Gröbner bases of maximal minors and critical point computations

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Let $\mathbb{k}$ be a field. Let $F = (f_1, \ldots, f_p) \subseteq \mathbb{k}[x_1, \ldots, x_n]$ be a set of polynomials in n variables, and let $g \in \mathbb{k}[x_1, \ldots, x_n]$ be a polynomial. The problem of computing the critical points of g restricted to $V(F)$ is a classical one in effective algebraic geometry and has applications in several scientific disciplines. The ideal $\mathcal{I}(g, F)$ corresponding to the variety consisting of the critical points of g restricted to $V(F)$ is generically zero-dimensional. A key step in solving the polynomial system defining $\mathcal{I}(g, F)$ using Gröbner bases is computing a grevlex Gröbner basis of $\mathcal{I}(g, F)$, which can be accomplished using a signature-based Gröbner basis algorithm such as F_5. Generically, the ideal $\mathcal{I}(g, F)$ cannot be generated by a regular sequence and thus F_5 encounters reductions to zero when computing its grevlex Gröbner basis. These reductions to zero arise entirely from the syzygies between the maximal minors of $\mathrm{jac}(g, F)$, the Jacobian matrix of $\{g, f_1, \ldots, f_p\}$. By analyzing the Eagon-Northcott complex associated to $\mathrm{jac}(g, F)$, we devise new criteria to detect and avoid more reductions to zero arising from these syzygies. We provide a complexity analysis of our algorithm which improves upon the best previously known complexity bound for computing a Gröbner basis of $\mathcal{I}(g, F)$ by a factor which is at least quadratic in n.
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