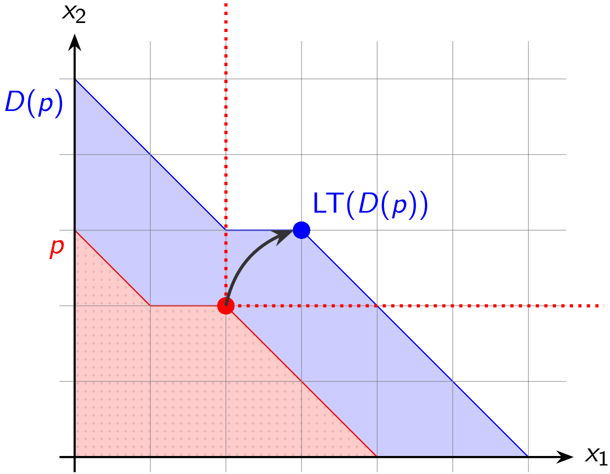
Polynomial reduction with LT(p) symbolic



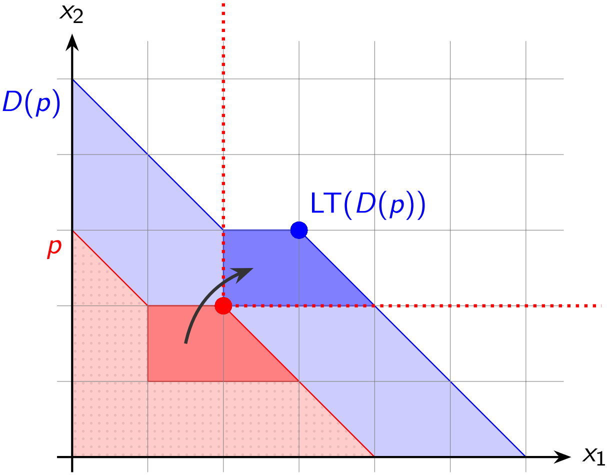
Polynomial reduction with LT(p) symbolic



Polynomial reduction with LT(p) symbolic

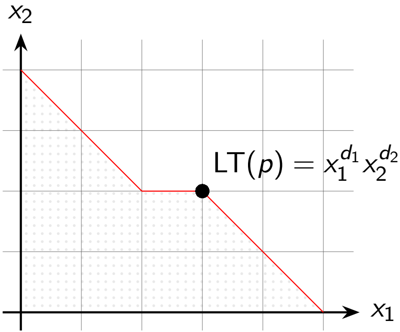


Polynomial reduction with LT(p) symbolic



Van der Pol

We fix $D = x_2\partial_1 + ((1 - x_1^2)x_2 - x_1)\partial_2$ and $\mathrm{LT}(p) = x_1^{d_1}x_2^{d_2}$.
Using the $\mathtt{DLex}_{x_2>x_1}$ order, the generic shape of p is:

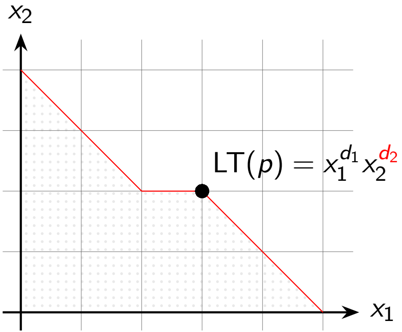


Generic Quotient for VDP

$$
\begin{aligned}
q = \; & -d_2x_1^2 \\
& -a_{d_1-2,d_2+1}x_2 \\
& +q_{x_1}x_1 \\
& +q_0
\end{aligned}
$$

Van der Pol

We fix $D = x_2\partial_1 + ((1 - x_1^2)x_2 - x_1)\partial_2$ and $\mathrm{LT}(p) = x_1^{d_1}x_2^{d_2}$.
Using the $\mathtt{DLex}_{x_2>x_1}$ order, the generic shape of p is:

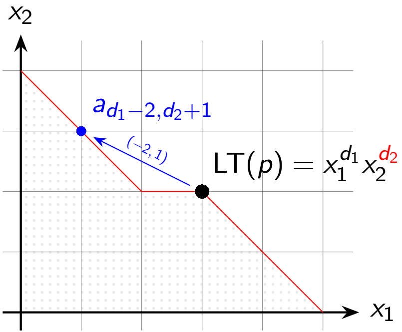


Generic Quotient for VDP

$$q = -d_2x_1^2 - a_{d_1-2,d_2+1}x_2 + q_{x_1}x_1 + q_0$$

Van der Pol

We fix $D = x_2\partial_1 + ((1 - x_1^2)x_2 - x_1)\partial_2$ and $\mathrm{LT}(p) = x_1^{d_1}x_2^{d_2}$.
Using the $\mathtt{DLex}_{x_2>x_1}$ order, the generic shape of p is:



Generic Quotient for VDP

$$q = -d_2x_1^2 - a_{d_1-2,d_2+1}x_2 + q_{x_1}x_1 + q_0$$

The coefficients of the generic quotient are polynomials in d_1, d_2 and coefficients of p that are local to $\mathrm{LT}(p)$

Impact of Monomial Orders

Definition

From a derivation $D = \sum_{i=1}^{n} f_i \partial_i$ we define the polynomial $s = \sum_{i=1}^{n} \lambda_i f_i \frac{x}{x_i}$ (where λ_i are symbolic coefficients)

Van der Pol

For $D = x_2 \partial_1 + ((1 - x_1^2)x_2 - x_1)\partial_2$, the associated polynomial s is

$$\lambda_1 x_2^2 + \lambda_2 (x_2 - x_1^2 x_2 - x_1) x_1$$

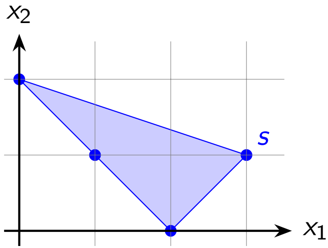
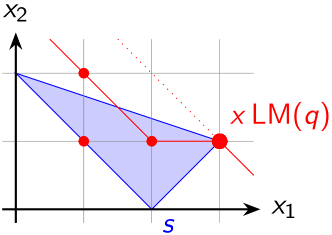


Figure: Newton polytope of s associated with VDP

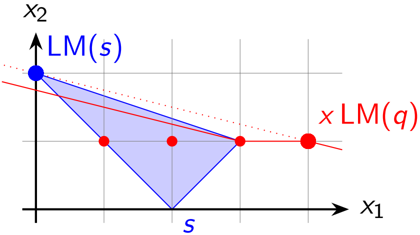
Proposition

Let q be the quotient of $D(p)$ w.r.t p for a monomial order $>$ and let s be the polynomial $s = \sum_{i=1}^{n} \lambda_i f_i \frac{x}{x_i}$ (where λ_i are symbolic coefficients). Then :

$$x\,\mathrm{LM}(q) \leq \mathrm{LM}(s)$$



(a) $\mathtt{DLex}_{x_2 > x_1}$

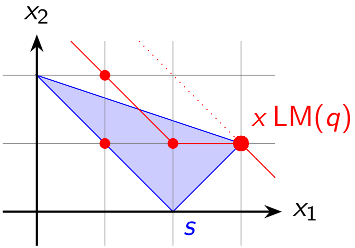


(b) $\mathtt{wLex}_{x_2 > x_1}$ weighted by $w = (1, 4)$

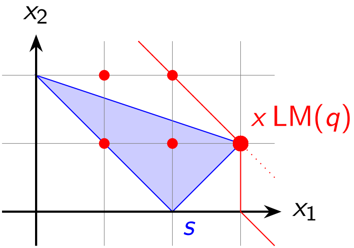
Figure: Support of q (in red, translated by $x_1 x_2$)

Minimize the support of q

($\mathcal{P}$) "Minimize the number of monomials $\leq \mathrm{LM}(s)$ over the monomials orders"



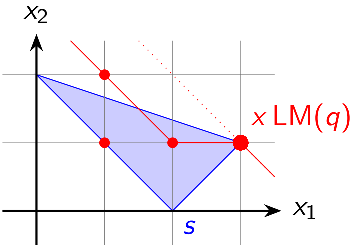
(a) $\mathtt{DLex}_{x_2 > x_1}$



(b) $\mathtt{DLex}_{x_1 > x_2}$

Minimize the support of q

$(\mathcal{P})$ "Minimize the number of monomials $\leq \mathrm{LM}(s)$ over the monomials orders"



(a) $\mathtt{DLex}_{x_2 > x_1}$

Solution for DLex

Suppose that $\mathrm{LM}(s) = x^{\beta}$ such that $\beta_1 \leq \cdots \leq \beta_n$. Then DLex with $x_1 > \cdots > x_n$ minimizes $(\mathcal{P})$ over the $n!$ variable orderings of DLex

This result can also be generalized to wLex

Proposition

Let q_1 (resp. q_2) be the quotient of $D(p)$ w.r.t p for a monomial order $>_1$ (resp. $>_2$). For p to be Darboux, it is necessary that

$$\text{support}(q_1) = \text{support}(q_2)$$

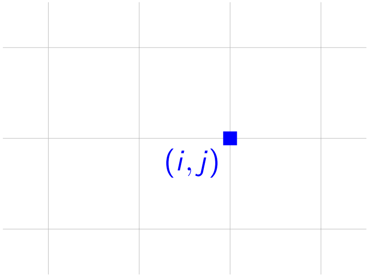
Reduced Quotient for Van der Pol

$q_{x_2} = -a_{d_1-2,d_2+1} = 0$ and thus

$$q = -d_2 x_1^2 + a_{d_1+1,d_2-1}(d_1+1) + d_2$$

Computing the Normal Form

Given D and q, we compute any coefficient of $r = \sum_{i,j} c_{i,j} x_1^i x_2^j$ using $r = D(p) - q.p$, tracking which coefficients of p contribute to $c_{i,j}$



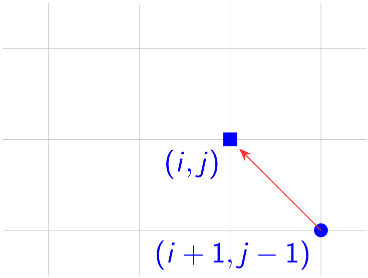
General Formula for VDP

For $D = x_2\partial_1 + (x_2 - x_1^2 x_2 - x_1)\partial_2$, we computed $q = -d_2 x_1^2 + q_0'$ and

$$c_{i,j} \quad =$$

Computing the Normal Form

Given D and q, we compute any coefficient of $r = \sum_{i,j} c_{i,j} x_1^i x_2^j$ using $r = D(p) - q.p$, tracking which coefficients of p contribute to $c_{i,j}$



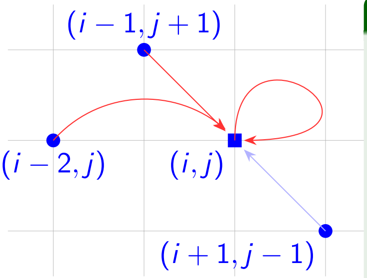
General Formula for VDP

For $D = x_2\partial_1 + (x_2 - x_1^2 x_2 - x_1)\partial_2$, we computed $q = -d_2 x_1^2 + q_0'$ and

$$c_{i,j} \quad = (i+1)a_{i+1,j-1}$$

Computing the Normal Form

Given D and q, we compute any coefficient of $r = \sum_{i,j} c_{i,j} x_1^i x_2^j$ using $r = D(p) - q.p$, tracking which coefficients of p contribute to $c_{i,j}$



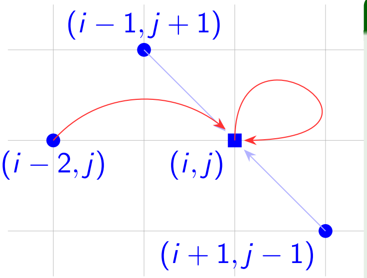
General Formula for VDP

For $D = x_2\partial_1 + (x_2 - x_1^2 x_2 - x_1)\partial_2$, we computed $q = -d_2 x_1^2 + q_0'$ and

$$\begin{aligned} c_{i,j} &= (i+1)a_{i+1,j-1} \\ &+(j)a_{i,j} \\ &-(j)a_{i-2,j} \\ &-(j+1)a_{i-1,j+1} \end{aligned}$$

Computing the Normal Form

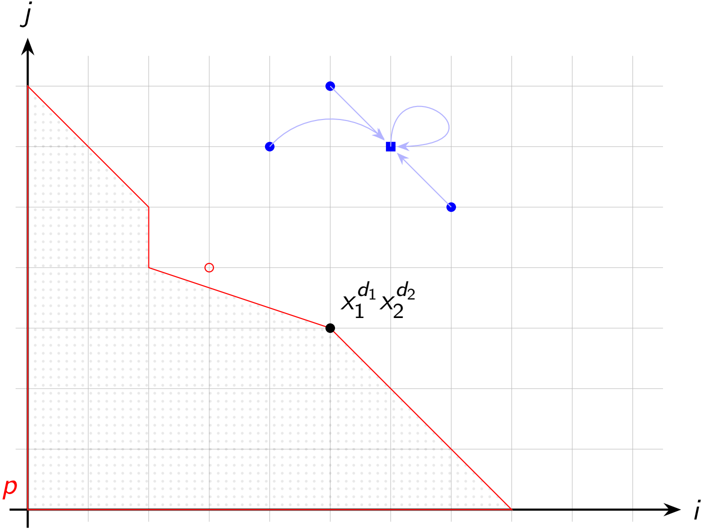
Given D and q, we compute any coefficient of $r = \sum_{i,j} c_{i,j} x_1^i x_2^j$ using $r = D(p) - q.p$, tracking which coefficients of p contribute to $c_{i,j}$

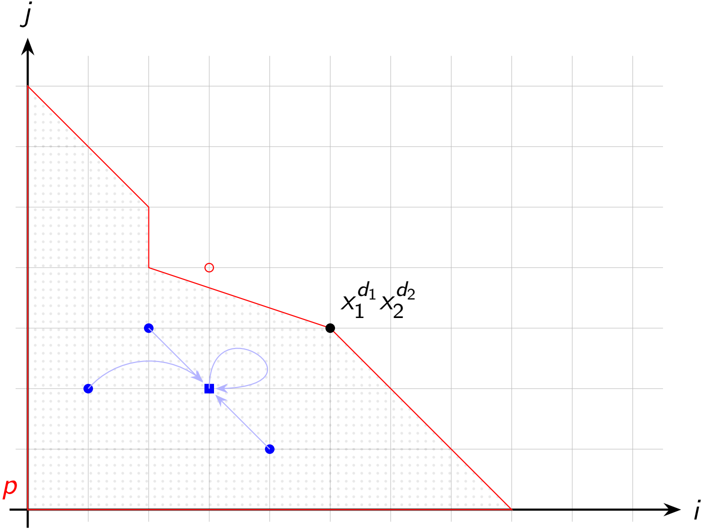


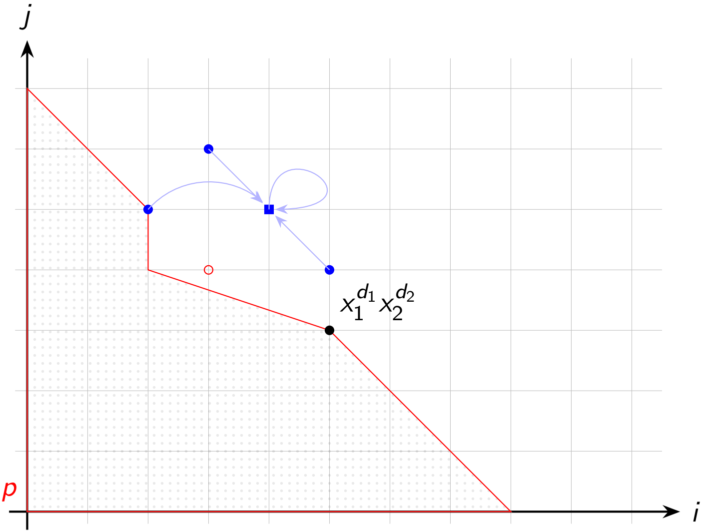
General Formula for VDP

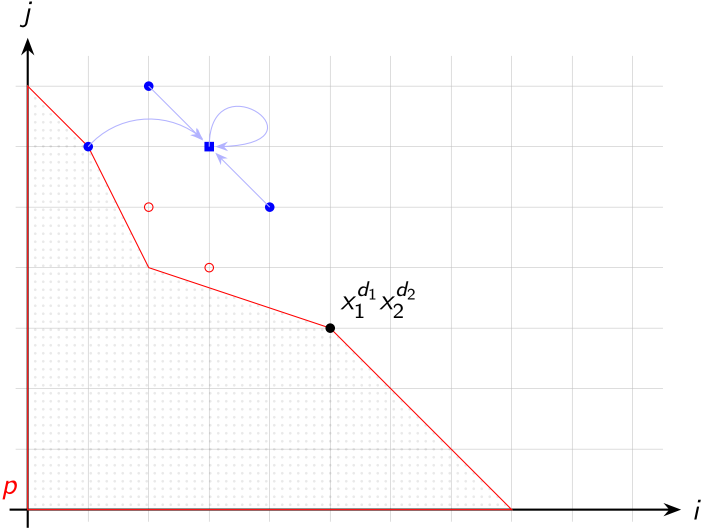
For $D = x_2\partial_1 + (x_2 - x_1^2 x_2 - x_1)\partial_2$, we computed $q = -d_2 x_1^2 + q_0'$ and

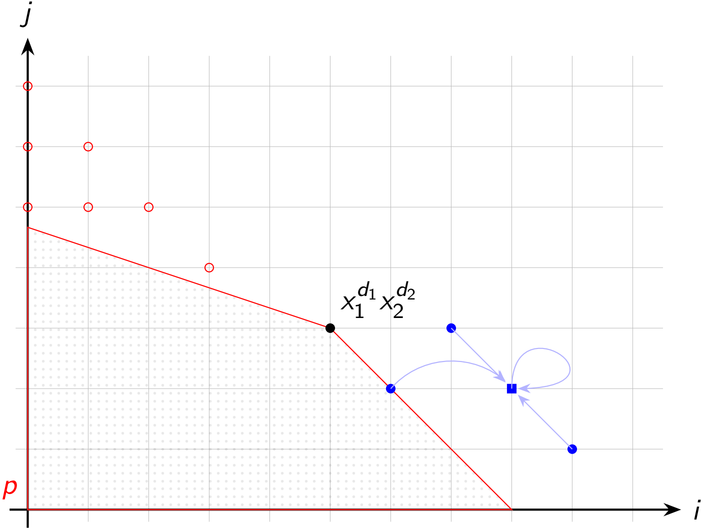
$$\begin{aligned} c_{i,j} &= (i+1)a_{i+1,j-1} \\ &+(j - q_0')a_{i,j} \\ &-(j - d_2)a_{i-2,j} \\ &-(j+1)a_{i-1,j+1} \end{aligned}$$

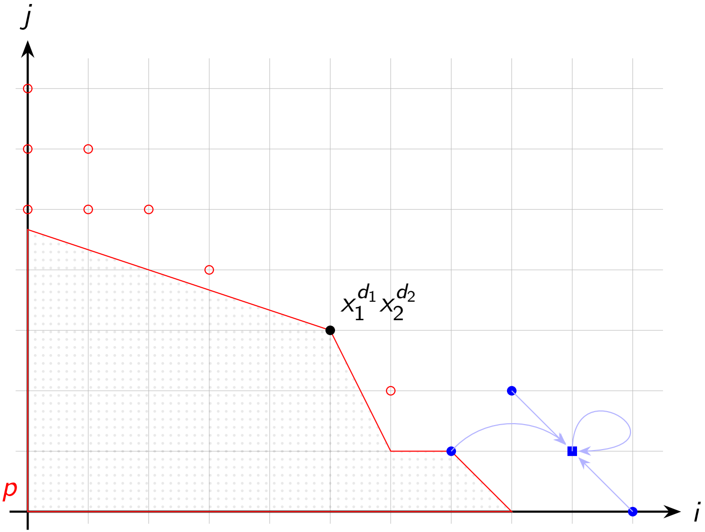

j
$x_1^{d_1} x_2^{d_2}$
p
i

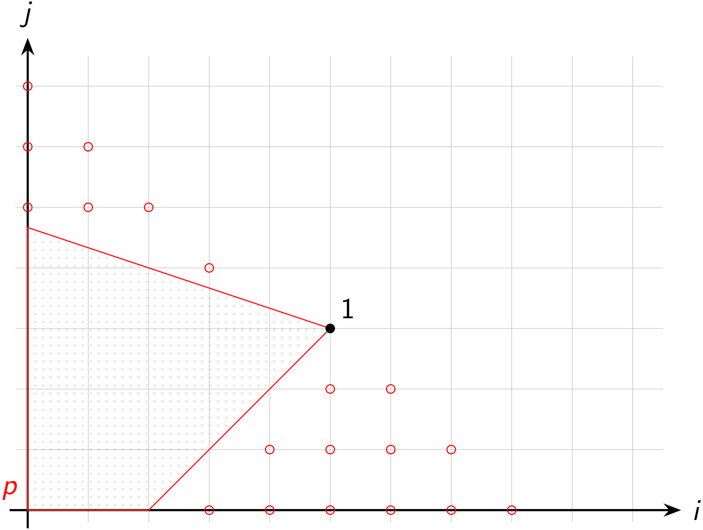

j
i
p
$x_1^{d_1} x_2^{d_2}$

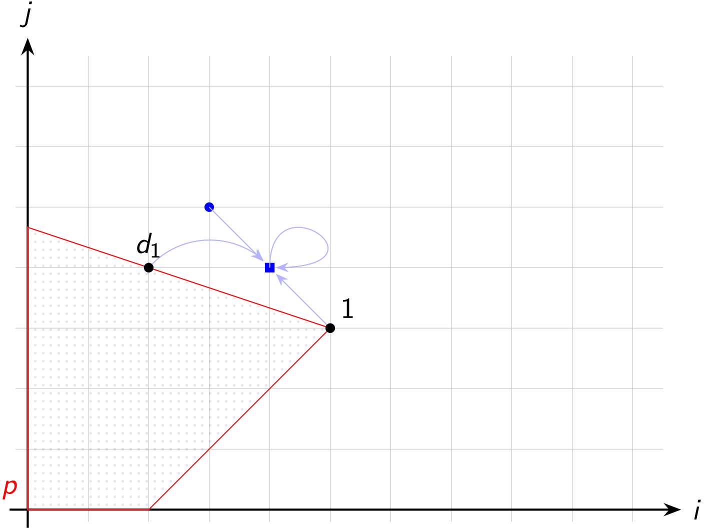

j
$x_1^{d_1} x_2^{d_2}$
p
i

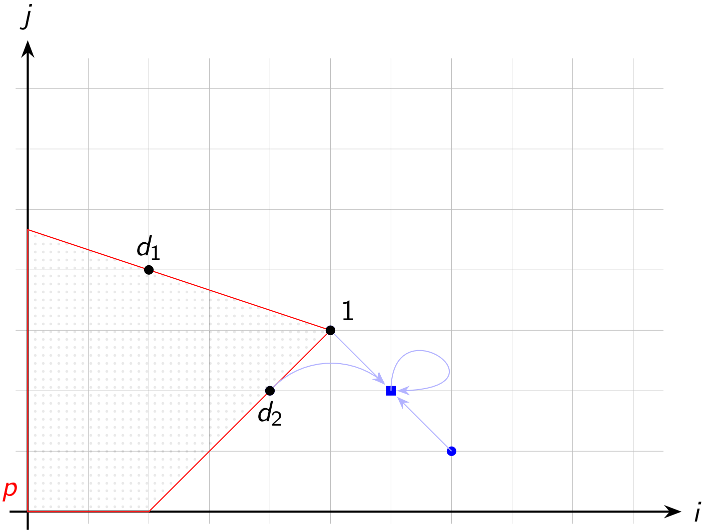

j
$x_1^{d_1} x_2^{d_2}$
p
i

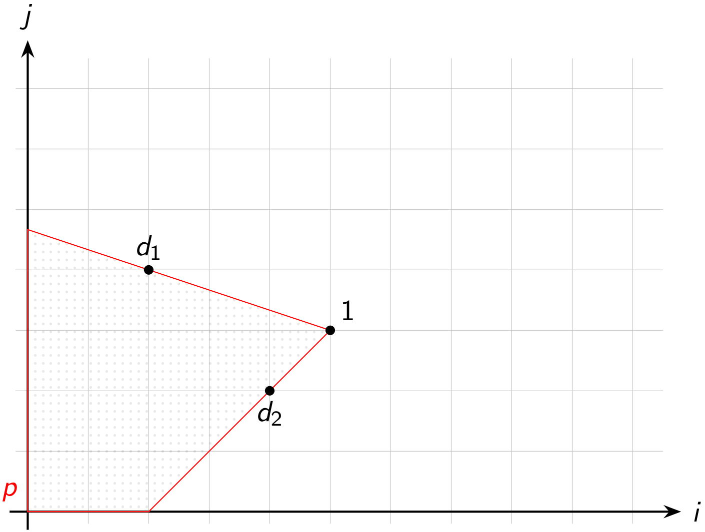

j
p
$x_1^{d_1} x_2^{d_2}$
i

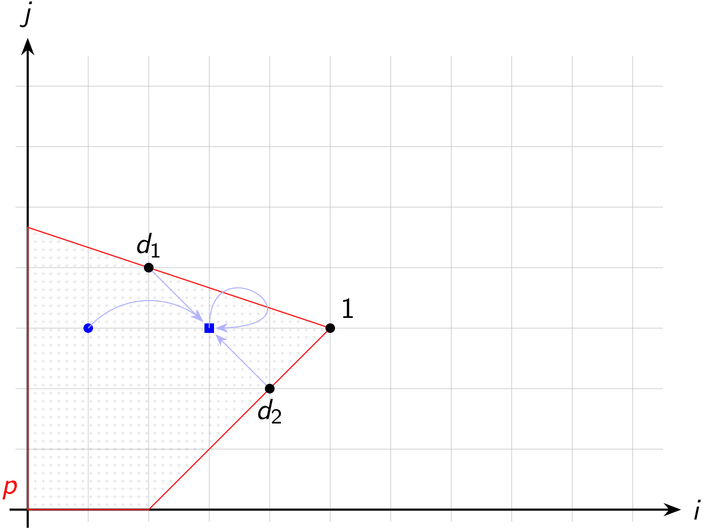

j
$x_1^{d_1} x_2^{d_2}$
p
i


j
1
p
i


j
d_1
1
p
i


j
i
p
d_1
d_2
1


j
d_1
1
d_2
p
i


j
d_1
1
d_2
p
i

Proposition

If p is a Darboux polynomial of leading term x^d for the Van der Pol oscillator, then $|d| = 0$

Corollary

The Van der Pol oscillator has no nontrivial Darboux polynomial over any field (of characteristic 0)

Liénard Dynamics

(Odani, 1981)

If a polynomial system of the following Liénard equation

$$\begin{cases} \dot{x_1} & = x_2 \\ \dot{x_2} & = -f(x_1)x_2 - g(x_1) \end{cases}$$

satisfies $f, g \neq 0, \deg f \geq \deg g$ and $\frac{f}{g} \neq \mathrm{const}$ then it has no algebraic solution curves

Using $\mathtt{DLex}_{x_2 > x_1}$, the reduced quotient q is $-d_2.f$.
If $\deg f > \deg g$, we can produce $d_1 = d_2 = 0$.

If $\deg f = \deg g = m$, we only get $d_1 = 0$ and reducing the rest leads to the following subset of equations :

$$\begin{cases} c_{d_1,d_2-1} & = d_2 \left(f_0 \frac{g_m}{f_m} - g_0 \right) = 0 \\ \dots \\ c_{d_1+m-1,d_2-1} & = d_2 \left(f_{m-1} \frac{g_m}{f_m} - g_{m-1} \right) = 0 \end{cases}$$

from which we retrieve the condition $\frac{f}{g} \neq \mathrm{const.}$

If $\deg f = \deg g = m$, we only get $d_1 = 0$ and reducing the rest leads to the following subset of equations :

$$\begin{cases} c_{d_1,d_2-1} & = d_2\left(f_0\frac{g_m}{f_m} - g_0\right) = 0 \\ \dots & \\ c_{d_1+m-1,d_2-1} & = d_2\left(f_{m-1}\frac{g_m}{f_m} - g_{m-1}\right) = 0 \end{cases}$$

from which we retrieve the condition $\frac{f}{g} \neq \mathrm{const}.$

Note that if $g = \lambda f$ then further solving yields that for all $N \in \mathbb{N}$, $(y+\lambda)^N$ is a Darboux polynomial of degree N and leading term y^N for $\mathtt{DLex}_{x_2 > x_1}$

Conclusion

Main Steps

Generic approach for generating conditions on the degree of the leading term x^d of a Darboux polynomial over any field

1. symbolic polynomial reduction (where $\mathrm{LT}(p) = x^d$ is symbolic)
2. discussed the impact of monomials orders on support(q)
3. on-demand computation of any coefficient of the normal form
4. zero propagation on the coefficients of p
5. generation of constraints on d

The steps that lead to finding $|d| = 0$ form a certificate

Complete Quotient

Van der Pol

For $D = x_1\partial_1 + ((1-x_1^2)x_2 - x_1)\partial_2$, using the $\texttt{DLex}_{x_2>x_1}$ order, $\mathrm{LM}(p) = x_1^{d_1}x_2^{d_2}$ and :

$$q = -d_2x_1^2 - (a_{d_1-2,d_2+1})x_2 + (a_{d_1-2,d_2+1}a_{d_1+1,d_2-1})x_1 + q_0 \quad (1)$$

where q_0 is the following constant coefficient

$$q_0 = a_{d_1+1,d_2-1}\left(-a_{d_1-2,d_2+1}a_{d_1-1,d_2} + d_1 + 1\right) + a_{d_1-2,d_2+1}a_{d_1,d_2-1} + d_2 \quad (2)$$

Generation Methods

Undetermined Coefficients

Let $p = \sum_\alpha a_\alpha x^\alpha$, $q = \sum_\alpha b_\alpha x^\alpha$ be template polynomials and solve $D(p) = q.p$ for the coefficients a_α and b_α

Extactic Curve (Pereira, 2001)

Darboux polynomials of total degree $\leq N$ are factors of

$$\mathcal{E}_{N,v}(D) = \det \begin{pmatrix} v_1 & v_2 & \dots & v_l \\ D(v_1) & D(v_2) & \dots & D(v_l) \\ \vdots & \vdots & & \vdots \\ D^{l-1}(v_1) & D^{l-1}(v_2) & \dots & D^{l-1}(v_l) \end{pmatrix}$$

where $v = \{v_1, \dots, v_l\}$ is a basis of $\mathbb{C}[x]_{\leq N}$

Related Works

(Chèze, 2014)

Let $D = \sum_{i=1}^{n} f_i \partial_i$, if p is a Darboux polynomial of cofactor q then

$$\mathcal{N}(q) \subseteq \mathbb{N}^n \cap \mathcal{N}\left(\sum_{i=1}^{n} \lambda_i \frac{f_i}{x_i}\right)$$

where $\mathcal{N}(q)$ denotes the Newton polytope of q