Solving parameter-dependent semi-algebraic systems with Hermite matrices

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This is a joint work with Mohab Safey El Din. We consider systems of polynomial equations and inequalities in $\mathbb{Q}[\boldsymbol{y}][\boldsymbol{x}]$ where $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_t)$. The $\boldsymbol{y}$ indeterminates are considered as *parameters* and we assume that when specialising them *generically*, the set of common complex solutions, to the obtained equations, is finite. We present an algorithm for the problem of real root classification of such parameter-dependent polynomial systems, *i.e.* identifying the possible number of real solutions depending on the values of the parameters and computing a description of the regions of the space of parameters over which the number of real roots remains invariant. Our algorithm uses a real root counting machinery based on so-called Hermite's quadratic form in some appropriate basis. The formulas it outputs enjoy a determinantal structure as they encode some constraints on the signature of *parametric Hermite matrices*. Under genericity assumptions, we show that its arithmetic complexity is polynomial in both the maximum degree d and the number s of the input inequalities and exponential in $nt + t^2$. The output formulas consist of polynomials of degree bounded by $(2s + n)d^{n+1}$. This is the first algorithm with such a singly exponential complexity.