Algebraic attacks for the Rank Decoding Problem

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NIST CALL FOR PROPOSALS

Post-Quantum Cryptography standardization process, 2017–2022–

- KEM + Signature.
- based on mathematical problems resistant to quantum computer.
- 4 Rounds since 2017.
- first selection for standardization in 07/2022:
  - 1 lattice-based KEM;
  - 2 lattice-based signatures;
  - 1 Hash-based signature.
- 3 code-based KEMs in the 4th Round.

Additional Digital Signature Schemes

- June 1, 2023. First Round ongoing.
- 40 submissions, with:
  - multivariate cryptography (12).
  - code-based cryptography (11).
  - Symmetric-based cryptography (4).
  - Lattice-based cryptography (7).
  - Other (6).

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Algebraic approaches are at the core of security assessment for multivariate and code-based cryptography.

Principle: write a Polynomial System

$$\begin{cases} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_m(x_1, \ldots, x_n) \end{cases}, \quad \deg(f_i) = d_i, f_i \in \mathbb{F}_q[x_1, \ldots, x_n].$$

such that finding the set of solutions gives (part of) the secret:

$$V(f_1, \ldots, f_m) = \left\{ (x_1, \ldots, x_n) \in \overline{\mathbb{F}_q}^n : f_i(x_1, \ldots, x_n) = 0, \forall i \in \{1..m\} \right\}$$

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- Key-recovery attack.
- Message-recovery attack.
- Signature forgery attack.

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- Cryptographic applications: always a finite number of solutions (one of them is enough).
- Often 0 or 1 solution, but sometimes m solutions over $\mathbb{F}_{q^m}$.

Signature forgery (or Message-recovery attack)

- Public key: a polynomial system, indistinguishable from a random system.

$$\begin{cases} f_1(x_1,\ldots,x_n) \\ \vdots \\ f_m(x_1,\ldots,x_n) \end{cases}, \quad \deg(f_i) = 2, \quad f_i \in \mathbb{F}_q[x_1,\ldots,x_n].$$

- $(y_1,\ldots,y_m)$ hash of the message to be signed (or ciphertext).
- signature (or cleartext) = $(x_1,\ldots,x_n)$ such that $(y_1,\ldots,y_m) = (f_1(\boldsymbol{x}),\ldots,f_m(\boldsymbol{x}))$
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How hard is it to solve a random system of algebraic equations?

Multivariate public-key cryptography

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How hard is it to solve a random system of algebraic equations?
How hard is it to solve a trapdoored system of algebraic equations?

Solving the algebraic system using Gröbner bases (object)

- A particular basis of the ideal

$$I = \langle f_1, \ldots, f_m \rangle = \left\{ \sum_{i=1}^{m} g_i f_i : g_i \in \mathbb{F}_q[x_1, \ldots, x_n] \right\}$$

  that solves the ideal-membership problem: $f \overset{?}{\in} I$.
- Depends on the choice of a monomial ordering.

Monomial ordering examples

$$x_1 \quad x_3 \quad 1 \quad x_3^3 \quad x_1x_3 \quad x_2^2 \quad x_1^2$$

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Lexicographical ordering $x_1 > \cdots > x_n$

$$x_1^{\alpha_1} \dots x_n^{\alpha_n} > x_1^{\beta_1} \dots x_n^{\beta_n} \text{ iff } \qquad \alpha_j = \beta_j \quad \forall j < i, \text{ and } \alpha_i > \beta_i.$$

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Graded Reverse Lexicographical ordering $x_1 > \cdots > x_n$

$$x_1^{\alpha_1} \ldots x_n^{\alpha_n} > x_1^{\beta_1} \ldots x_n^{\beta_n} \text{ iff } \begin{cases} \deg(\boldsymbol{x}^{\alpha}) > \deg(\boldsymbol{x}^{\beta}) \\ \text{or } \alpha_j = \beta_j \ \forall j > i, \text{ and } \alpha_i < \beta_i. \end{cases}$$

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$$x_3^3 > x_1^2 > x_2^2 > x_1x_3 > x_1 > x_3 > 1$$

Different monomial orderings have different properties

- the *lex* order (Lexicographical): in Shape Position, for a zero-dimension ideal, the (reduced) lex basis is

$$\begin{cases} x_1 - & g_1(x_n), \\ x_2 - & g_2(x_n), \\ \vdots & \\ x_{n-1} - & g_{n-1}(x_n), \\ & g_n(x_n), \end{cases}$$

  with $\deg(g_n) = D$ the number of solutions to the system.
- the *grevlex* order (Graded Reverse Lexicographical): usually the best one w.r.t. the complexity.

The (reduced) grevlex and lex bases are the same:

- If the system has no solution:

$$\langle 1 \rangle .$$

Systems with 0 or 1 solution

The (reduced) grevlex and lex bases are the same:

- If the system has no solution:

$$\langle 1 \rangle .$$

- If the system has 1 solution:

$$\begin{cases} x_1 - a_1, \\ \vdots \\ x_n - a_n, \end{cases}$$

where $(a_1, \ldots, a_n) \in \mathbb{F}_q^n$ is the solution.

For zero-dimensional systems:

- The FGLM (J.-C. Faugère, Gianni, Daniel Lazard, and Mora (1993)) Algorithm performs a change of ordering in complexity

$$O(nD^3),$$

n number of variables, $n \to \infty$, D degree of the ideal (number of solutions).

- Complexity for grevlex to lex (Shape position) (J.-C. Faugère, Gaudry, Huot, and Renault (2014)):

$$O(\log_2(D)(D^\omega + n\log_2(D)D)).$$

ω coefficient of linear algebra.

Change of ordering

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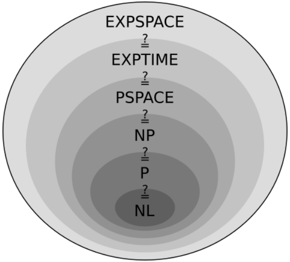
We focus on the grevlex ordering

Complexity classes

A Gröbner basis solves the Ideal Membership problem.

A hard problem

- Ideal Membership testing is EXPSPACE-complete,
- Existence of solutions to a system of polynomial equations over a finite field is NP-complete (Fraenkel and Yesha (1979)),



For cryptographic applications

- We need precise estimates for concrete parameters.
- Asymptotic estimates are also appreciated.
- The security levels are $2^{143}, 2^{207}$ and 2^{272} bits operations.
- Take the best algorithm (combinatorial, algebraic, hybrid, ...).

General algorithms, for any input system:

- Buchberger (1965);
- F4 from J.-C. Faugère (1999);

The algorithms will always terminate and give the Gröbner basis.
But the time is hard to predict for *any* instance.

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But the time is hard to predict for *any* instance.

Specific algorithms, for a particular class of systems:

- The algorithms will terminate in a predictable time.
- The result is not always a Gröbner basis of the system.
- For random instances in the specific class, the result is a Gröbner basis.

$$\text{System} \begin{cases} f_1(x_1,\ldots,x_n) \\ \vdots \\ f_m(x_1,\ldots,x_n) \end{cases}, \quad \deg(f_i) = d_i, f_i \in \mathbb{F}_q[x_1,\ldots,x_n].$$

- Macaulay Matrices Macaulay (1902):

$$\mathscr{M}_d(\{f_1,\ldots,f_m\}) = \begin{array}{c} \vdots \\ (\boldsymbol{x}^\alpha, i) \\ \vdots \end{array} \overset{\boldsymbol{x}^\beta}{\begin{pmatrix} & & \\ & \mathrm{coeff}(\boldsymbol{x}^\alpha f_i, \boldsymbol{x}^\beta) & \\ & & \end{pmatrix}}$$

$\deg(\boldsymbol{x}^\alpha f_i) = d = \deg(\boldsymbol{x}^\beta)$.

Example: 3 quadratic equations in 3 variables, $\mathbb{F}_5$

$$
\begin{cases}
x_1^2 + 3x_1x_2 + x_2^2 + x_1x_3 + 2x_2x_3 + 2x_3^2, & (f_1) \\
x_1^2 + 4x_1x_2 + 3x_2^2 + 4x_1x_3 + 3x_3^2, & (f_2) \\
x_1^2 + 2x_2^2 + 4x_2x_3 + 3x_3^2. & (f_3)
\end{cases}
$$

EXAMPLE: 3 QUADRATIC EQUATIONS IN 3 VARIABLES, $\mathbb{F}_5$

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\begin{cases}
x_1^2 + 3x_1x_2 + x_2^2 + x_1x_3 + 2x_2x_3 + 2x_3^2, & (f_1)\\
x_1^2 + 4x_1x_2 + 3x_2^2 + 4x_1x_3 + 3x_3^2, & (f_2)\\
x_1^2 + 2x_2^2 + 4x_2x_3 + 3x_3^2. & (f_3)
\end{cases}
$$

$$
\mathscr{M}_2 = \begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \end{array}
\begin{array}{cccccc} x_1^2 & x_1x_2 & x_2^2 & x_1x_3 & x_2x_3 & x_3^2 \end{array}
\begin{pmatrix}
{\color{red}1} & 3 & 1 & 1 & 2 & 2\\
{\color{blue}1} & 4 & 3 & 4 & 0 & 3\\
{\color{blue}1} & 0 & 2 & 0 & 4 & 3
\end{pmatrix}
$$

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$$\begin{cases} x_1^2 + 3x_1x_2 + x_2^2 + x_1x_3 + 2x_2x_3 + 2x_3^2, & (f_1) \\ x_1^2 + 4x_1x_2 + 3x_2^2 + 4x_1x_3 + 3x_3^2, & (f_2) \\ x_1^2 + 2x_2^2 + 4x_2x_3 + 3x_3^2. & (f_3) \end{cases}$$

$$Ech(\mathscr{M}_2) = \begin{array}{c} \\ \tilde{f}_1 \\ \tilde{f}_2 \\ \tilde{f}_3 \end{array} \begin{array}{cccccc} x_1^2 & x_1x_2 & x_2^2 & x_1x_3 & x_2x_3 & x_3^2 \\ \hline 1 & & & 2 & 3 & 4 \\ 0 & 1 & & & 2 & 2 \\ 0 & & 1 & 4 & 3 & 2 \end{array}$$

$$\left\{\begin{array}{ccccccccccc}
x_1^2 & + & 3x_1x_2 & + & x_2^2 & + & x_1x_3 & + & 2x_2x_3 & + & 2x_3^2, \\
x_1^2 & + & 4x_1x_2 & + & 3x_2^2 & + & 4x_1x_3 & & & + & 3x_3^2, \\
x_1^2 & & & + & 2x_2^2 & & & + & 4x_2x_3 & + & 3x_3^2.
\end{array}\right.$$

$\mathscr{M}_3 =$

	x_1^3	$x_1^2x_2$	$x_1x_2^2$	x_2^3	$x_1^2x_3$	$x_1x_2x_3$	$x_2^2x_3$	$x_1x_3^2$	$x_2x_3^2$	x_3^3
x_3f_1					1	3	1	1	2	2
x_2f_1		1	3	1		1	2		2	
x_1f_1	1	3	1		1	2		2		
x_3f_2					1	4	3	4	0	3
x_2f_2		1	4	3		4	0		3	
x_1f_2	1	4	3		4	0		3		
x_3f_3					1	0	2	0	4	3
x_2f_3		1	0	2		0	4		3	
x_1f_3	1	0	2		0	4		3		

$$
\begin{cases}
x_1^2 & & & + & 2x_1x_3 & + & 3x_2x_3 & + & 4x_3^2, \\
& x_1x_2 & & & & + & 2x_2x_3 & + & 2x_3^2, \\
& & x_2^2 & + & 4x_1x_3 & + & 3x_2x_3 & + & 2x_3^2.
\end{cases}
$$

$$
\mathscr{M}_3 = \begin{array}{c} \\ x_3\tilde{f}_1 \\ x_2\tilde{f}_1 \\ x_1\tilde{f}_1 \\ x_3\tilde{f}_2 \\ x_2\tilde{f}_2 \\ x_1\tilde{f}_2 \\ x_3\tilde{f}_3 \\ x_2\tilde{f}_3 \\ x_1\tilde{f}_3 \end{array}
\begin{array}{c}
\begin{array}{cccccccccc} x_1^3 & x_1^2x_2 & x_1x_2^2 & x_2^3 & x_1^2x_3 & x_1x_2x_3 & x_2^2x_3 & x_1x_3^2 & x_2x_3^2 & x_3^3 \end{array} \\
\begin{pmatrix}
 & & & & {\color{red}1} & & & 2 & 3 & 4 \\
 & {\color{red}1} & & & & 2 & 3 & & 4 & \\
{\color{red}1} & & & & 2 & 3 & & 4 & & \\
 & & & & & {\color{red}1} & & & 2 & 2 \\
 & & {\color{red}1} & & & & 2 & & 2 & \\
 & {\color{blue}1} & & & & 2 & & 2 & & \\
 & & & & & & {\color{red}1} & 4 & 3 & 2 \\
 & & & {\color{red}1} & & 4 & 3 & & 2 & \\
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\end{pmatrix}
\end{array}
$$

$$\left\{\begin{array}{ccccccccc}
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\end{array}\right.$$

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\begin{array}{c}
\begin{array}{cccccccccc} x_1^3 & x_1^2x_2 & x_1x_2^2 & x_2^3 & x_1^2x_3 & x_1x_2x_3 & x_2^2x_3 & x_1x_3^2 & x_2x_3^2 & x_3^3 \end{array} \\
\begin{pmatrix}
 & & & & 1 & & & & & 4 \\
 & 1 & & & & & & & & 4 \\
1 & & & & & & & & & 4 \\
 & & & & & 1 & & & & 4 \\
 & & 1 & & & & & & & 4 \\
 & 0 & & & & & & 1 & & 4 \\
 & & & & & & 1 & & & 4 \\
 & & & 1 & & & & & & 4 \\
 & & 0 & & & & & & 1 & 4
\end{pmatrix}
\end{array}$$

Gröbner basis via linear algebra

$$\text{Gröbner Basis} = \begin{cases} x_1x_3^2 + 4x_3^3, & (x_1f_2) \\ x_2x_3^2 + 4x_3^3, & (x_1f_3) \\ x_1^2 + 2x_1x_3 + 3x_2x_3 + 4x_3^2, & (f_1) \\ x_1x_2 + 2x_2x_3 + 2x_3^2, & (f_2) \\ x_2^2 + 4x_1x_3 + 3x_2x_3 + 2x_3^2 & (f_3). \end{cases}$$

One projective solution: $(1, 1, 1)$.

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$$\mathcal{M}_2 = \begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \end{array} \begin{array}{c} \begin{array}{cccccc} x_1^2 & x_1x_2 & x_2^2 & x_1x_3 & x_2x_3 & x_3^2 \end{array} \\ \begin{pmatrix} 1 & 3 & 1 & 1 & 2 & 2 \\ 1 & 4 & 3 & 4 & 0 & 3 \\ 1 & 0 & 2 & 0 & 4 & 1 \end{pmatrix} \end{array}$$

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$$\begin{cases} x_1^2 + 3x_1x_2 + x_2^2 + x_1x_3 + 2x_2x_3 + 2x_3^2, \\ x_1^2 + 4x_1x_2 + 3x_2^2 + 4x_1x_3 + 3x_3^2, \\ x_1^2 + 2x_2^2 + 4x_2x_3 + 1x_3^2. \end{cases}$$

$$Ech(\mathscr{M}_2) = \begin{array}{c} \\ \tilde{f}_1 \\ \tilde{f}_2 \\ \tilde{f}_3 \end{array} \begin{array}{cccccc} x_1^2 & x_1x_2 & x_2^2 & x_1x_3 & x_2x_3 & x_3^2 \\ \end{array} \begin{pmatrix} 1 & & & 2 & 3 & 4 \\ 0 & 1 & & & 2 & 4 \\ 0 & & 1 & 4 & 3 & 2 \end{pmatrix}$$

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$Ech(\mathscr{M}_3) =$

	x_1^3	$x_1^2x_2$	$x_1x_2^2$	x_2^3	$x_1^2x_3$	$x_1x_2x_3$	$x_2^2x_3$	$x_1x_3^2$	$x_2x_3^2$	x_3^3
$x_3\tilde{f}_1$					1					1
$x_2\tilde{f}_1$		1								1
$x_1\tilde{f}_1$	1									3
$x_3\tilde{f}_2$						1				1
$x_2\tilde{f}_2$			1							0
$x_1\tilde{f}_2$		0						1		3
$x_3\tilde{f}_3$							1			2
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$$
Ech(\mathscr{M}_3) = \begin{array}{c} \\ x_3\tilde{f}_1 \\ x_2\tilde{f}_1 \\ x_1\tilde{f}_1 \\ x_3\tilde{f}_2 \\ x_2\tilde{f}_2 \\ x_1\tilde{f}_2 \\ x_3\tilde{f}_3 \\ x_2\tilde{f}_3 \\ x_1\tilde{f}_3 \end{array}
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 & & & 1 & & & & & & 1 \\
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$$

x_1f_3 vs x_3f_3: need to go to degree $D = 4$ to get the Gröbner Basis.

At $D = 4$:

- $\binom{6}{4} = 15$ monomials of degree 4,
- $3\binom{4}{2} = 18$ rows tf_i of degree 4,
- $\mathscr{M}_4$ has rank $15 \to 3$ rows reduce to 0 ($x_1^2 f_2, x_1 x_2 f_3, x_1^2 f_3$), 1 new polynomial ($x_1 x_3 f_3$).

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First system:

- $\mathscr{M}_4$ has rank $14 \to 4$ rows reduce to 0, no new polynomial.

Do we need to compute the Gröbner basis?

- easy to recover the value of all variables from the evaluation of all monomials of degree D.
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$$f_i = \sum_{i,j} c_{i,j} x_i y_j \in \mathbb{F}_q[\boldsymbol{x}, \boldsymbol{y}].$$

Macaulay matrix at bi-degree (d_1, d_2) = the vector space $\langle \boldsymbol{x}^\alpha \boldsymbol{y}^\beta f_i \rangle$ with $\deg(\boldsymbol{x}^\alpha) = d_1 - 1$, $\deg(\boldsymbol{y}^\beta) = d_2 - 1$.

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Gröbner basis via linear algebra

Rows of Macaulay matrices:

- Describes the vector space $\langle tf_i : \deg(tf_i) = d \rangle_{\mathbb{F}_q}$.
- D. Lazard (1983); Giusti (1984): linear algebra on the Macaulay matrices up to degree $D \rightarrow$ Gröbner basis.
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Main challenges to get complexity estimates for Gröbner Basis computations

- Estimate D.
- Estimate the cost of linear algebra.

$\mathbb{C}$ of linear algebra. Jeannerod, Pernet, and Storjohann (2013)

Matrix $\mathscr{M}$ with N rows, Mon columns, rank Rk, and δ non-zero elements per row. Echelon Form can be computed in:

$$C_\omega \times \mathrm{N} \times \mathrm{Mon} \times \mathrm{Rk}^{\omega-2} + o(\mathrm{N}\,\mathrm{Mon}\,\mathrm{Rk}^{\omega-2}), \quad \mathrm{N}, \mathrm{Mon}, \mathrm{Rk} \to \infty,$$

For instance:

- $(\omega, C_\omega) = (3, 1)$ for Gaussian Elimination;
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These are upper bounds.

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⚠ we cannot remove rows at random ⚠

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- then D is bounded by the smallest value such that

$$\mathrm{Rk}_d = \mathrm{Mon}_d - 1 \qquad \text{(resp. } \mathrm{Rk}_d = \mathrm{Mon}_d\text{)}.$$

Hilbert Series (homogeneous system)

$I \subset R = \mathbb{F}_q[x_1, \ldots, x_n], \quad R = \oplus_d R_d, \quad I_d = R_d \cap I.$

$$HS_{R/I}(z) \stackrel{\text{def}}{=} \sum_{d \in \mathbb{N}} \dim(R_d/I_d) z^d.$$

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c-ex: there is no boolean semi-regular quadratic system of 1 polynomial in $n > 6$ variables. Hodges, Molina, and Schlather (2017).
More generally, if $n \gg m$ there is no boolean semi-regular sequence of m polynomials of degree $d_1, \ldots, d_m \geq 2$.

- $m = n$ regular system: $D \leq n+1$, $\mathrm{Mon}_D = \binom{n+D-1}{D}$

Quadratic systems in different classes

- $m = n$ regular system: $D \leq n + 1$, $\mathrm{Mon}_D = \binom{n+D-1}{D}$
- $m = n + 1$ semi-regular system: $D \leq \lceil \frac{n+2}{2} \rceil$, $\rightarrow$ hybrid approach 🙂

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- $m = n$ regular over $\mathbb{F}_2$: $D \leq 0.0900n + o(n^{1/3})$, but $\text{Mon}_D = \binom{n}{D}$.

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  - The complexity can be smaller or larger 🤔 ⁉

3 AFFINE QUADRATIC EQUATIONS IN 2 VARIABLES, $\mathbb{F}_5$

$$\left\{\begin{array}{lllllllll} x_1^2 & & & + & 2x_1 & + & 3x_2 & + & 4, & (f_1) \\ & x_1x_2 & & & & + & 2x_2 & + & 2 (\text{or } 4), & (f_2) \\ & & x_2^2 & + & 4x_1 & + & 3x_2 & + & 2 (\text{or } 1). & (f_3) \end{array}\right.$$

- $D^{top} = 2$, not enough to get linear equations.

3 AFFINE QUADRATIC EQUATIONS IN 2 VARIABLES, $\mathbb{F}_5$

$$\begin{cases} x_1^2 + 2x_1 + 3x_2 + 4, & (f_1) \\ x_1x_2 + 2x_2 + 2 \text{(or } 4), & (f_2) \\ x_2^2 + 4x_1 + 3x_2 + 2 \text{(or } 1). & (f_3) \end{cases}$$

- $D^{top} = 2$, not enough to get linear equations.
- $D^h = 3$ (or 4)

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- second case: need another $D = 2$ matrix to get $I = \langle 1 \rangle$.

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- determine the generic relations between rows in the Macaulay matrices = syzygies,
- compute the rank of the Macaulay matrices for generic systems,
- deduce the maximal degree $D \rightarrow$ complexity estimates,
- design a specific Gb algorithm that is more efficient.

LET'S PLAY A GAME

Some important parameters to estimate the complexity of solving a polynomial system:

- the number of variables,
- the number of equations,
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But not sufficient!
Given a polynomial system of equations, what can you say "a priori" about its complexity?

Complexity of solving a system

$$\begin{cases} x_1 + 2x_5 + 2x_6 + 1, \\ x_1 + x_5 + x_6 + 2, \\ x_1 + 2x_2 + 2x_3 + 2x_4 + x_6 + 1, \\ x_1 + x_2 + x_4 + 2x_5 + x_6 + 1, \\ x_1 + x_2 + 2x_3 + x_4 + x_5 + x_6, \\ 2x_1 + 2x_2 + x_3 + x_4 + x_5 + 1 \end{cases}$$

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Linear system, polynomial time complexity.

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Linear system, polynomial time complexity.
Number of solutions? ($\mathbb{F}_3$)

Example (Bayer-Stillman 1988)

$$\mathscr{S}_{ex} = \begin{cases} f_0 c_{0,\ell} b_{0,\ell}^2 + s_0 c_{0,\ell}, & \\ s_i c_{i,1} + s_{i+1}, & \\ s_i c_{i,4} + f_{i+1}, & \\ f_i c_{i,1} + s_i c_{i,2}, & i \in \{0..2\} \\ s_i c_{i,3} + f_i c_{i,4}, & \ell \in \{1..4\} \\ f_i c_{i,2} b_{i,1} + f_i c_{i,3} b_{i,4}, & \\ s_i c_{i,2} + s_i c_{i,3}, & \\ f_i c_{i,2} b_{i,3} c_{i+1,\ell} b_{i+1,\ell} + f_i c_{i,\ell} c_{i,2} b_{i,2}, & \end{cases}$$

$\mathscr{S}_{ex} \in \mathbb{F}_2[f_i, s_i, c_{i,\ell}, b_{i,\ell}]$ for $i \in \{0..3\}, \ell \in \{1..4\}$.
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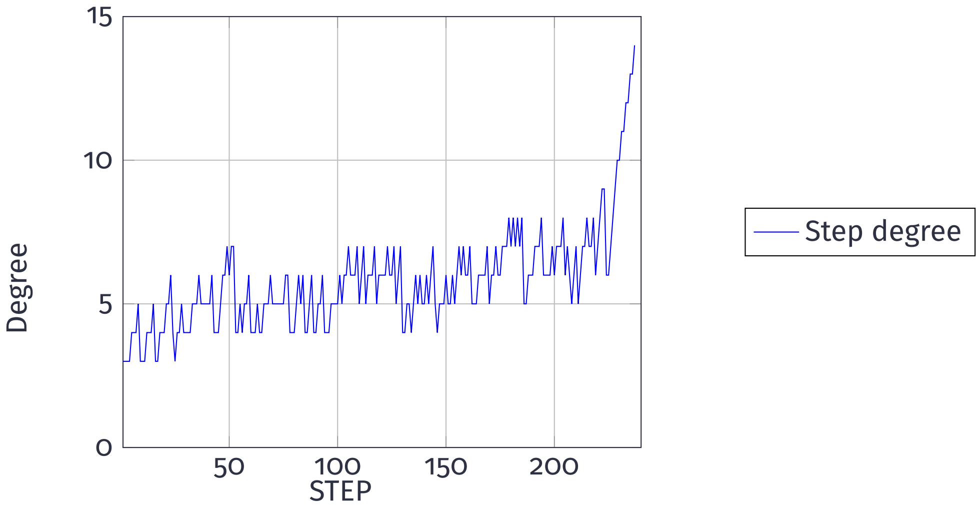
40 variables, 34 polynomials of degrees `2:15, 3:3, 4:4, 5:12`.
$D = 82$ for regular systems

Step Degrees during the grevlex computation for $\mathscr{S}_{ex}$ (`magma V2.28-2`)



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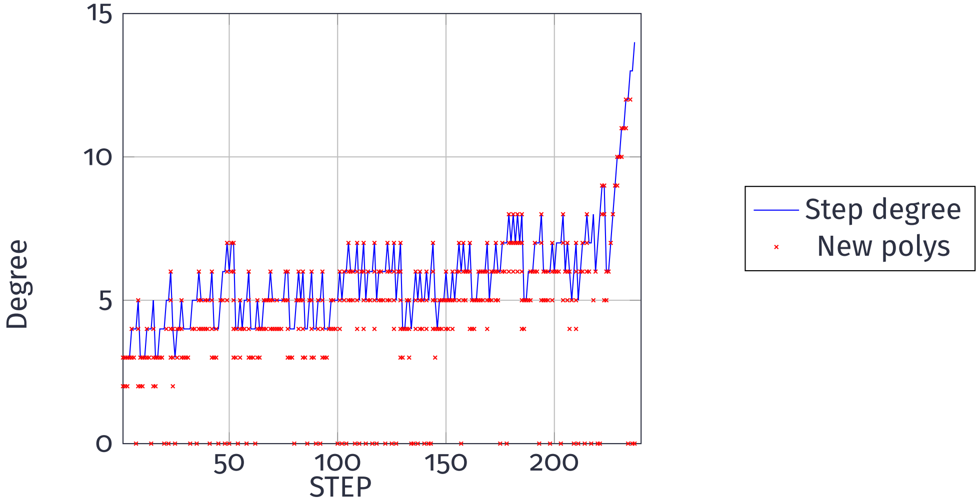
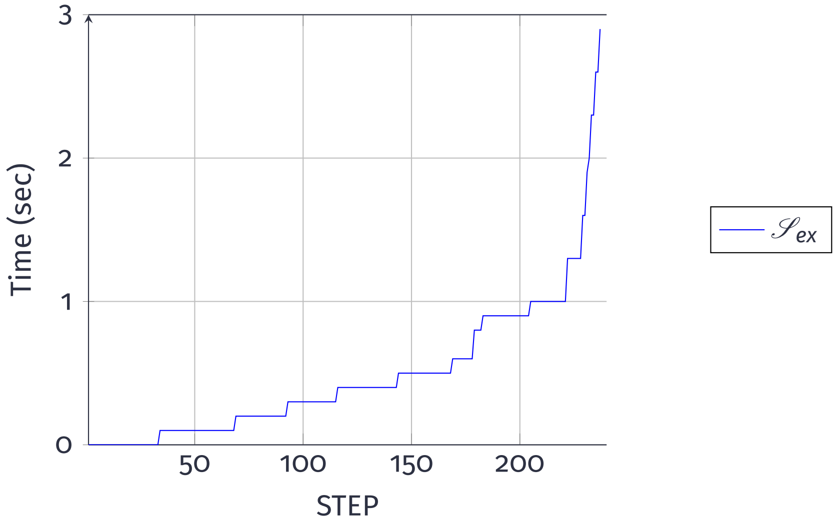


Time of the computation (in sec) for $\mathscr{S}_{ex}$ (`magma V2.28-2`)



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$\mathscr{S}_{ex} \in \mathbb{F}_2[f_i, s_i, c_{i,\ell}, b_{i,\ell}]$ for $i \in \{0..3\}, \ell \in \{1..4\}$.

40 variables, 34 polynomials of degrees `2:15, 3:3, 4:4, 5:12`.

$\mathscr{S}_{ex}$ solved in 3.3 seconds.

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$\mathscr{S}_{ex} \in \mathbb{F}_2[f_i, s_i, c_{i,\ell}, b_{i,\ell}]$ for $i \in \{0..3\}, \ell \in \{1..4\}$.
40 variables, 34 polynomials of degrees `2:15, 3:3, 4:4, 5:12`.
$\mathscr{S}_{ex}$ solved in seconds.

Example (Bayer-Stillman 1988)

$$\mathscr{S}_{bs} = \begin{cases} f_0 c_{0,\ell} b_{0,\ell}^2 + s_0 c_{0,\ell}, & \\ s_i c_{i,1} + s_{i+1}, & \\ s_i c_{i,4} + f_{i+1}, & \\ f_i c_{i,1} + s_i c_{i,2}, & i \in \{0..2\} \\ s_i c_{i,3} + f_i c_{i,4}, & \ell \in \{1..4\} \\ f_i c_{i,2} b_{i,1} + f_i c_{i,3} b_{i,4}, & \\ s_i c_{i,2} + s_i c_{i,3}, & \\ f_i c_{i,2} b_{i,3} c_{i+1,\ell} b_{i+1,\ell} + f_i c_{i+1,\ell} c_{i,2} b_{i,2}, & \end{cases}$$

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$\mathscr{S}_{bs}$ solved in seconds.

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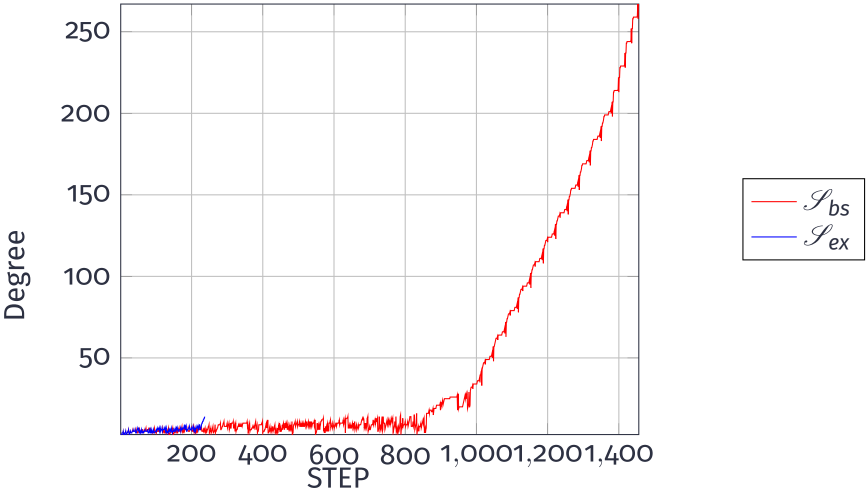
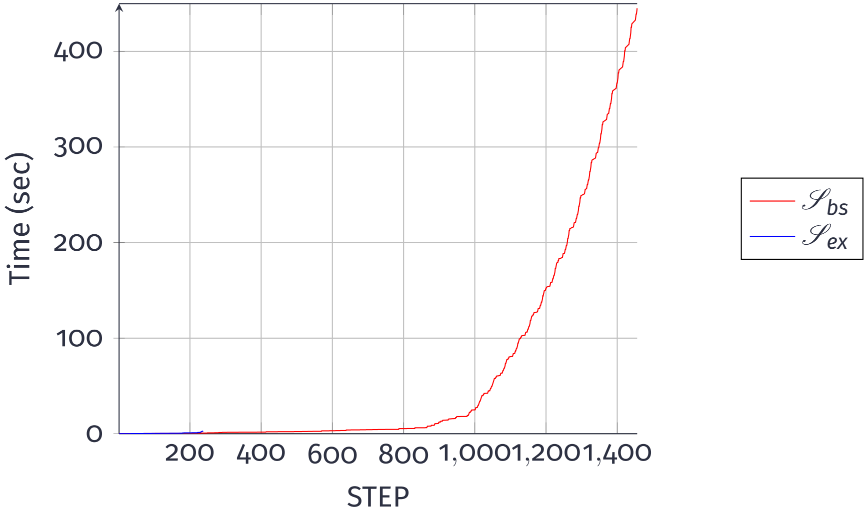
$\mathscr{S}_{bs} \in \mathbb{F}_2[f_i, s_i, c_{i,\ell}, b_{i,\ell}]$ for $i \in \{0..3\}, \ell \in \{1..4\}$.
40 variables, 34 polynomials of degrees `2:15, 3:3, 4:4, 5:12`. $D = 82$? 🤔
$\mathscr{S}_{bs}$ solved in 448.5 seconds.

Step Degrees during the computation for $\mathscr{S}_{bs}$ and $\mathscr{S}_{ex}$ (`magma V2.28-2`)



Time of the computation (in sec) for $\mathscr{S}_{bs}$ and $\mathscr{S}_{ex}$ (`magma V2.28-2`)



Bayer and Stillman (1988) example

- parameter m,
- $10m+4$ equations (degrees `2:5m, 3:m, 4:4, 5:4m`),
- $10(m+1)$ variables.
- the Gröbner basis contains polynomials of degree $2^{2^m}+2$.
- the example was $m=3$: maximal degree $2^{2^3}+2=258$.

EX vs BS example $m = 4$

- 703 STEPS vs > 40770
- max degree 14 vs 65538
- time 27.5 sec vs > 1131 seconds (segfault...)

80 QUADRATIC EQUATIONS 80 VARIABLES IN $\mathbb{F}_{16}$

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- regular?

80 quadratic equations 80 variables in $\mathbb{F}_{16}$

- regular? yes!

80 QUADRATIC EQUATIONS 80 VARIABLES IN $\mathbb{F}_{16}$

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- Complexity?

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80 QUADRATIC EQUATIONS 80 VARIABLES IN $\mathbb{F}_{16}$

- regular? yes!
- Complexity? $D = 81$, $\mathrm{Mon}_{81} = 2^{156}$
- my system:

$$\begin{cases} x_1^2, \\ x_2^2, \\ \vdots \\ x_{80}^2. \end{cases}$$

<beamer>

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