Eleanor Archer

GHP convergence via GP convergence and lower mass bound

Background

There are many strategies to prove GHP convergence of metric-measure spaces; I would like to discuss one which could be useful in a range of different settings and seems to be fairly under-used. In the paper [ALW16], the authors showed that distributional GHP convergence of a sequence $(X_n, d_n, \mu_n)_{n\geq 1}$ to a limit (X, d, μ) is equivalent to the following two conditions.

(a) Convergence in distribution of the relevant k-point functions for all $k \geq 2$ (w.r.t. to the product topology). (When (X, d, μ) is compact and μ has full support this is equivalent to Gromov weak or Gromov-Prohorov convergence.)

(b) The lower mass bound, that is, if for all $\delta > 0$ and $n \geq 1$ we set

$$m_n(\delta) := \inf_{x\in X_n} \mu(B(x, \delta)),$$

then for all $\delta > 0$ the sequence $(m_n(\delta)^{-1})_{n\geq 1}$ is a tight sequence in $\mathbb{R}$.

The strategy of proving GHP convergence via conditions (a) and (b) is particularly useful in the case where there is no canonical coupling between X_n and X that can be used to construct a correspondence with vanishing distortion. Establishing GHP convergence via (a) and (b) above has been carried out in [BDvdHS20, BDvdHS22] for the critical configuration models, in [BvdHS18, BS20] for the multiplicative coalescent and heavy-tailed mimimum spanning trees, in [Fri21] for coalescence structures and in [PR04, ANS21, AS23] for high-dimensional uniform spanning trees.

In [ANS21] we introduced a general bootstrapping strategy which reduces (b) to a local tail bound for the volume of a ball around a uniform vertex. We anticipate that this should be useful for proving convergence of other mm-spaces.

Some other examples of graphs for which it may be possible to prove GHP scaling limits in this way are as follows. (However, I have not checked all the details and it is possible that these strategies are too optimistic.)

1. Minimal spanning tree (MST) of the hypercube. Here the main challenge should be proving (a). Take a graph G and endow each edge with an i.i.d. $\mathsf{Uniform}([0,1])$ edge weight. $MST(G)$ is the spanning tree T of G that minimises the sum of the edge weights of T. MSTs can be sampled using greedy algorithms such as Prim's algorithm or Kruskal's algorithm and in fact this leads to nice connections between MSTs and critical percolation clusters on G. On the complete graph this connection was exploited to establish the GHP scaling limit of $MST(K_n)$ [ABBGM17, ABBG12]. In [vdHN17] van der Hofstad and Nachmias showed that paths in percolation clusters on the hypercube look like non-backtracking random walks. Taking a bit of a leap, this suggests that paths in the MST look like non-backtracking random walks. Since the mixing times of these non-backtracking random walks are smaller than the scaling factors, it may therefore be possible to couple (segments of) non-backtracking random walks on the hypercube with non-backtracking random walks on the complete graph two show that the k-point functions have the same scaling limit.
2. Uniform spanning tree (UST) of 4d torus. This programme has already been initiated by Schweinsberg who proved (a) in [Sch09]. To prove (b) one would have to fine-tune the strategy used for five dimensions and higher in [ANS21]. This would require one to obtain precise control of 1) length and capacity of loop-erased random walks, and possibly 2) finer control of dependencies in different parts of the UST. Note that four is the critical dimension for USTs and the proofs are likely to be much more technical than in the high dimensional case.

Résumé

I would like to discuss the general strategy of proving GHP convergence in this way and explain (at least informally) why (a) and (b) give GHP convergence. I will also outline the bootstrap method and resulting local tail bound condition. Then I will explain a bit of background about the models above and why their scaling limits may be amenable to this strategy.

Bibliography

[ABBG12] L. Addario-Berry, N. Broutin, and C. Goldschmidt. The continuum limit of critical random graphs. *Probability Theory and Related Fields*, 152(3):367–406, 2012.

[ABBGM17] L. Addario-Berry, N. Broutin, C. Goldschmidt, and G. Miermont. The scaling limit of the minimum spanning tree of the complete graph. *The Annals of Probability*, 45(5):3075–3144, 2017.

[ALW16] S. Athreya, W. Löhr, and A. Winter. The gap between Gromov-vague and Gromov-Hausdorff-vague topology. *Stochastic Process. Appl.*, 126(9):2527–2553, 2016.

[ANS21] E. Archer, A. Nachmias, and M. Shalev. The GHP scaling limit of uniform spanning trees in high dimensions. *arXiv preprint arXiv:2112.01203*, 2021.

[AS23] E. Archer and M. Shalev. The GHP scaling limit of uniform spanning trees of dense graphs. *arXiv preprint arXiv:2301.00461*, 2023.

[BDvdHS20] S. Bhamidi, S. Dhara, R. van der Hofstad, and S. Sen. Universality for critical heavy-tailed network models: Metric structure of maximal components. *Electronic Journal of Probability*, 25:1 – 57, 2020.

[BDvdHS22] S. Bhamidi, S. Dhara, R. van der Hofstad, and S. Sen. Global lower mass-bound for critical configuration models in the heavy-tailed regime. *Electronic Journal of Probability*, 27:1 – 29, 2022.

[BS20] S. Bhamidi and S. Sen. Geometry of the minimal spanning tree in the heavy-tailed regime: new universality classes. *arXiv preprint arXiv:2009.10696*, 2020.

[BvdHS18] S. Bhamidi, R. van der Hofstad, and S. Sen. The multiplicative coalescent, inhomogeneous continuum random trees, and new universality classes for critical random graphs. *Probability Theory and Related Fields*, 170(1):387–474, 2018.

[Fri21] N. Frilet. *Metric coalescence of homogeneous and inhomogeneous random graphs*. Theses, Université Grenoble Alpes [2021], December 2021.

[PR04] Y. Peres and D. Revelle. Scaling limits of the uniform spanning tree and loop-erased random walk on finite graphs. *arXiv preprint arXiv:0410430*, 2004.

[Sch09] J. Schweinsberg. The loop-erased random walk and the uniform spanning tree on the four-dimensional discrete torus. *Probability Theory and Related Fields*, 144(3-4):319–370, 2009.

[vdHN17] R. van der Hofstad and A. Nachmias. Hypercube percolation. *Journal of the European Mathematical Society*, 19(3):725–814, 2017.