Functional equations and combinatorics II

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Background

Bell numbers, $B_n :=$ number of partitions of a set of cardinality $n \geq 1$

defined by $\sum_{n\geq 0} \frac{B_n}{n!} t^n := \exp(e^t - 1)$

$B(t) := 1 + \sum_{n\geq 1} B_n t^n = 1 + t + 2t + 5t^3 + 15t^4 + 52t^5 + 203t^6 + 877t^6 + 4140t^7 + 21147t^8 + 115975t^9 + \cdots \in \mathbb{Z}[[t]]$

$B\left(\frac{t}{1+t}\right) = tB(t) + 1$

Theorem (Klazar 2003)

$B(t)$ is differentially transcendental over the field of meromorphic functions at 0, i.e. is not solution of an algebraic differential equation with coefficients meromophic at 0.

RMK. $z(t) := \Gamma\left(\frac{1}{t}\right)^{-1}$ solution of $z\left(\frac{t}{t+1}\right) = tz(t)$

homogeneous eq. associated to $B\left(\frac{t}{t+1}\right) = tB(t) + 1$

General result on D-transcendence (Adamczewski-Dreyfus-Hardouin, 2019)

Theorem (Adamczewski-Dreyfus-Hardouin, 2019)

Let $f \in \mathbb{C}((t))$ satisfy

$$\alpha_0 y + \alpha_1 \tau(y) + \cdots + \alpha_n \tau^n(y) = 0,$$

where $\alpha_i \in \mathbb{C}(t)$ and τ is one of the following operators:

- $\tau(f(t)) = f\left(\frac{t}{t+1}\right)$;
- $\tau(f(t)) = f(qt)$ for some $q \in \mathbb{C}^*$, not a root of unity;
- $\tau(f(t)) = f(t^m)$ for some $m \in \mathbb{Z}_{>1}$.

Then either $f \in \mathbb{C}(t)$ or it is D-transcendental over $\mathbb{C}(t)$.

Conjecture (Pak-Yeliussizov, 2018)

$\sum_{n\geq 0} a_n t^n, \sum_{n\geq 0} a_n \frac{t^n}{n!} \in \mathbb{C}[[t]]$ are D-algebraic over $\mathbb{C}(t)$

$\Rightarrow$ both are D-finite

[$\Leftrightarrow$ both $(a_n)_{n\geq 0}$ and $\left(\frac{a_n}{n!}\right)_{n\geq 0}$ satisfiy a lin. recurrence with polynomial coeff. in n]

Generating series coming from combinatorics and functional equations *à la Klazar*

$\hat{\cdot}, \Phi_\tau : \mathbb{C}[[t]] \to \mathbb{C}[[t]]$:

$(\sum_{n\geq 0} g_n t^n)\hat{} := \sum_{n\geq 0} g_n \frac{t^n}{n!}$ $\qquad$ $\Phi_\tau(f) := \frac{1}{1-t} \cdot f\left(\frac{t}{1-t}\right)$

$\forall f, g \in C[[t]]$: $\quad \Phi_\tau(f) = g \Leftrightarrow \hat{g} = \hat{f} \cdot e^t$; $\quad \frac{d}{dt}(\hat{f}) = \left(\frac{f(t)-f(0)}{t}\right)\hat{}$.

Proposition (Bostan, D.V., Raschel)

Let $f \in \mathbb{C}[[t]]$. If $\exists a_0(t), \ldots, a_r(t), P(t) \in C[t]$, s.t.:

$$a_0(e^t)\hat{f} + a_1(e^t)(\hat{f})' + \cdots + a_r(e^t)(\hat{f})^{(r)} = P(t),$$

then f satisfies a linear inhomog. τ-eq., with $\tau(t) := \frac{t}{t+1}$
(of order at most $\max_i(\deg a_i)$, with coefficients in $C[t]$ of degree at most $\max_i(\deg a_i, \deg P)$).

Example

$B(t)$ OGF of Bell numbers

$\hat{B}(t) := \exp(e^t - 1)$ EGF

$\Rightarrow \frac{d}{dt}(\hat{B}) - e^t \cdot \hat{B} = 0$

$\Rightarrow \frac{B-1}{t} = \Phi_\tau(B)$

$\Rightarrow \tau(B) = tB + 1$

polynomial $P_n(x)$	EGF $\sum_{n\geq 0} P_n(x)\frac{t^n}{n!}$	a	b
Bernoulli $B_n(x)$	$\frac{t}{\mathrm{e}^t-1}\cdot\exp(xt)$	$1+t$	$-\frac{(1+t)t}{(xt-t-1)^2}$
Glaisher $U_n(x)$	$\frac{t}{\mathrm{e}^t+1}\cdot\exp(xt)$	$1+t$	$\frac{(1+t)t}{(xt-t-1)^2}$
Apostol-Bernoulli $A_n^{(\gamma)}(x)$	$\frac{t}{\gamma\,\mathrm{e}^t-1}\cdot\exp(xt)$	$\gamma(1+t)$	$-\frac{(1+t)t}{(xt-t-1)^2}$
Imschenetsky $S_n(x)$	$\frac{t}{\mathrm{e}^t-1}\cdot(\exp(xt)-1)$	$1+t$	$\frac{t^2x(xt-2t-2)}{(1+t)(xt-t-1)^2}$
Euler $E_n(x)$	$\frac{2}{\mathrm{e}^t+1}\cdot\exp(xt)$	$-(1+t)$	$\frac{2(1+t)}{1+t-xt}$
Genocchi $G_n(x)$	$\frac{2t}{\mathrm{e}^t+1}\cdot\exp(xt)$	$-(1+t)$	$\frac{2(1+t)t}{(1+t-xt)^2}$
Carlitz $C_n^{(\gamma)}(x)$	$\frac{1-\gamma}{1-\gamma\,\mathrm{e}^t}\cdot\exp(xt)$	$\gamma(1+t)$	$\frac{(1-\gamma)(1+t)}{1+t-xt}$
Fubini $F_n(x)$	$1/(1-x(\mathrm{e}^t-1))$	$\frac{x}{x+1}\cdot(1+t)$	$\frac{1}{x+1}$
Bell-Touchard $\phi_n(x)$	$\exp\left(x(\mathrm{e}^t-1)\right)$	xt	1
Mahler $s_n(x)$	$\exp\left(x(1+t-\mathrm{e}^t)\right)$	$\frac{x(1+t)t}{xt-t-1}$	$\frac{1+t}{1+t-xt}$
Toscano's actuarial $a_n^{(\gamma)}(x)$	$\exp\left(-x\mathrm{e}^t+\gamma t+x\right)$	$\frac{x(1+t)t}{\gamma t-t-1}$	$\frac{1+t}{1+t-\gamma t}$

$$\rightsquigarrow \tau(y) = ay + b$$

Theorem (Bostan, D.V.,Raschel)

Let $a, b \in \mathbb{C}(t)$, with $a \neq 0$, and let $w \in \mathbb{C}((t)) \setminus \mathbb{C}(t)$ verify the difference equation $w\left(\frac{t}{1+t}\right) = aw + b$. Then w is differentially transcendental over $\mathbb{C}(\{t\})$.

The proof relies on difference Galois theory....

Franke (1963), van der Put-Singer (1997)

An example of walks in the quarter plane

Walks in $\mathbb{N}^2$ starting at $(0,0)$, with steps $\mathcal{S} = \{\searrow, \swarrow, \nwarrow\}$

$q_{\mathcal{S}}(i,j;n) = $ # walks of length n ending at (i,j); $\quad Q_{\mathcal{S}}(x,y;t) = \sum_{i,j,n=0}^{\infty} q_{\mathcal{S}}(i,j;n) x^i y^j t^n \in \mathbb{Z}[[x,y,t]]$

$$K(x,y,t)Q(x,y;t) = xy - tx^2 Q(x,0;t) - ty^2 Q(0,y;t)$$

$K(x,y) = xy - t\,(y^2 + x^2y^2 + x^2)$, with $t = \frac{v}{1+v^2}$ $\quad$ $t \in (0,1/2) \Leftrightarrow v \in (0,1)$

Parametrization of K: $\left(x_0(s) = \frac{(1-v^2)s}{v(s^2+1)},\ y_0(s) = \frac{(1-v^2)s}{v^2s^2+1}\right)$ $\left(\widetilde{x_0}(s) = \frac{(1-v^2)v\,s}{v^4s^2+1}, y_0(s)\right)$,

with $\widetilde{x_0}(s) = x_0(v^2 s)$

For $v \in (0,1)$: $G_0(v^2s) - G_0(s) = \frac{(v^2-1)}{v}\left(\frac{1}{s^2+1} - \frac{2}{v^2s^2+1} + \frac{1}{v^4s^2+1}\right)$

$\Rightarrow Q(x,0,t)$ is D-transc (and also $Q(0,y,t)$, $Q(x,y,t)$) for any $t \in (0,1/2)$.

[Dreyfus-Hardouin-Raschel-Singer 2020, Bostan-DV-Raschel 2021]

 $\rightsquigarrow K(x,y) = xy - \frac{1}{2}\left(y^2 + x^2y^2 + x^2\right)$ $t = 1/2 \Leftrightarrow v = 1$

$$K(x,y) = \frac{1}{2}\left(ix - iy + xy\right)\left(ix - iy - xy\right) \Rightarrow \left(x, y(x) := \frac{ix}{1+ix}\right) \in \mathbb{C}(x)^2$$

$G_1(x) := \frac{x^2}{2} Q\left(x, 0, \frac{1}{2}\right)$ verifies the functional equation

$$G_1\left(\tau(x)\right) = G_1(x) + \frac{ix^2}{1+ix}, \text{ with } \tau(x) = \frac{ix}{1+ix}.$$

$\rightarrow Q(x, 0, 1/2)$ is D-transc (and also $Q(0, y, 1/2)$, $Q(x, y, 1/2)$) is D-transc. over the meromorphic functions at the origin.

Theorem (Bostan-DV-Raschel)

$$Q(x, 0; \pm 1/2) = 2\sum_{n \geq 0} (2^{2n+2} - 1)\frac{(-1)^n}{n+1} B_{2n+2} x^{2n},$$

where $(B_n) = \left(1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, \ldots\right)$ is the sequence of Bernoulli numbers.

Iterative functional equations

$(\mathcal{R})$ $R(t) \in \mathbb{C}(t),\ R(0) = 0,\ R'(0) \in \{0, 1, \text{roots of unity}\}$, but no iteration of $R(t)$ is equal to the identity.

Theorem (DV-Fernandes-Mishna)

Let $R(t)$ satisfy assumption $(\mathcal{R})$. We suppose that there exist $a, b \in \mathbb{C}(t)$, and $f \in \mathbb{C}((t))$ such that $f(R(t)) = a(t)f(t) + b(t)$. Then either f is f is D-transc. over $\mathbb{C}(t)$ or:

1. if $b = 0$, $\exists N \in \mathbb{Z}$, $N \geq 1$, such that $f^N \in \mathbb{C}(t)$;
2. if $a = 1$, $f \in \mathbb{C}(t)$;
3. in all the cases there exists $\alpha, \beta \in \mathbb{C}(t)$ such that $f' = \alpha f + \beta$.

A permutation $\sigma \in \mathfrak{S}_n$ is said to avoid the consecutive pattern 1423 if there is no $1 \leq i \leq n-4$ such that $\sigma(i) < \sigma(i+4) < \sigma(i+2) < \sigma(i+3)$.

$\widehat{P}(t)$ = EGF for permutations that avoid the consecutive pattern 1423 [OEISA201692]

$\widehat{P}(t) = \frac{1}{2-\widehat{S}(t)}$ such that $S(t) = S\left(\frac{t}{1+t^2}\right)\frac{t}{1+t} + 1$ [Elizalde and Noy, 2012]

$S(t)$ has infinite number of singularities $\Rightarrow$ not D-finite [Beaton, Conway and Guttmann, 2017]

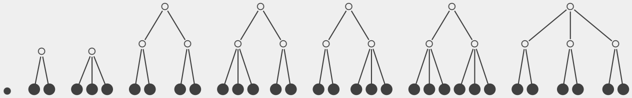
Corollary

$S(t)$ is D-transc. over $\mathbb{C}(t)$

Question

The theorem above gives a potentially simpler path to establish that $S(t)$ is not D-finite (and indeed the even stronger conclusion that it is differentially transcendental) since you would just need to show that $S(t)$ is not solution of an inhomogeneous linear differential equation of order 1.

Complete $\{2,3\}$-trees



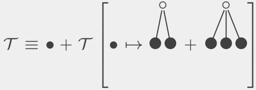


Figure: All complete $\{2,3\}$-trees up to size 6

$t_n = \#$ trees with n leaves $\rightsquigarrow T(t) = \sum_{n\geq 1} t_n t^n$ OGS

$\rightsquigarrow \quad T(t) = t + T(t^2 + t^3)$ $\qquad T(t) = t + t^2 + t^3 + t^4 + 2t^5 + 2t^6 + O(t^7)$ (OEISA014535)

Corollary. $T(t)$ is D-trasc. over $\mathbb{C}(t)$

On easily proves that it cannot be rational!

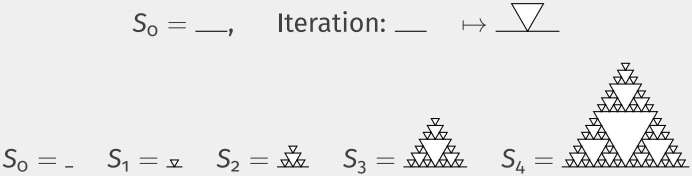


Figure: Initial iterates defining the Sierpiński graph.

Green function = probability generating function which describes the n-step displacement starting and returning to a certain origin vertex

Green function $G(t)$ for walks that return to their origin on the Sierpiński graph satisfies the functional equation: [Grabner and Woess 1997]

$$G\left(\frac{t^2}{4-3t}\right) = \frac{(2+t)(4-3t)}{(4+t)(2-t)}\,G(t). \qquad (1)$$

$\Rightarrow$ D-transc. over $\mathbb{C}(t)$.

Galois theory of functional equations

K of characteristic zero field + an automorphism $\tau : K \to K$
$C := K^\tau := \{f \in K : \tau(f) = f\}$ = algebraically closed field (=the "constants of the theory")

EXAMPLE.

$K = \mathbb{C}((t))$ and $\tau(f(t)) := f\left(\frac{t}{t+1}\right), \forall f \in \mathbb{C}((t)) \Rightarrow \mathbb{C}((t))^\tau = \mathbb{C}$

$\tau(\vec{y}) = A\vec{y}$, where $A \in GL_\nu(K)$

RMK. $\vec{y}$ is a vector of unknowns and τ acts on vectors (and later also on matrices) componentwisely.

A Picard-Vessiot ring for $\tau(\vec{y}) = A\vec{y}$ over K is a K-algebra R plus an automorphism extending the action of τ:

1. R is τ-simple;
2. $\exists Y \in \mathrm{GL}_\nu(R)$ s.t. $\tau(Y) = AY$ and $R = K[Y, \det Y^{-1}]$.

RMK. A Picard-Vessiot ring always exists.

EXAMPLE

Let $a \in K$, $a \neq 0$ and R be the Picard-Vessiot ring of $\tau(y) = ay$. Then there exists $z \in R$ such that $\tau(z) = az$ and $R = K[z, z^{-1}]$.

The Galois group G of $\tau(\vec{y}) = A\vec{y}$ over K:

$\mathrm{Aut}^{\tau}(R/K) =$ automorphisms of rings $\varphi : R \to R$ that commute with τ and s.t. $\varphi_{|K} = id$

$\varphi(Y) \in \mathrm{GL}_{\nu}(R)$ solution of $\tau(\vec{y}) = A\vec{y}$.
$\tau(Y^{-1}\varphi(Y)) = Y^{-1}\varphi(Y)$ and hence that $Y^{-1}\varphi(Y) \in \mathrm{GL}_{\nu}(C)$.
$\rightsquigarrow$ a natural group morphism $G \to \mathrm{GL}_{\nu}(C)$.

THEOREM

1. $G \to \mathrm{GL}_{\nu}(C)$, $\varphi \mapsto Y^{-1}\varphi(Y)$ is an injective group morphism.
2. $\mathrm{tr.deg}_K R = \dim_C G$.

EXAMPLE. $\tau(y) = ay$, $a \in K$

$\forall\varphi$ of G $\varphi(z)$ is another solution of $\tau(y) = ay$: $\varphi(z) = c_{\varphi}z$, for some $c_{\varphi} \in C \Rightarrow G \subset C^*$

z is transcendental $/K$ if and only if $G = C^*$.
Since the only algebraic subgroups of C^* are the groups of roots of unity,
z is algebraic over K if and only if G is a group of roots of unity,
i.e., if and only if there exists a positive integer N s.t. $z^N \in K$.

$\tau(y) = ay + f$, $a, b \in K \rightsquigarrow \tau(\vec{y}) = \begin{pmatrix} a & f \\ 0 & 1 \end{pmatrix} \vec{y}$

$Y = \begin{pmatrix} z & w \\ 0 & 1 \end{pmatrix}$, with $\tau(z) = az$ and $\tau(w) = aw + f$

$\Rightarrow R = K[z, z^{-1}, w]$ (Picard-Vessiot ring)

$\forall \varphi \in G$, therefore there exist $c_\varphi, d_\varphi \in C$ such that $\varphi(z) = c_\varphi z$ and $\varphi(w) = w + d_\varphi z$

$$\varphi \begin{pmatrix} z & w \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} z & w \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_\varphi & d_\varphi \\ 0 & 1 \end{pmatrix}$$

$$\widetilde{G} := \left\{ \begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix} : c, d \in C, c \neq 0 \right\} \subset \mathrm{GL}_2(C)$$

According to whether z and w are algebraically dependent or not, either G will be a proper linear algebraic subgroup of $\widetilde{G}$, or $G = \widetilde{G}$.

$R = \mathbb{R}_1 \oplus \cdots \oplus \mathbb{R}_r$ is a direct sum of domains

$\mathbb{L} = \mathbb{L}_1 \oplus \cdots \oplus \mathbb{L}_r$, where $\mathbb{L}_i = Frac(R_i)$ and $\tau(\mathbb{L}_i) = \mathbb{L}_{i+1}$

$\mathbb{L}$ = the total Picard-Vessiot ring of $\tau(\vec{y}) = A\vec{y}$

The action of the Galois group $\mathrm{Aut}^{\tau}(R/K)$ naturally extends from R to $\mathbb{L}$.

Porposition

$\mathbb{L}$ is uniquely determined (up to an isomorphism) by the following properties:

1. $\mathbb{L}$ has no nilpotent elements and any non-zero divisor of $\mathbb{L}$ is invertible.
2. $\mathbb{L}^{\tau} = C$.
3. $\exists Y \in \mathrm{GL}_{\nu}(\mathbb{L})$ solution of $\tau(\vec{y}) = A\vec{y}$.
4. $\mathbb{L}$ is minimal with respect to the inclusion and the three previous properties.

Any τ-ring satisfying the $1, 2, 3$ above contains a copy of R.

$$\mathcal{F} = \{\tau\text{-stable rings } K \subset F \subset \mathbb{L}, \text{ s.t. } \forall f \in F \text{ is either a zero divisor or a unit in } F\}$$

$$\forall F \in \mathcal{F}, H_F := \{\varphi \in \mathcal{G} : \varphi(f) = f \text{ for all } f \in F\}.$$

$$\mathcal{G} = \{\text{linear algebraic subgroups of } G\}$$

$$\forall H \in \mathcal{G}, \mathbb{L}^H = \{f \in \mathbb{L} : \varphi(f) = f \text{ for all } \varphi \in H\}$$

Theorem. The following two maps are each other's inverses:

$$\begin{array}{ccc} \mathcal{G} & \to & \mathcal{F} \\ H & \mapsto & \mathbb{L}^H \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{F} & \to & \mathcal{G} \\ F & \mapsto & H_F \end{array}.$$

In particular, $\mathbb{L}^H = K$ if and only if $H = G$.

There exists a derivation ∂ on K commuting with τ.

EXAMPLE: for $\tau(f(t)) = f\left(\frac{t}{1+t}\right)$, we can take $\partial := t^2 \frac{d}{dt}$

Existence-definition of ∂-Picard-Vessiot ring (Wibmer)

There exists a K-algebra $\mathcal{R}$, equipped with an extension of τ and of ∂, preserving the commutation, such that:

1. there exists $Y \in \mathrm{GL}_\nu(\mathcal{R})$ such that $\tau(Y) = AY$;
2. $\mathcal{R}$ is generated over K by the entries of Y, $\frac{1}{\det(Y)}$ and all their derivatives;
3. $\mathcal{R}$ is τ-simple.

Moreover, the total ring of fractions of $\mathcal{R}$ contains the total Picard-Vessiot ring L.

Applying ∂^n to the system $\tau(\vec{y}) = A\vec{y}$ for any positive integer n, we can consider the difference system:

$$\tau(\vec{y}) = \begin{pmatrix} A & \partial(A) & \cdots & \frac{\partial^n}{n}(A) \\ 0 & A & \ddots & \vdots \\ \vdots & \ddots & \ddots & \partial(A) \\ 0 & \cdots & 0 & A \end{pmatrix} \vec{y}, \quad \text{with solution} \begin{pmatrix} \frac{\partial^n}{n!}(Y) & \frac{\partial^{n-1}}{(n-1)!}(Y) & \cdots & Y \\ 0 & \frac{\partial^n}{n!}(Y) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{\partial^{n-1}}{(n-1)!}(Y) \\ 0 & \cdots & 0 & \frac{\partial^n}{n!}(Y) \end{pmatrix}.$$

F will be a K-algebra s.t.:

1. with no nilpotent elements;
2. any element is either a zero divisor, or invertible;
3. $\exists\ \tau$ and of ∂ on F, preserving the commutation;
4. $F^\tau = C$

Proposition

Let $f \in K$, and let $w \in F$ be such that $\tau(w) = w + f$. Then the following assertions are equivalent:

1. w is differentially algebraic over K.
2. $\exists$ integer $n \geq 0$, $\alpha_0, \ldots, \alpha_n \in C$ (not all zero) and $g \in K$ s.t. $\alpha_0 f + \alpha_1 \partial(f) + \cdots + \alpha_n \partial^n(f) = \tau(g) - g$.
3. $\exists$ integer $n \geq 0$, $\alpha_0, \ldots, \alpha_n \in C$ (not all zero) s.t. $g := \sum_{i=0}^{n} a_i \partial^i(w) \in K$.

We consider the OGF of the family of Bernoulli polynomials,

$$\tau(B) = (1+t)\cdot B - \frac{t(1+t)}{(1+t-tx)^2},$$

$\widetilde{B}(x,t) := tB(x,t) \rightsquigarrow \tau(\widetilde{B}) = \widetilde{B} - \left(\frac{t}{1+t-tx}\right)^2$

$\widetilde{B}$ is differentially transcendental over $\mathbb{C}(t)$, $\forall\, x \in \mathbb{C}$

By contradiction, $\exists n \geq$, $\alpha_0, \dots, \alpha_n$ (not all zero) and $g \in \mathbb{C}(t)$ such that

$\alpha_0 b + \alpha_1 \partial(b) + \cdots + \alpha_n \partial^n(b) = \tau(g) - g$, with $\partial := t^2 \frac{d}{dt}$ and $b = \left(\frac{t}{1+t-tx}\right)^2$.

$\partial^k(b) = (k+1)! \left(\frac{t}{1+t-tx}\right)^{k+2} \quad \forall k \geq 1$

$x \neq 1 \Rightarrow$ the left-hand side of has a unique pole at $t_0 = \frac{1}{x-1}$...

$x = 1 \Rightarrow$, the left-hand side is a non-zero polynomial with no constant term... one shows that g can only be a constant...

Let b and a be non-zero elements of K, and let us consider the difference equation $\tau(y) = ay + b$.

Theorem

$\tau(y) = ay + b$, with $a, b \in K$, such that $a \neq 0, 1$ and $b \neq 0$.
Let F/K be a field extension such that there exists $w \in F \setminus K$ satisfying the equation $\tau(w) = aw + b$.
Moreover, let F_a be a K-algebra as above, such that there exists $z \in F_a$ satisfying the equation $\tau(z) = az$.
If z is differentially transcendental over K, then w is differentially transcendental over K.

Thanks!

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