量子物理与光量子信息教育部重点实验室
Key Laboratory of Quantum Physics and Photonic Quantum Information,
Ministry of Education, University of Electronic Science and Technology of China

Foundations of Quantum Physics and the Marginal Problem

Zizhu Wang

Joint work with

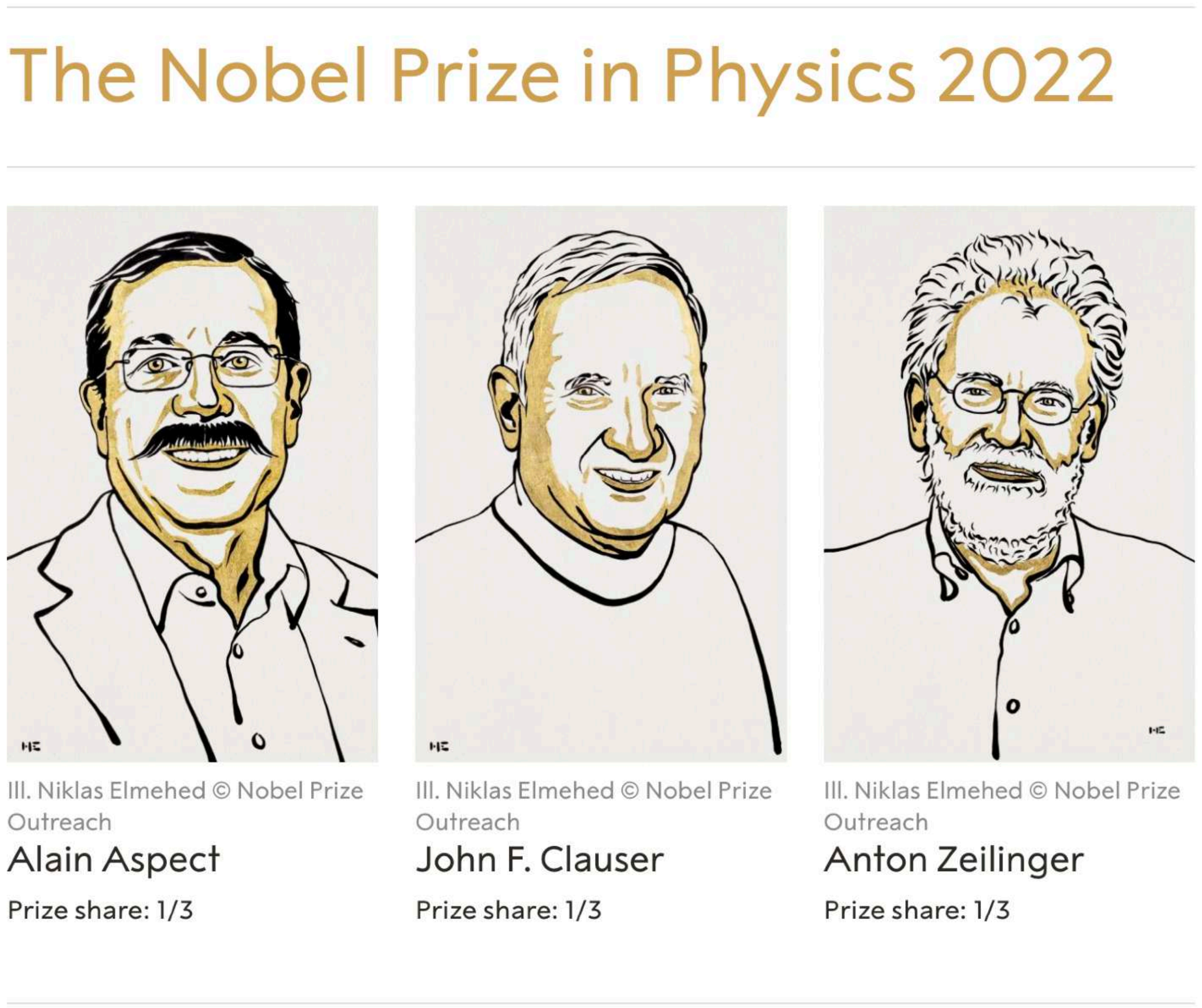
Miguel Navascués & Xiao Zeng


AND NOW FOR
SOMETHING
COMPLETELY
DIFFERENT

What's in This Talk?

- What is quantum nonlocality? The story behind the 2022 Nobel prize in physics.
- The marginal problem in 1D and its applications in quantum information.
- The 2D marginal problem: results, progress and open questions.

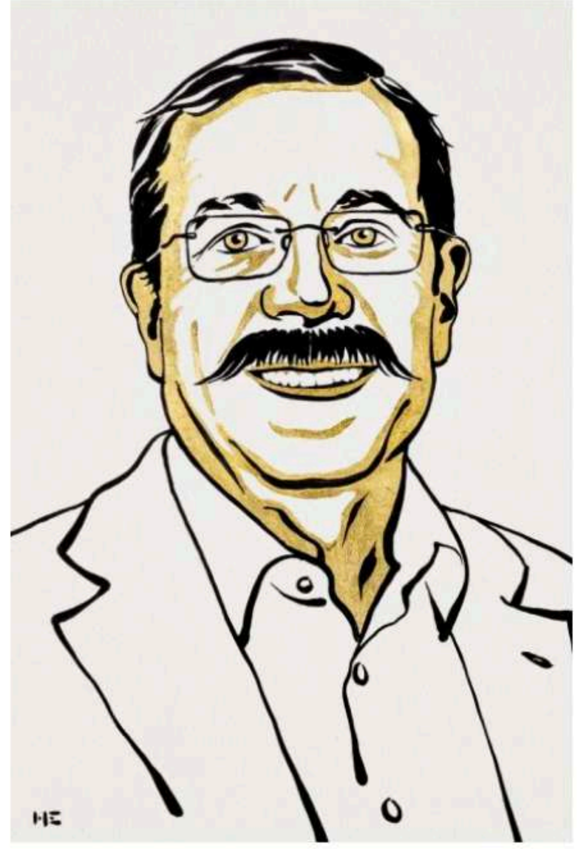
And the Award Goes To...



The Nobel Prize in Physics 2022 was awarded jointly to Alain Aspect, John F. Clauser and Anton Zeilinger "for experiments with entangled photons, establishing the violation of Bell inequalities and pioneering quantum information science"

And the Award Goes To...

The Nobel Prize in Physics 2022



Ill. Niklas Elmehed © Nobel Prize Outreach

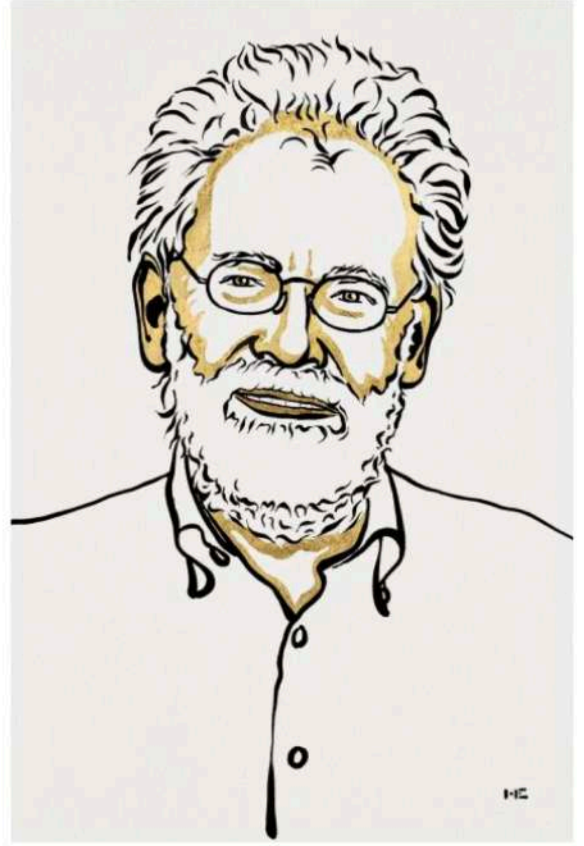
Alain Aspect

Prize share: 1/3

Ill. Niklas Elmehed © Nobel Prize Outreach

John F. Clauser

Prize share: 1/3



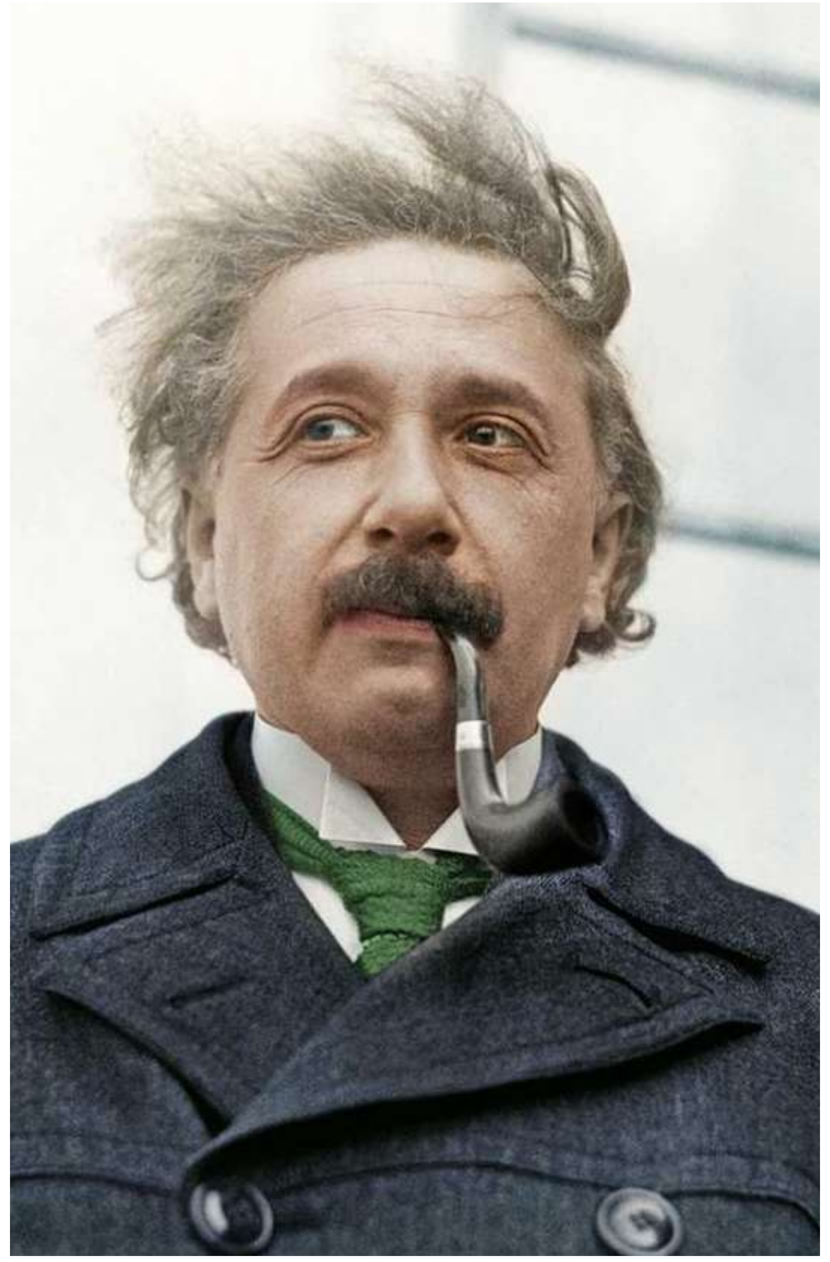
Ill. Niklas Elmehed © Nobel Prize Outreach

Anton Zeilinger

Prize share: 1/3

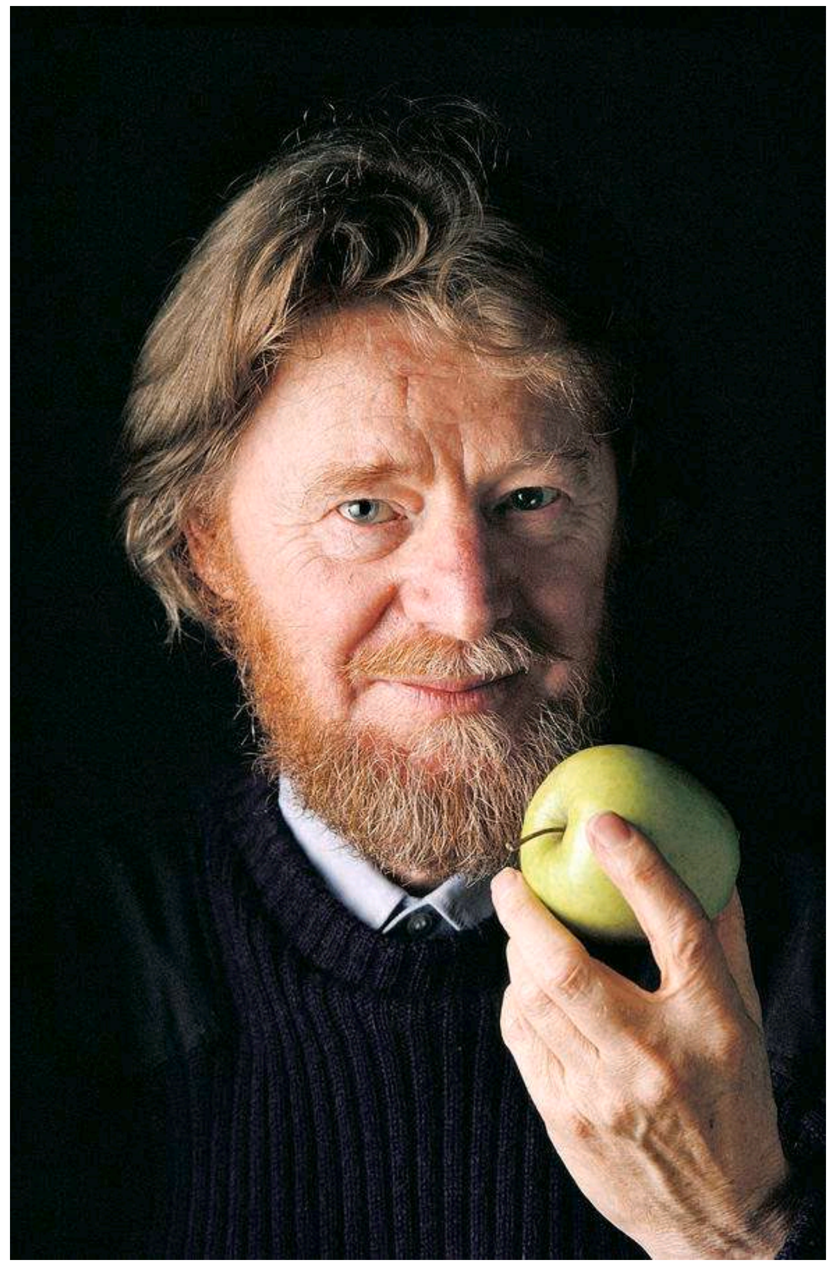
The Nobel Prize in Physics 2022 was awarded jointly to Alain Aspect, John F. Clauser and Anton Zeilinger "for experiments with entangled photons, establishing the violation of Bell inequalities and pioneering quantum information science"

What Are “Bell Inequalities”?



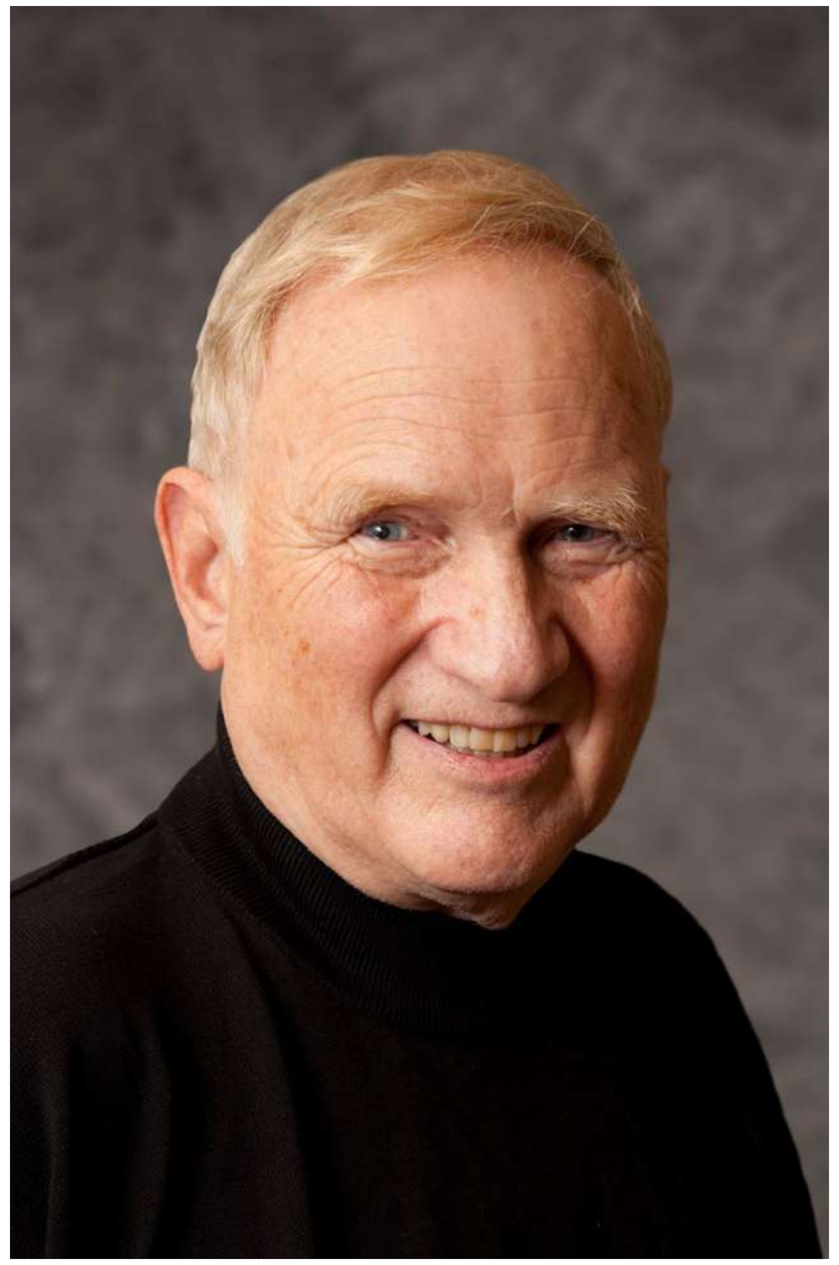
Albert Einstein

EPR Paradox



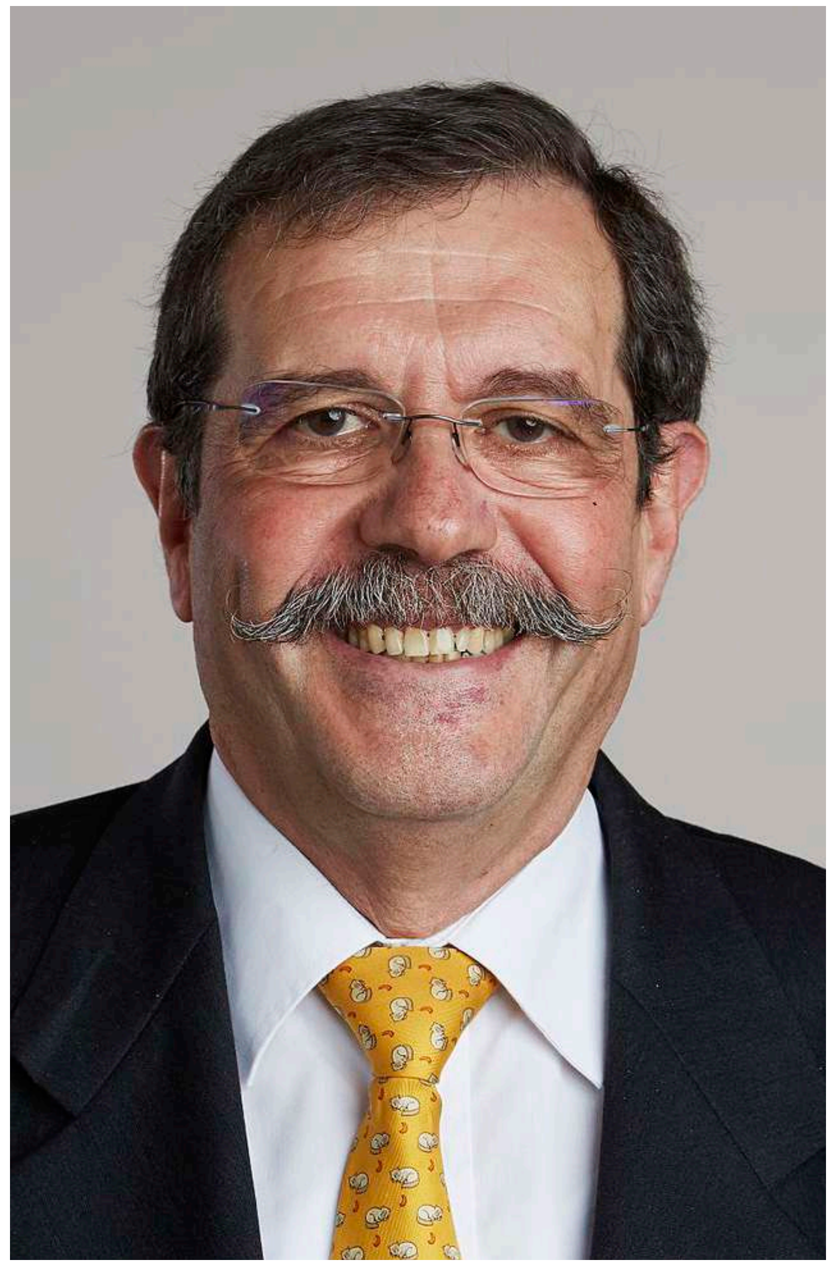
John S. Bell

Formalize LHV / Bell Inequality



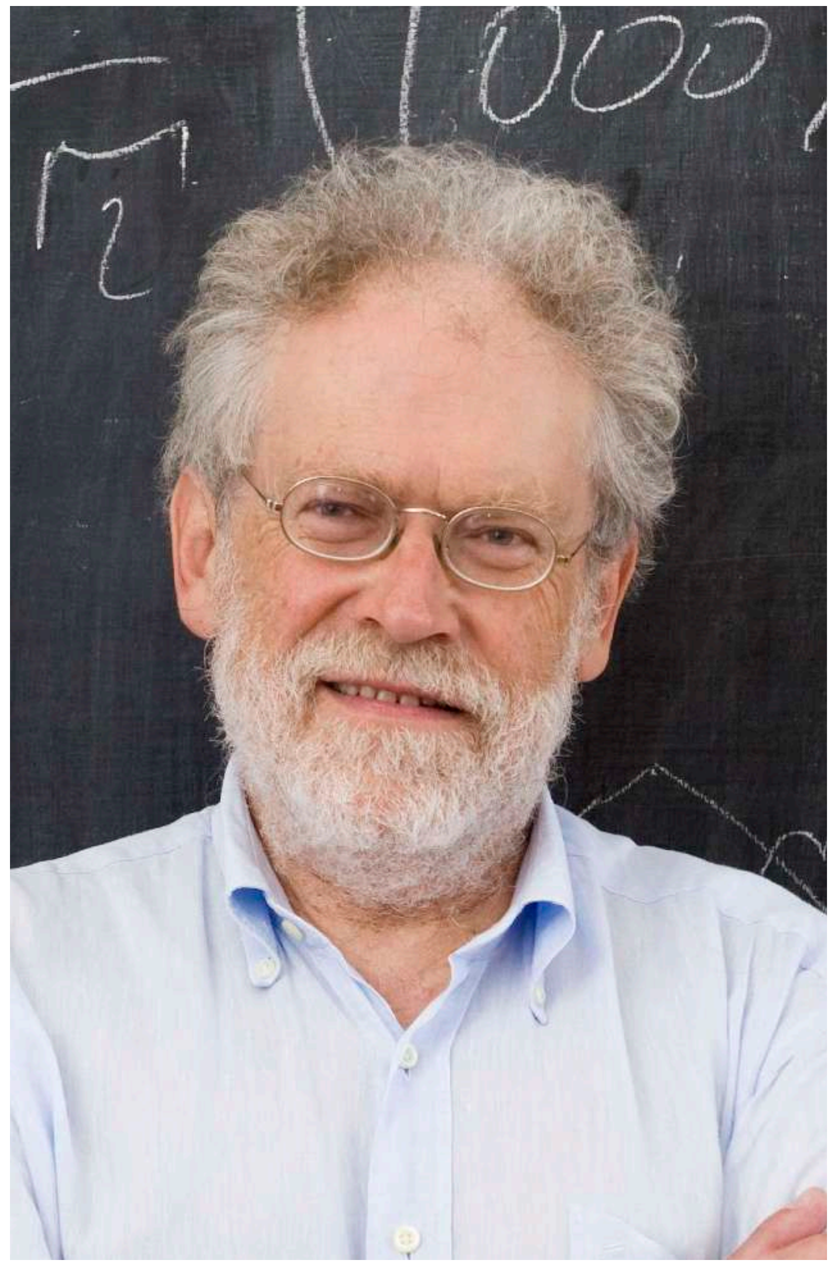
John F. Clauser

1st Bell Test / CH-CHSH Inequalities



Alain Aspect

1st Bell Test with Entangled Photons

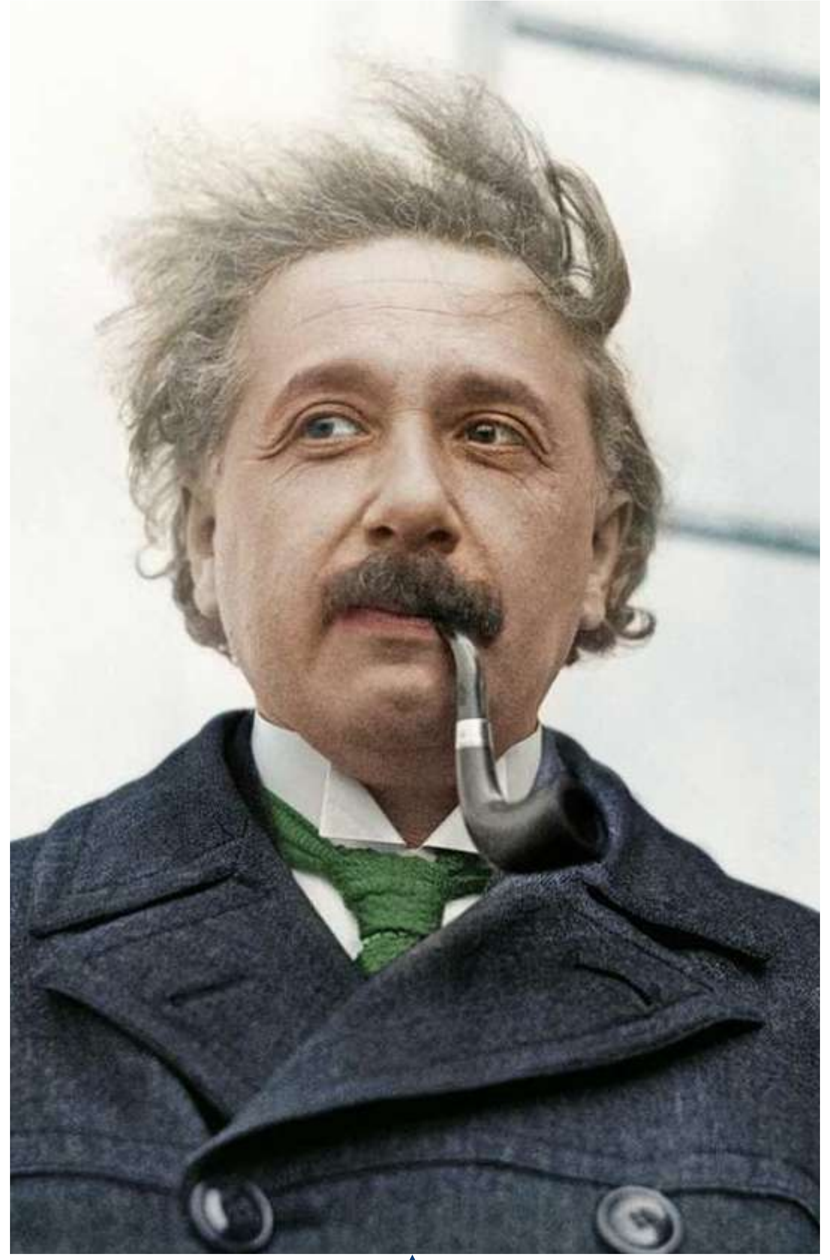


Anton Zeilinger

Loophole-free Bell Test

It all Began in 1935...

EPR Paradox



Albert Einstein

MAY 15, 1935 PHYSICAL REVIEW VOLUME 47

Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?

A. EINSTEIN, B. PODOLSKY AND N. ROSEN, *Institute for Advanced Study, Princeton, New Jersey*

(Received March 25, 1935)

In a complete theory there is an element corresponding to each element of reality. A sufficient condition for the reality of a physical quantity is the possibility of predicting it with certainty, without disturbing the system. In quantum mechanics in the case of two physical quantities described by non-commuting operators, the knowledge of one precludes the knowledge of the other. Then either (1) the description of reality given by the wave function in quantum mechanics is not complete or (2) these two quantities cannot have simultaneous reality. Consideration of the problem of making predictions concerning a system on the basis of measurements made on another system that had previously interacted with it leads to the result that if (1) is false then (2) is also false. One is thus led to conclude that the description of reality as given by a wave function is not complete.

What Is “Real”?

“Physical Reality”

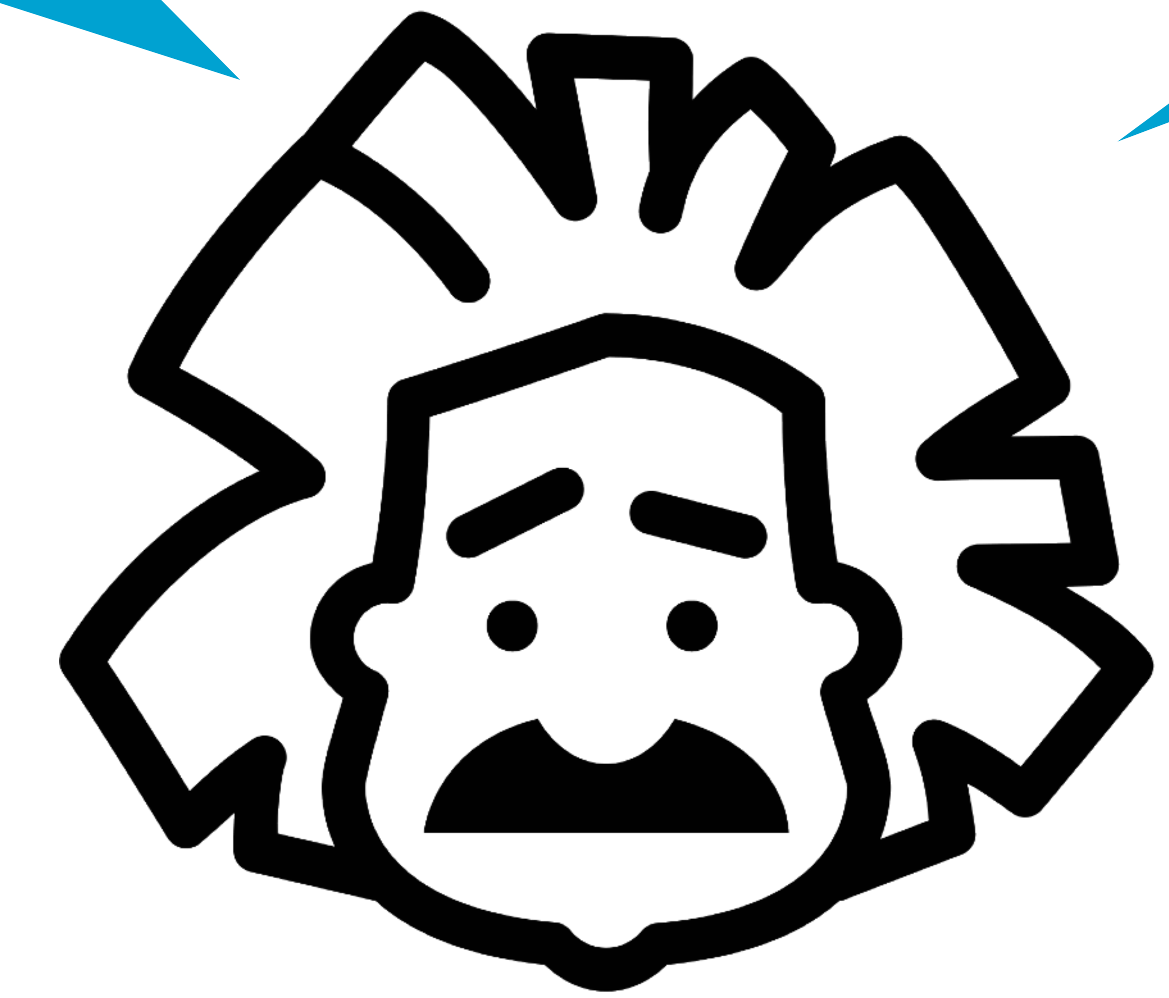
What Is “Real”?

“Physical Reality”

What Is “Real”?

The possibility of predicting its value with certainty, without disturbing the system.

Every element of the physical reality must have a counterpart in the physical theory.



What Is “Real”?

The EPR Argument

1. Consider position and momentum: x and p.
2. In quantum mechanics, they correspond to operators $\hat{X}$ and $\hat{P}$.
3. These operators do not commute: $\hat{X}\hat{P} \neq \hat{P}\hat{X}$.
4. Therefore the measurement of one precludes the knowledge of the other.
5. Therefore they can not have simultaneous reality.

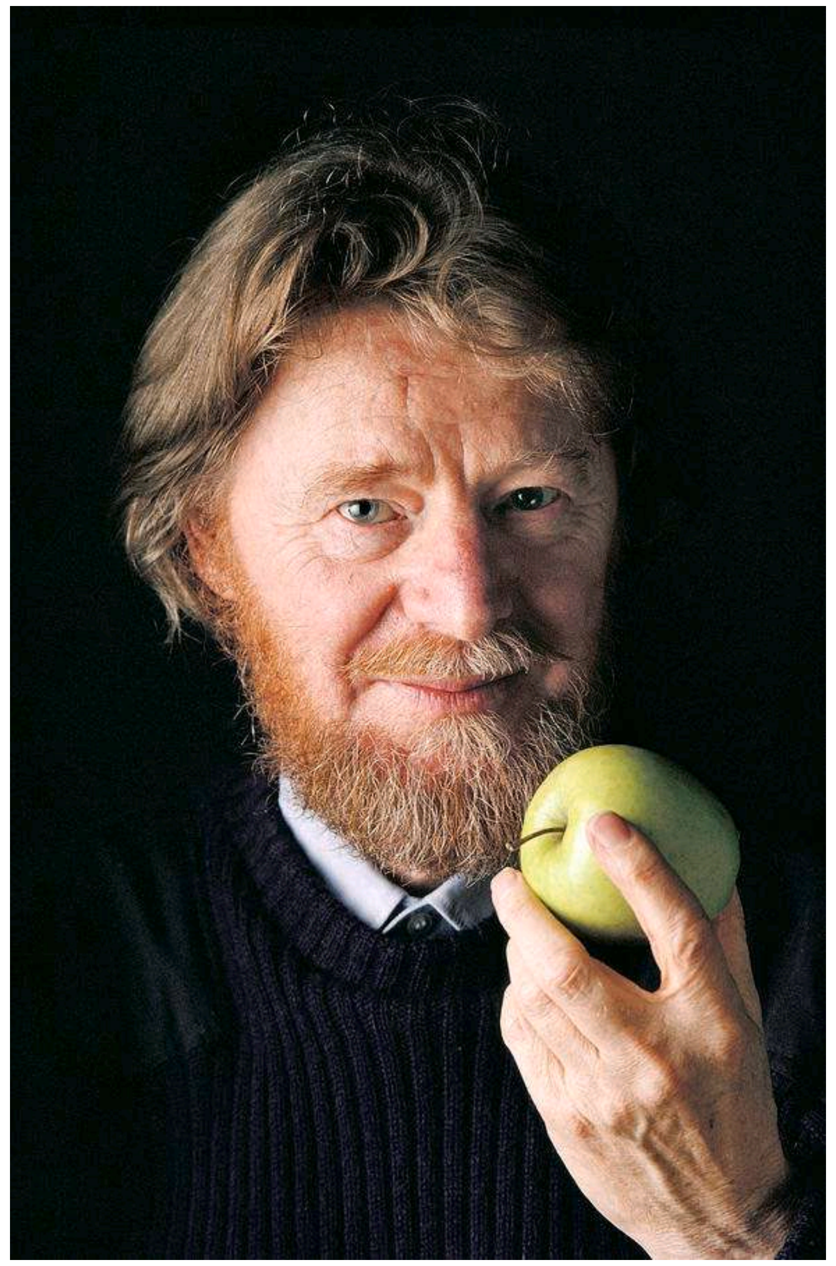
What Is “Real”?

Something is missing…

- Are wave functions real?
- Even if they are real, how can we measure them?
- If the wave function does not give a complete description of physical reality, what does?
- What if there’s some “hidden variable”, which can not be experimentally revealed, but nevertheless dictates the outcomes of our experiments?

Think Outside the Box

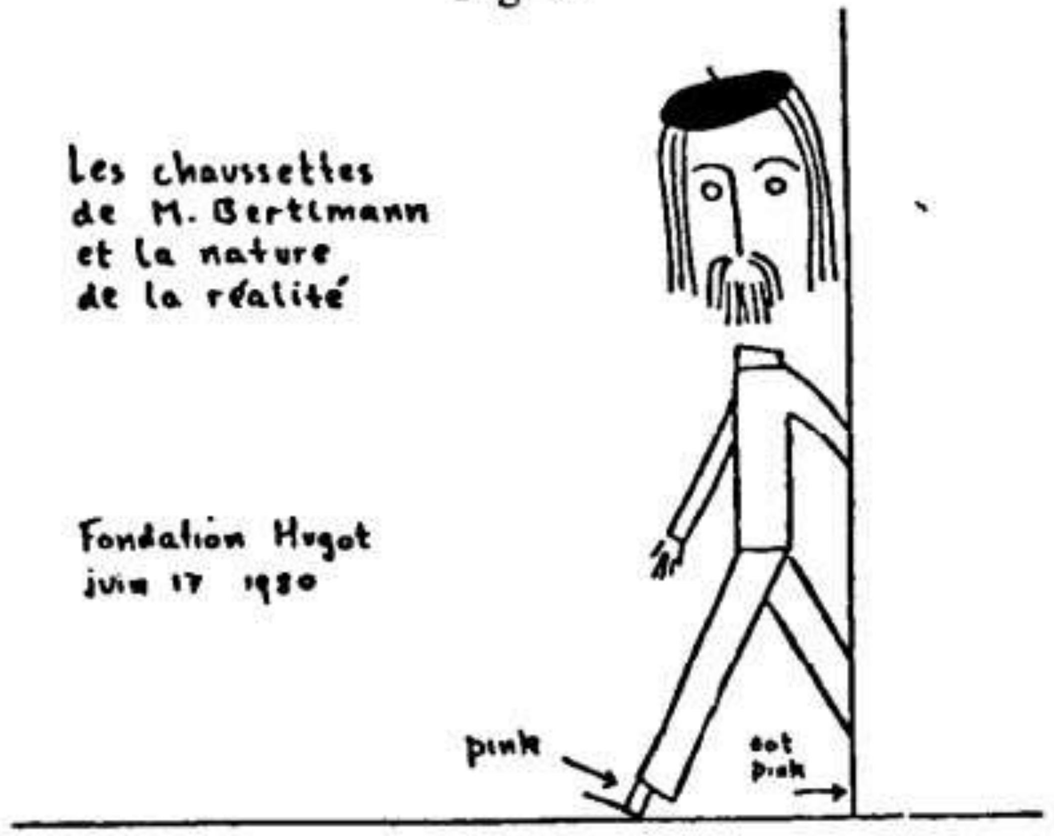
Enter John Bell



John Bell

- "Real job" is a particle physicist working at CERN.
- Thinking about quantum foundations is his "secret hobby".
- Only a few close friends at CERN knew what he was doing. One of them is Reinhold Bertlmann, whose colorful socks were made famous by Bell.
- Died from brain hemorrhaging in 1990, the year when he was considered for a Nobel prize.
- Propose the first "Bell's inequality" exactly 60 year ago!

Fig. 1.



Think Outside the Box

Bell's Insight

We need a theory-independent framework to compare quantum and classical physics.

1. Allow both quantum and classical physics to make predictions.
2. General enough to incorporate "unknown variables".
3. Able to express the notions of "independence" and "disturbance".

Use probability!

Think Outside the Box

Morpheus

Neo

m

n

$$P(a, b \mid m, n) = \int P(a \mid m, \lambda) P(b \mid n, \lambda) P(\lambda) \, d\lambda$$

Think Outside the Box

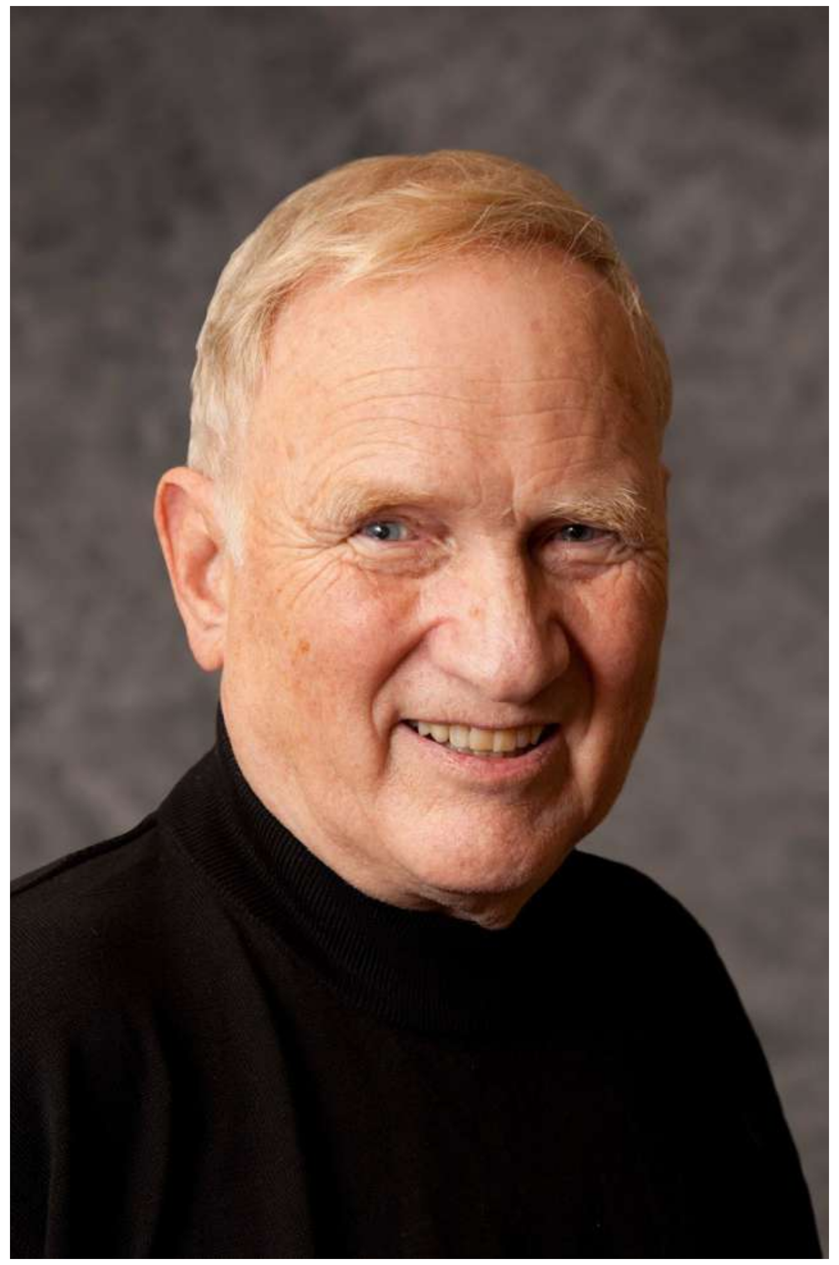
Local Hidden Variable Theory

$$P(a, b \mid m, n) = \int P(a \mid m, \lambda) P(b \mid n, \lambda) P(\lambda)\, d\lambda$$

1. Physical reality means probability = 1.
2. “No disturbance” guaranteed by special relativity.
3. Hidden influence also can not travel faster than light.
4. Predictions in LHV may differ from those made in quantum theory!

Make It Testable

John Clauser, the capital C



John Clauser

- Bell's original formulation is not very experiment-friendly.
- EPR and Bell all used the singlet as the quantum mechanical model.
- To actually test LHV using quantum particles, Clauser had to invent new "inequalities".
- The most famous "Bell inequalities" are called CH and CHSH.
- Clauser is the first "C" in both of them.

Think Outside the Box

Morpheus

Neo

$m \in \{0,1\}$

$n \in \{0,1\}$

$P(a\,|\,m)$

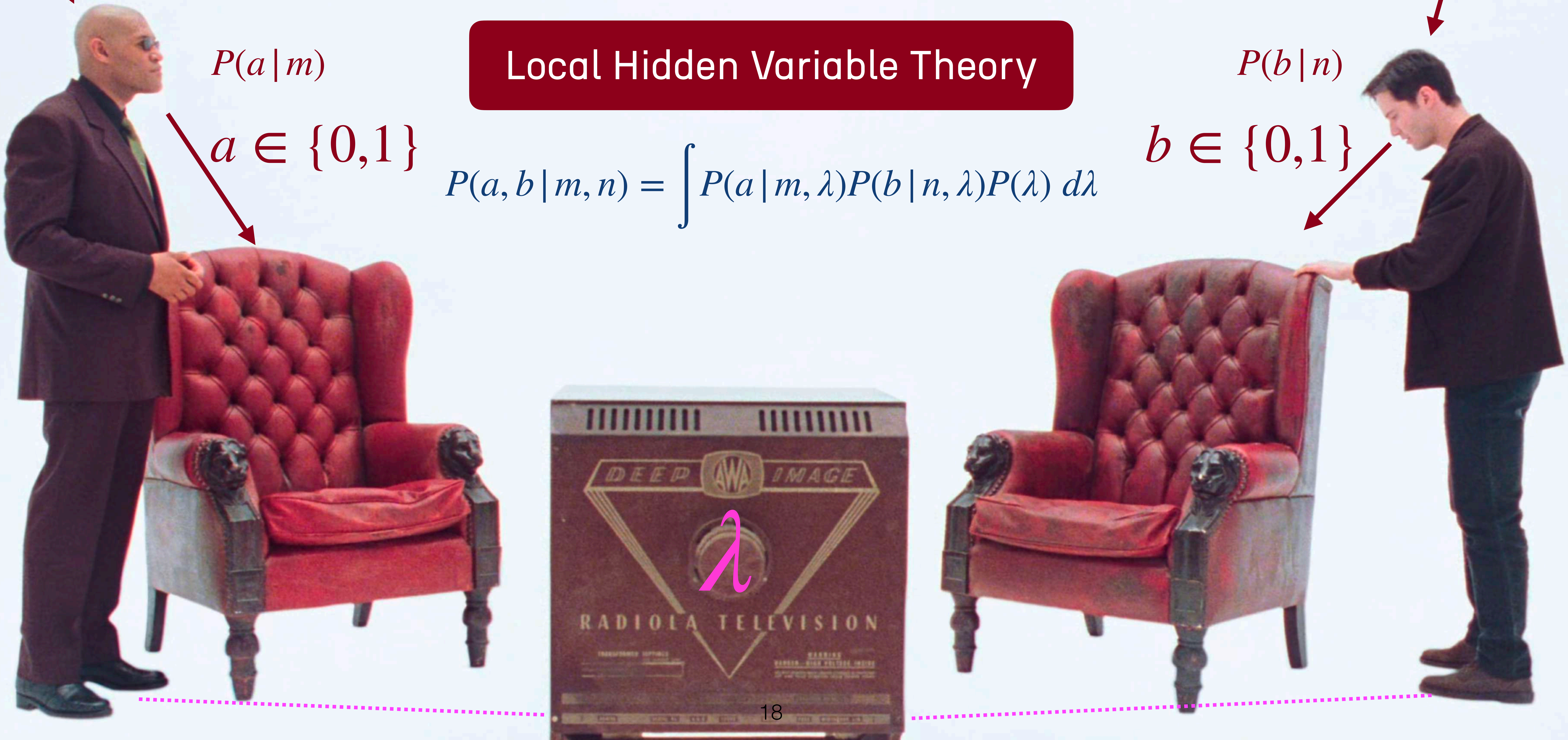
Local Hidden Variable Theory

$P(b\,|\,n)$

$a \in \{0,1\}$

$b \in \{0,1\}$

$$P(a,b\,|\,m,n) = \int P(a\,|\,m,\lambda)P(b\,|\,n,\lambda)P(\lambda)\; d\lambda$$



Think Outside the Box

Local Hidden Variable Theory

$$P(a, b \mid m, n) = \int P(a \mid m, \lambda) P(b \mid n, \lambda) P(\lambda)\, d\lambda$$

$$+$$

$$E(mn) = P(00 \mid mn) + P(11 \mid mn) - P(01 \mid mn) - P(10 \mid mn)$$

$$P(00 \mid 00) - P(00 \mid 01) - P(00 \mid 10) - P(11 \mid 11) \leq 0$$

$$-2 \leq E(00) + E(01) + E(10) - E(11) \leq 2$$

CH Inequality

CHSH Inequality

Quantum Theory Is Incompatible With LHV

In LHV theory

$$P(00|00) - P(00|01) - P(00|10) - P(11|11) \leq 0$$

CH Inequality

$$-2 \leq E(00) + E(01) + E(10) - E(11) \leq 2$$

CHSH Inequality

In quantum theory

For almost all pure states, the inequality > 0

$$\max_{QM}(CHSH) = 2\sqrt{2}$$

That's Fine

Fine's Theorem

$P(a \mid m)$

$P(\lambda)$

2 inputs/outputs

$P(0 \mid 0)$

$P(1 \mid 0)$

$P(0 \mid 1)$

$P(1 \mid 1)$

Bijective map between the values of the hidden variable and the local response functions

$P(\lambda = 00)$

$P(\lambda = 01)$

$P(\lambda = 10)$

$P(\lambda = 11)$

4 values (2 bits)

A. Fine, Phys. Rev. Lett., 48, 291 (1982)

That's Fine

Fine's Theorem

Which bit to look at

$$P(a\,|\,m)$$

The value of the bit

$$
\begin{aligned}
P(0\,|\,0) &= P(\lambda = 00) + P(\lambda = 01) \\
P(1\,|\,0) &= P(\lambda = 10) + P(\lambda = 11) \\[1em]
P(0\,|\,1) &= P(\lambda = 00) + P(\lambda = 10) \\
P(1\,|\,1) &= P(\lambda = 01) + P(\lambda = 11)
\end{aligned}
$$

That's Fine

Fine's Theorem

$$P(a, b \,|\, m, n)$$

$$P(01 \,|\, 10) = P_N(0 \,|\, 1) \cdot P_M(1 \,|\, 0)$$

$$= P(\lambda_N = 00)P(\lambda_M = 10) + P(\lambda_N = 00)P(\lambda_M = 11) + P(\lambda_N = 10)P(\lambda_M = 10) + P(\lambda_N = 10)P(\lambda_M = 11)$$

$$\begin{aligned} P(1 \,|\, 0) &= P(\lambda = 10) + P(\lambda = 11) \\ P(0 \,|\, 1) &= P(\lambda = 00) + P(\lambda = 10) \end{aligned}$$

Local Hidden Variable models are convex polytopes!

The First Experimental Test

Experimental Test of Local Hidden-Variable Theories*

Stuart J. Freedman and John F. Clauser
Department of Physics and Lawrence Berkeley Laboratory, University of California, Berkeley, California 94720
(Received 4 February 1972)

We have measured the linear polarization correlation of the photons emitted in an atomic cascade of calcium. It has been shown by a generalization of Bell's inequality that the existence of local hidden variables imposes restrictions on this correlation in conflict with the predictions of quantum mechanics. Our data, in agreement with quantum mechanics, violate these restrictions to high statistical accuracy, thus providing strong evidence against local hidden-variable theories.

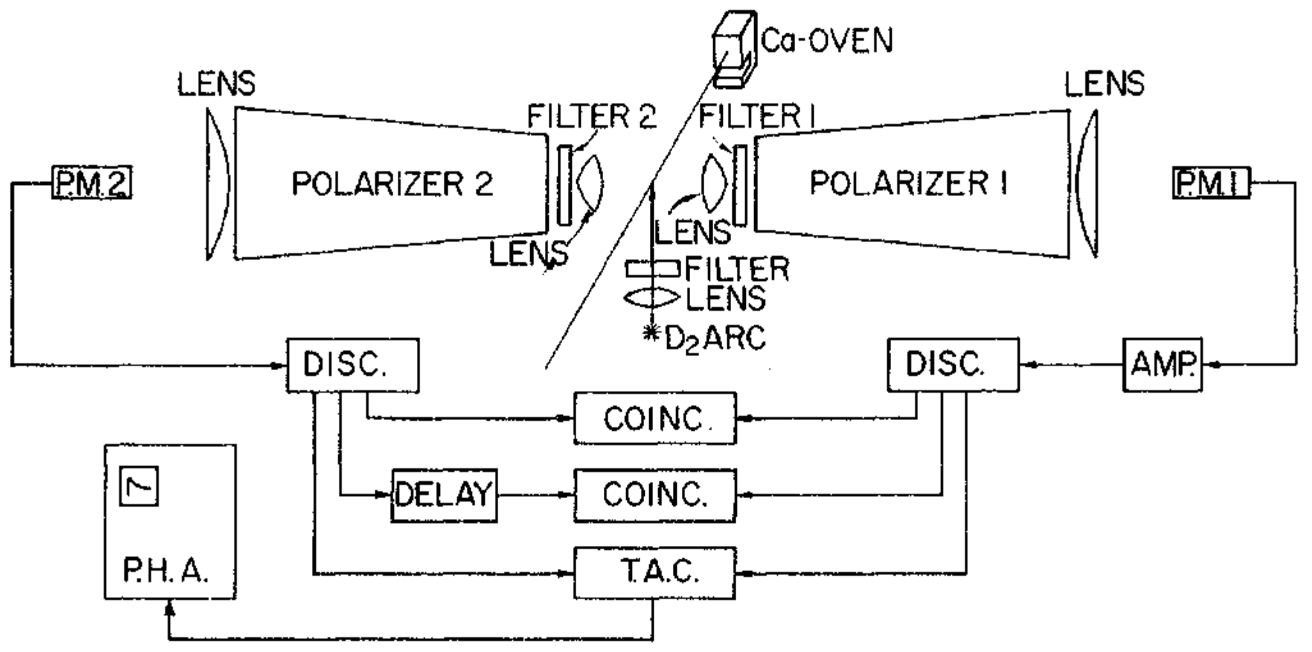


FIG. 1. Schematic diagram of apparatus and associated electronics. Scalers (not shown) monitored the outputs of the discriminators and coincidence circuits during each 100-sec count period. The contents of the scalers and the experimental configuration were recorded on paper tape and analyzed on an IBM 1620-II computer.

The Loopholes

Locality Loophole

- LHV assumes no communication between Morpheus and Neo.
- The "ultimate physical bound" to enforce this is given by the speed of light.

- Enforce physical separation on the order of dozens of meters.

Detection Loophole

- Morpheus and Neo can cheat by refusing to give answers sometimes.
- This will "bias" the inequality to show a violation.

- Detector efficiency must be sufficiently high: $2\sqrt{2}-2 \approx 82.84\,\%$

Memory Loophole

- The experimental rounds are assumed to be i.i.d.
- This is not a problem when the experiment is infinitely long, but is a problem with finite data.

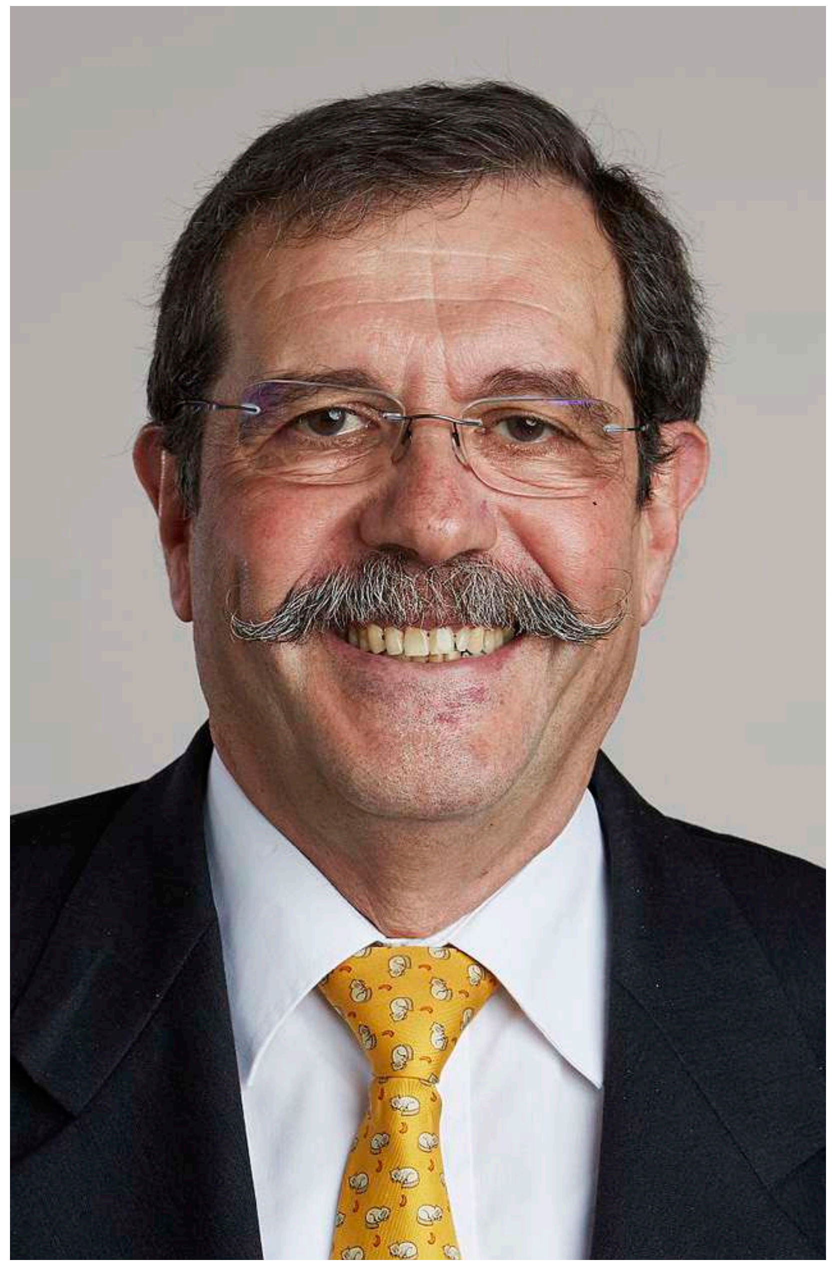
- Use non-i.i.d. estimators and sophisticated statistical analysis.

Free-will Loophole

- Morpheus and Neo can cheat if they have information about the inputs.
- They can collude to produce outputs which violate the inequality.

- Fast switching of inputs with good random number generators.

The Road to Loophole-Free Tests



Alain Aspect

Volume 49, Number 2 PHYSICAL REVIEW LETTERS 12 July 1982

Experimental Realization of Einstein-Podolsky-Rosen-Bohm *Gedankenexperiment*: A New Violation of Bell's Inequalities

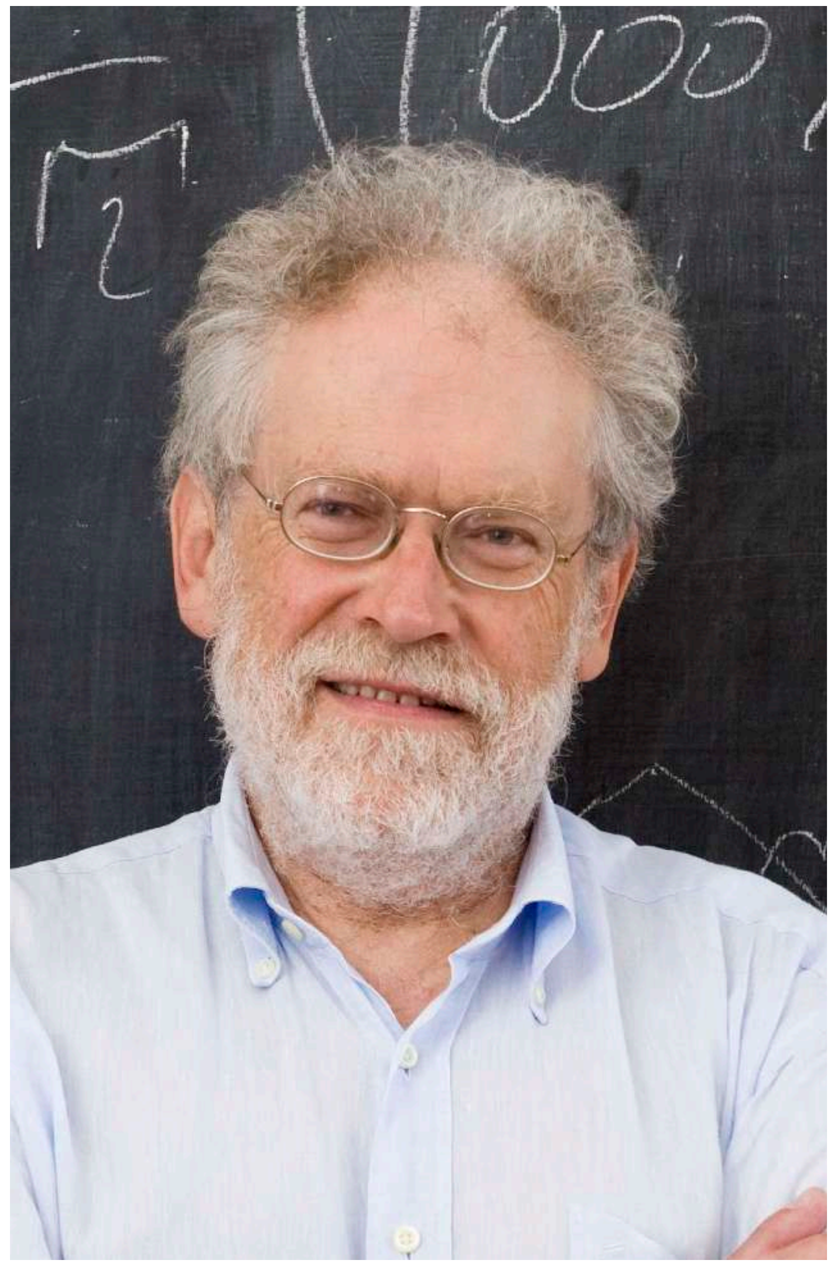
Alain Aspect, Philippe Grangier, and Gérard Roger
Institut d'Optique Théorique et Appliquée, Laboratoire associé au Centre National de la Recherche Scientifique, Université Paris-Sud, F-91406 Orsay, France
(Received 30 December 1981)

The linear-polarization correlation of pairs of photons emitted in a radiative cascade of calcium has been measured. The new experimental scheme, using two-channel polarizers (i.e., optical analogs of Stern-Gerlach filters), is a straightforward transposition of Einstein-Podolsky-Rosen-Bohm *gedankenexperiment*. The present results, in excellent agreement with the quantum mechanical predictions, lead to the greatest violation of generalized Bell's inequalities ever achieved.

PACS numbers: 03.65.Bz, 35.80.+s

- Fully optical. Uses entangled photons.
- No locality loophole!

The Road to Loophole-Free Tests



Anton Zeilinger

PHYSICAL REVIEW
LETTERS

VOLUME 81 | 7 DECEMBER 1998 | NUMBER 23

Violation of Bell's Inequality under Strict Einstein Locality Conditions

Gregor Weihs, Thomas Jennewein, Christoph Simon, Harald Weinfurter, and Anton Zeilinger
Institut für Experimentalphysik, Universität Innsbruck, Technikerstraße 25, A-6020 Innsbruck, Austria
(Received 6 August 1998)

Locality Loophole, 1998

Violation of local realism with freedom of choice

Thomas Scheidl[a], Rupert Ursin[a], Johannes Kofler[a,b,1], Sven Ramelow[a,b], Xiao-Song Ma[a,b], Thomas Herbst[b], Lothar Ratschbacher[a,2], Alessandro Fedrizzi[a,3], Nathan K. Langford[a,4], Thomas Jennewein[a,5], and Anton Zeilinger[a,b,1]

[a]Institute for Quantum Optics and Quantum Information, Austrian Academy of Sciences, Boltzmanngasse 3, 1090 Vienna, Austria; and [b]Faculty of Physics, University of Vienna, Boltzmanngasse 5, 1090 Vienna, Austria

Edited by William D. Phillips, National Institute of Standards and Technology, Gaithersburg, MD, and approved September 15, 2010 (received for review March 4, 2010)

Free-will Loophole, 2010

LETTER

doi:10.1038/nature12012

Bell violation using entangled photons without the fair-sampling assumption

Marissa Giustina[1,2]*, Alexandra Mech[1,2]*, Sven Ramelow[1,2]*, Bernhard Wittmann[1,2]*, Johannes Kofler[1,3], Jörn Beyer[4], Adriana Lita[5], Brice Calkins[5], Thomas Gerrits[5], Sae Woo Nam[5], Rupert Ursin[1] & Anton Zeilinger[1,2]

Detection Loophole, 2013

Experimental Proof That Reality Is Nonlocal

Selected for a Viewpoint in *Physics*
PRL **115,** 250401 (2015) PHYSICAL REVIEW LETTERS week ending 18 DECEMBER 2015

Significant-Loophole-Free Test of Bell's Theorem with Entangled Photons

Marissa Giustina,[1,2,*] Marijn A. M. Versteegh,[1,2] Sören Wengerowsky,[1,2] Johannes Handsteiner,[1,2] Armin Hochrainer,[1,2] Kevin Phelan,[1] Fabian Steinlechner,[1] Johannes Kofler,[3] Jan-Åke Larsson,[4] Carlos Abellán,[5] Waldimar Amaya,[5] Valerio Pruneri,[5,6] Morgan W. Mitchell,[5,6] Jörn Beyer,[7] Thomas Gerrits,[8] Adriana E. Lita,[8] Lynden K. Shalm,[8] Sae Woo Nam,[8] Thomas Scheidl,[1,2] Rupert Ursin,[1] Bernhard Wittmann,[1,2] and Anton Zeilinger[1,2,†]

LETTER

doi:10.1038/nature15759

Loophole-free Bell inequality violation using electron spins separated by 1.3 kilometres

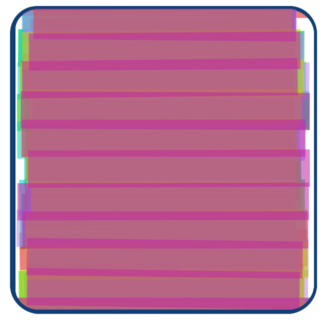
B. Hensen[1,2], H. Bernien[1,2]†, A. E. Dréau[1,2], A. Reiserer[1,2], N. Kalb[1,2], M. S. Blok[1,2], J. Ruitenberg[1,2], R. F. L. Vermeulen[1,2], R. N. Schouten[1,2], C. Abellán[3], W. Amaya[3], V. Pruneri[3,4], M. W. Mitchell[3,4], M. Markham[5], D. J. Twitchen[5], D. Elkouss[1], S. Wehner[1], T. H. Taminiau[1,2] & R. Hanson[1,2]

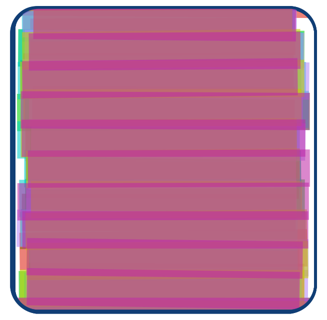
Selected for a Viewpoint in *Physics*
PRL **115,** 250402 (2015) PHYSICAL REVIEW LETTERS week ending 18 DECEMBER 2015

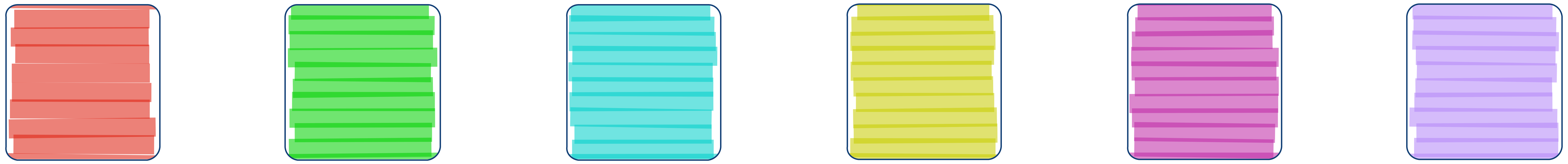
Strong Loophole-Free Test of Local Realism*

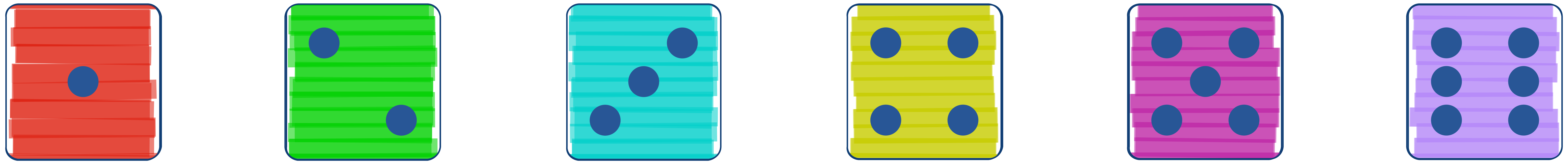
Lynden K. Shalm,[1,†] Evan Meyer-Scott,[2] Bradley G. Christensen,[3] Peter Bierhorst,[1] Michael A. Wayne,[3,4] Martin J. Stevens,[1] Thomas Gerrits,[1] Scott Glancy,[1] Deny R. Hamel,[5] Michael S. Allman,[1] Kevin J. Coakley,[1] Shellee D. Dyer,[1] Carson Hodge,[1] Adriana E. Lita,[1] Varun B. Verma,[1] Camilla Lambrocco,[1] Edward Tortorici,[1] Alan L. Migdall,[4,6] Yanbao Zhang,[2] Daniel R. Kumor,[3] William H. Farr,[7] Francesco Marsili,[7] Matthew D. Shaw,[7] Jeffrey A. Stern,[7] Carlos Abellán,[8] Waldimar Amaya,[8] Valerio Pruneri,[8,9] Thomas Jennewein,[2,10] Morgan W. Mitchell,[8,9] Paul G. Kwiat,[3] Joshua C. Bienfang,[4,6] Richard P. Mirin,[1] Emanuel Knill,[1] and Sae Woo Nam[1,‡]

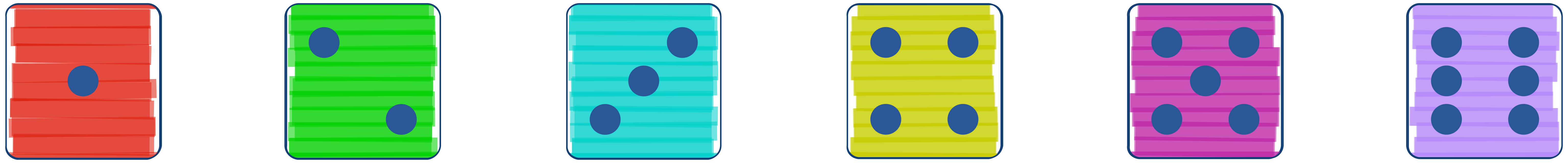
The Marginal Problem in 1D











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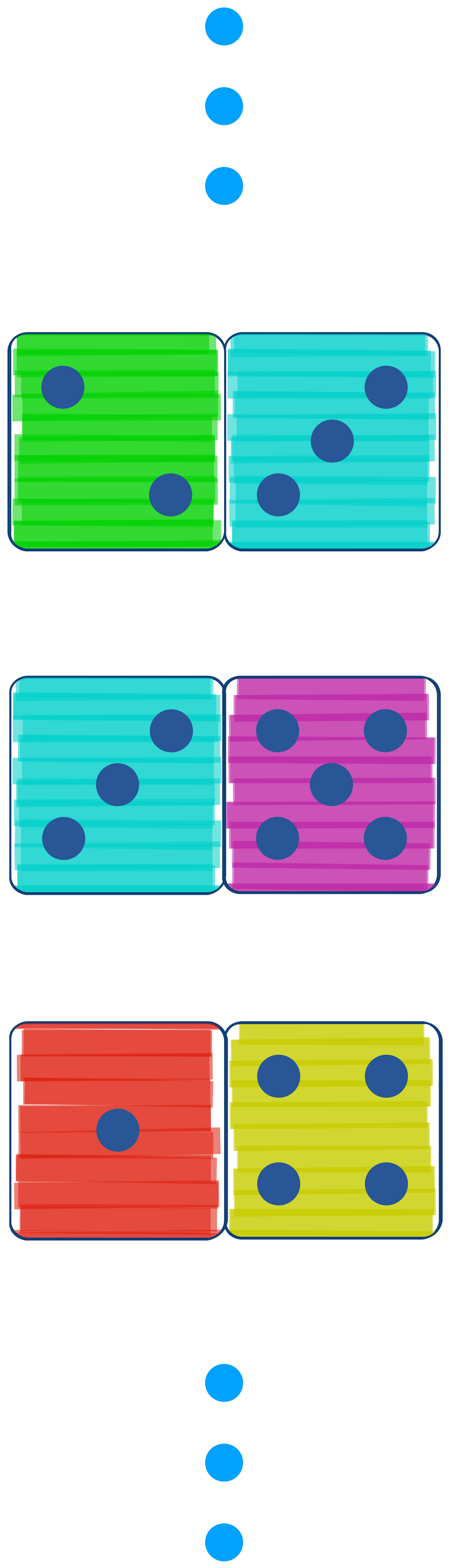
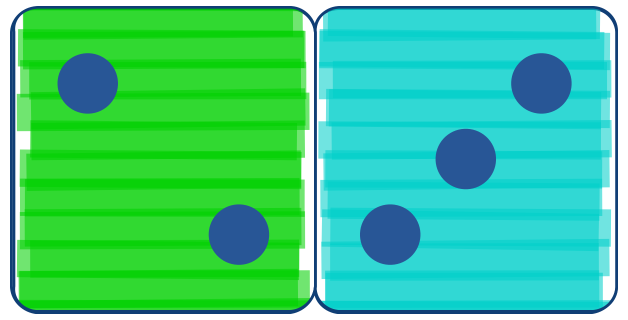
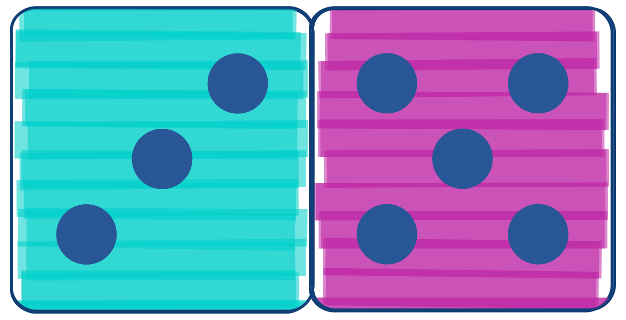
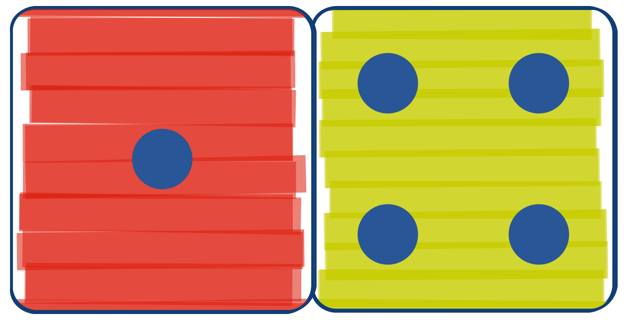




$$\frac{1}{36}$$

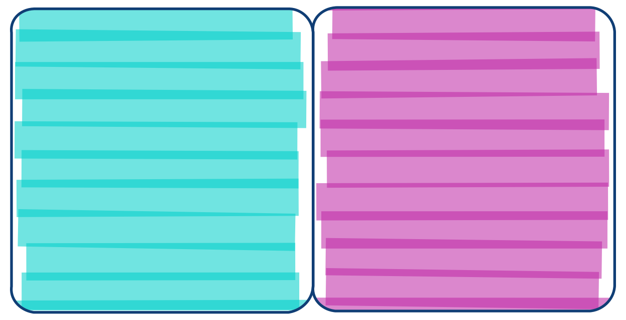


$$\frac{1}{36}$$

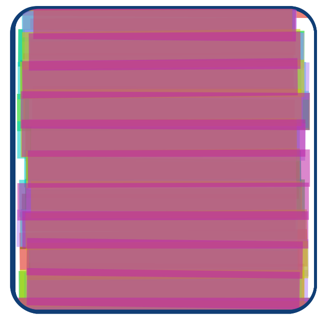


$$\frac{1}{36}$$

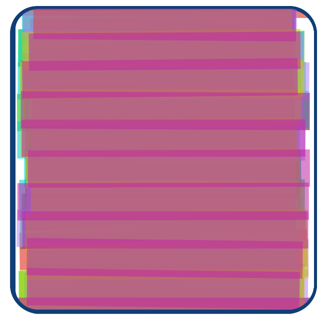
Necessary Condition for Translation Invariance



Necessary Condition for Translation Invariance



Necessary Condition for Translation Invariance



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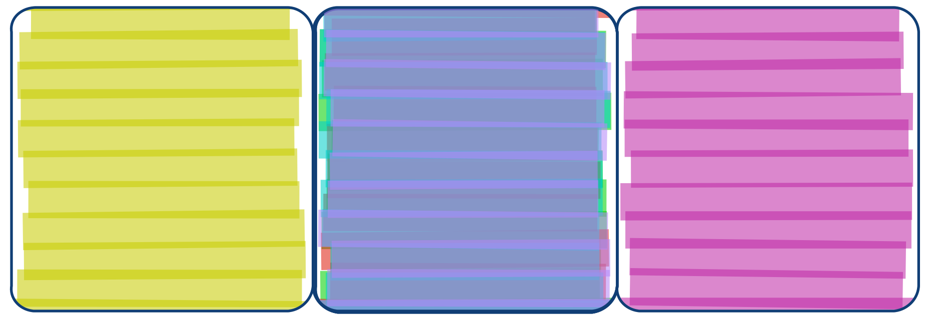

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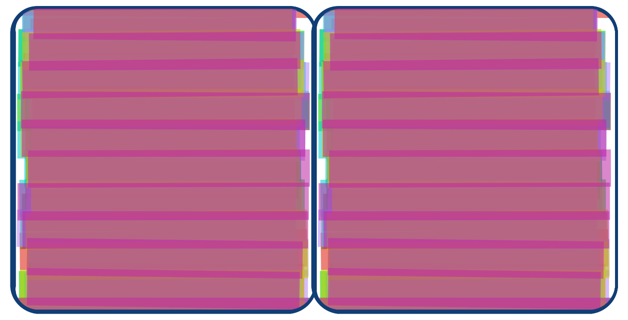

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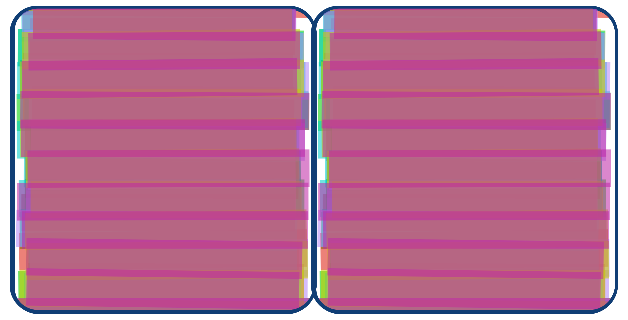
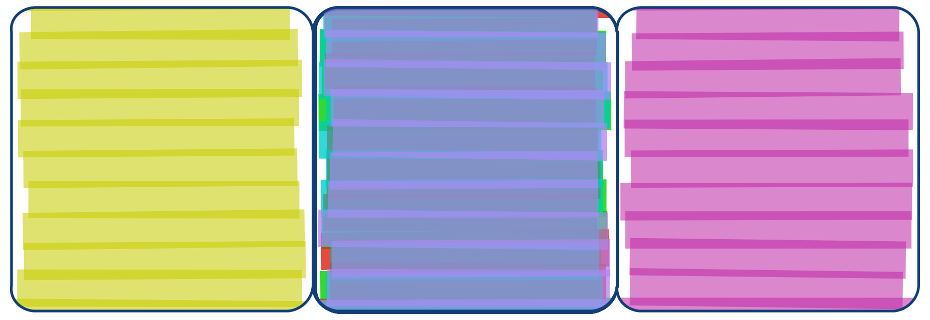
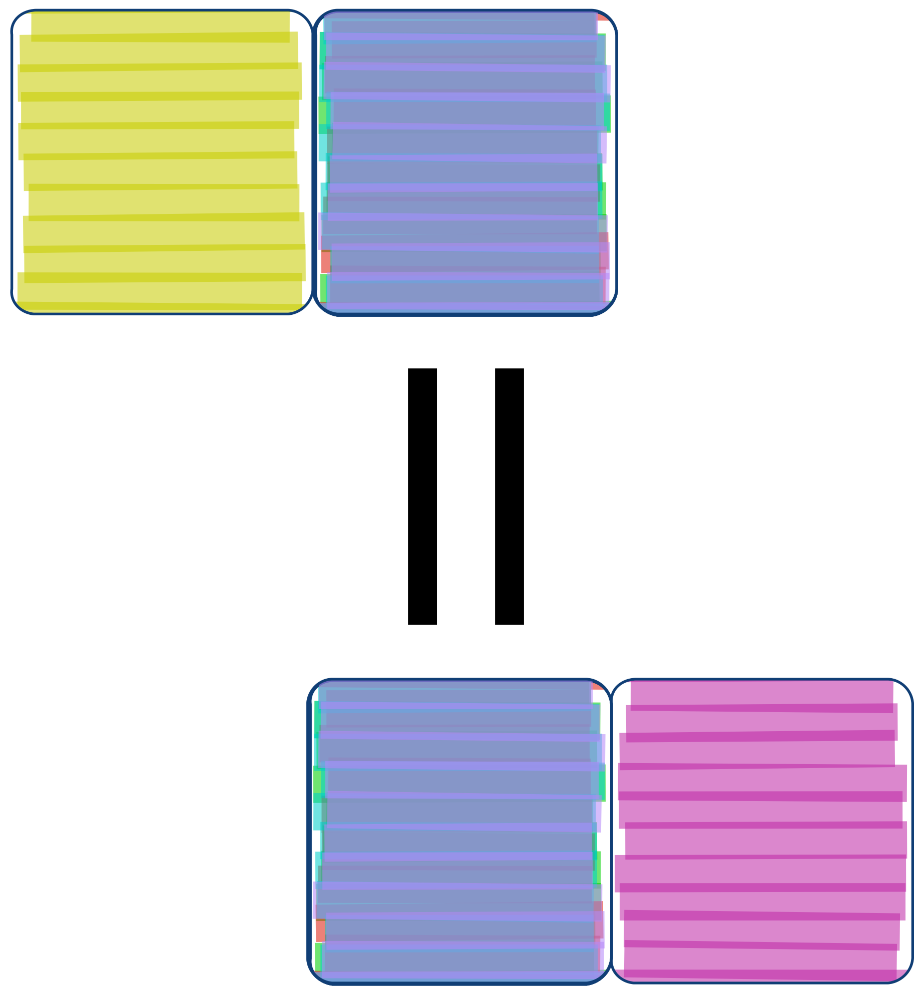


Necessary Condition for Translation Invariance

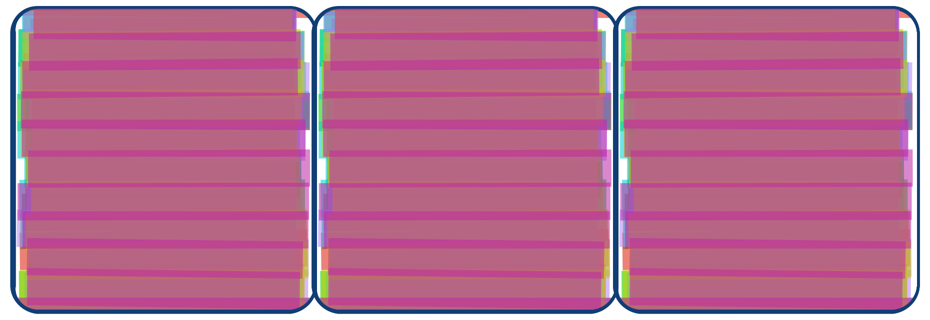


Necessary Condition for Translation Invariance



Is the Necessary Condition Also Sufficient?

Local Translation Invariance (LTI)

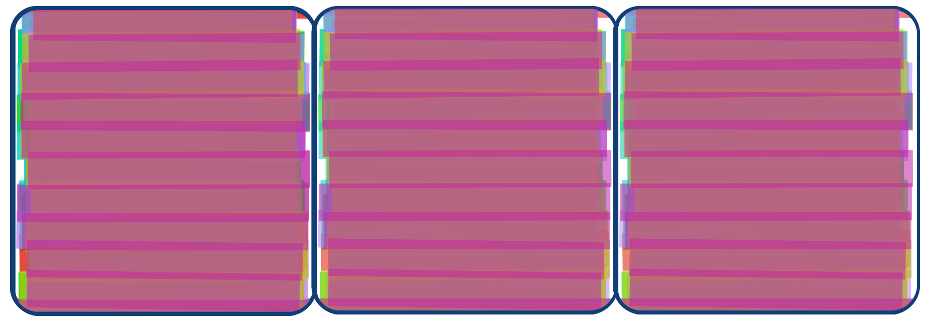


$$P(A_0,A_1,A_2)$$

$$P(A_0,A_1) \equiv \sum_{A_2} P(A_0,A_1,A_2) = \sum_{A_0} P(A_0,A_1,A_2) \equiv P(A_1,A_2)$$

Is the Necessary Condition Also Sufficient?

Local Translation Invariance (LTI)

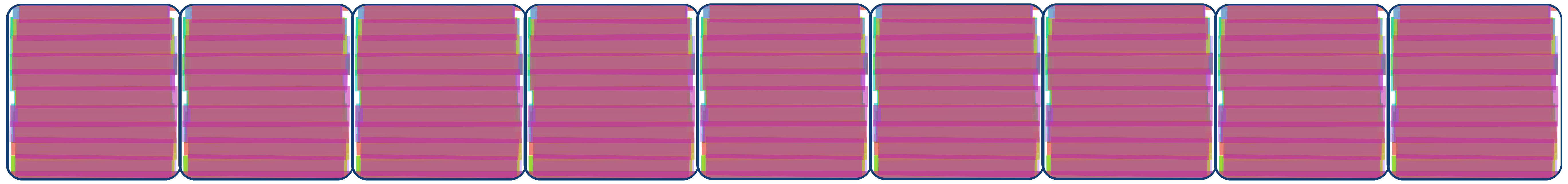


$$P(A_{-1},A_0,A_1,A_2) = P(A_0,A_1,A_2)P(A_{-1}|A_0,A_1)$$

$$P(\mathbf{A_0},\mathbf{A_1})$$

$$P(A_0,A_1,A_2)P(A_{-1}|A_0,A_1) = P(A_0,A_1,A_2)\frac{P(A_{-1},A_0,A_1)}{P(A_0,A_1)}$$

Is the Necessary Condition Also Sufficient?



Is the Necessary Condition Also Sufficient?

Yes!

LTI reduces a global property to a local one.

Is the Necessary Condition Also Sufficient?

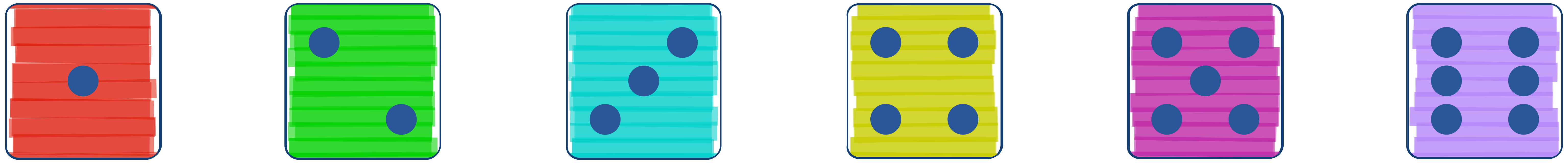
Bonus

$$P(A_0,A_1) \equiv \sum_{A_2} P(A_0,A_1,A_2) = \sum_{A_0} P(A_0,A_1,A_2) \equiv P(A_1,A_2)$$

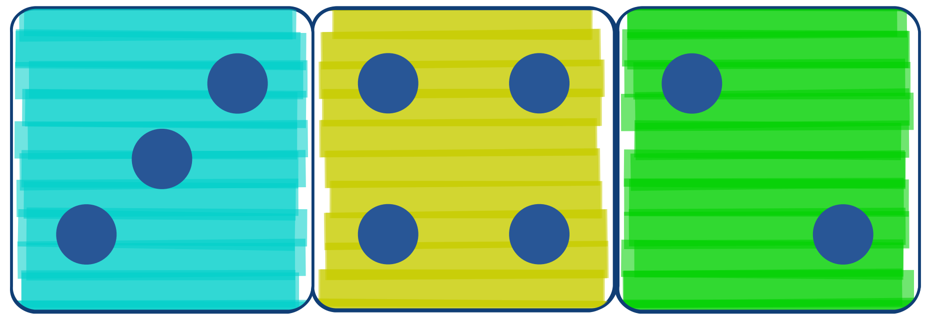
$$\sum_{A_0,A_1,A_2} P(A_0,A_1,A_2) = 1$$

It's a convex polytope!

Extreme Points of the LTI Polytope

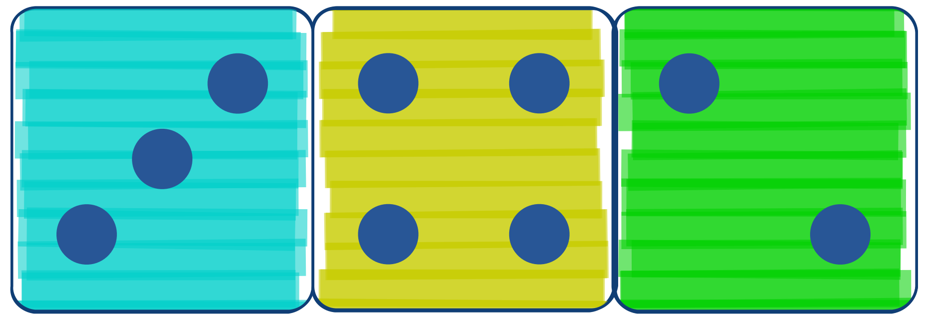


Extreme Points of the LTI Polytope

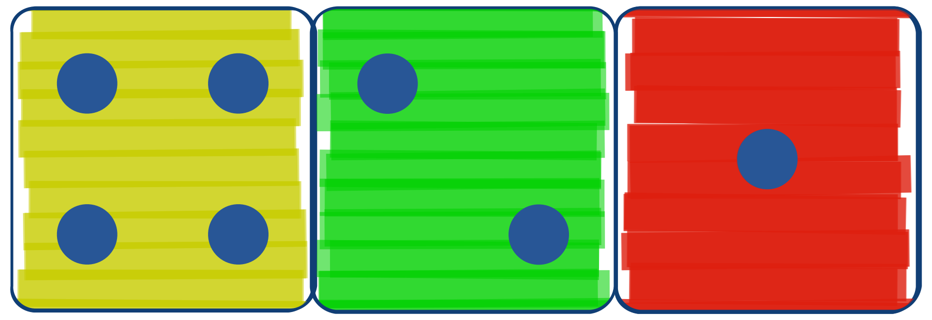


Extreme Points of the LTI Polytope

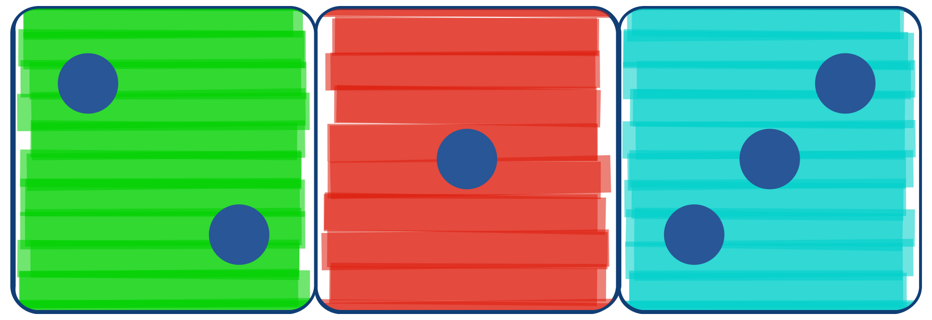
$$P(A_0 = 3, A_1 = 4, A_2 = 2) = \frac{1}{4}$$



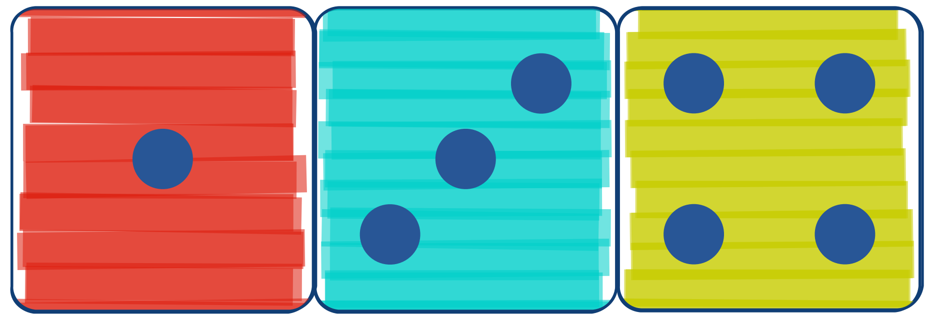
$$P(A_0 = 4, A_1 = 2, A_2 = 1) = \frac{1}{4}$$



$$P(A_0 = 2, A_1 = 1, A_2 = 3) = \frac{1}{4}$$

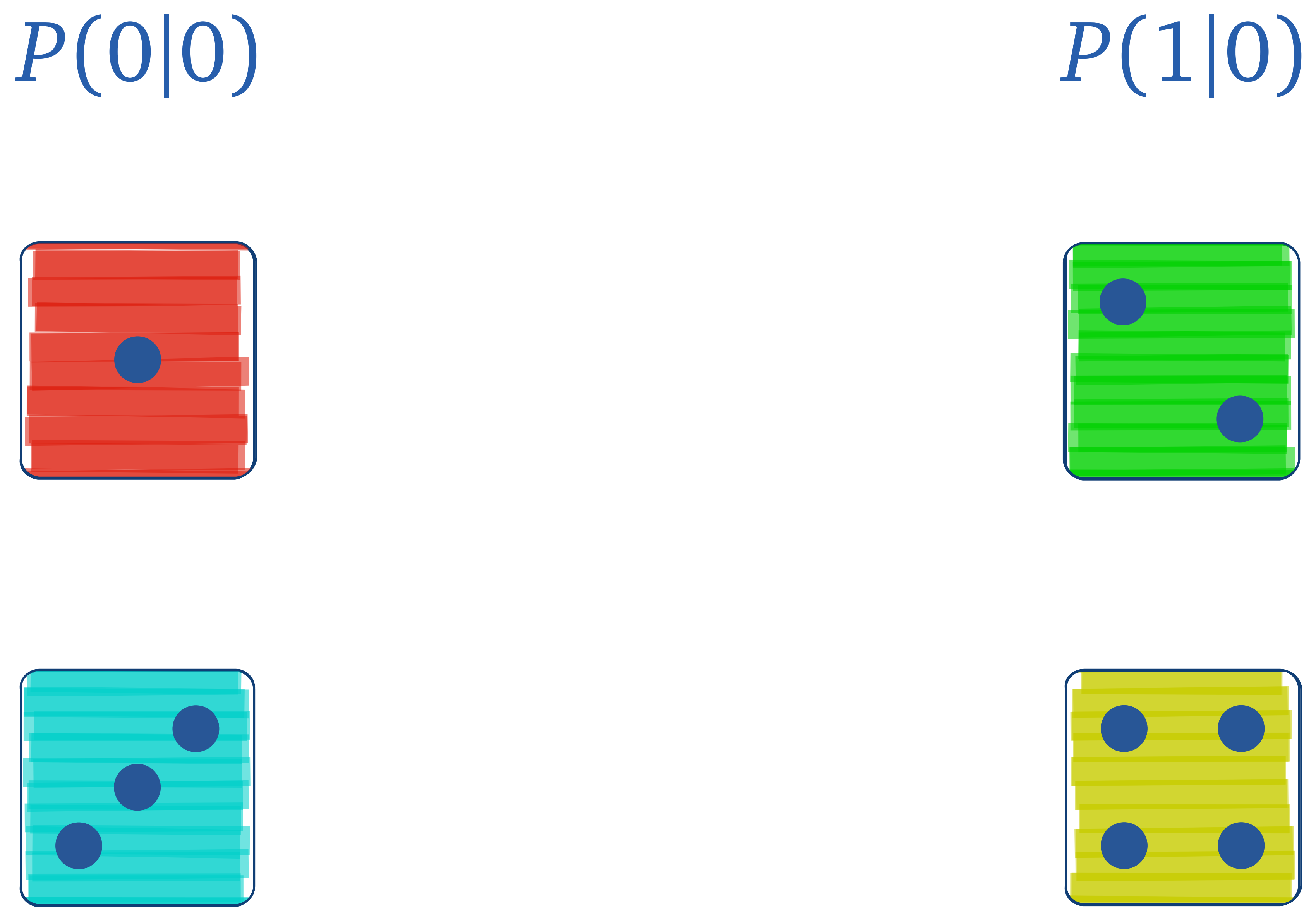


$$P(A_0 = 1, A_1 = 3, A_2 = 4) = \frac{1}{4}$$



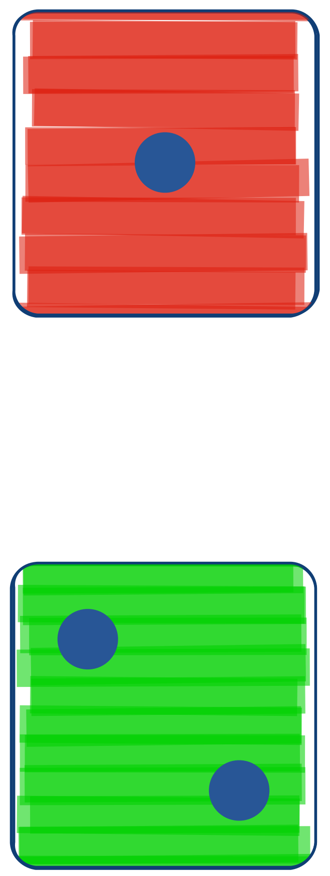
Extreme Points of the Bell-LTI Polytope

Extreme Points of the Bell-LTI Polytope

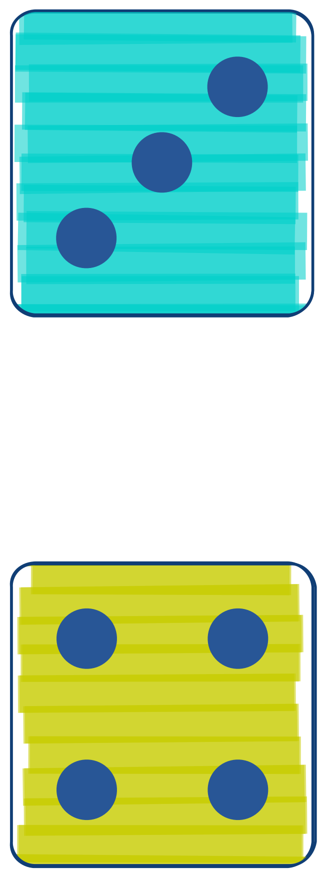


Extreme Points of the Bell-LTI Polytope

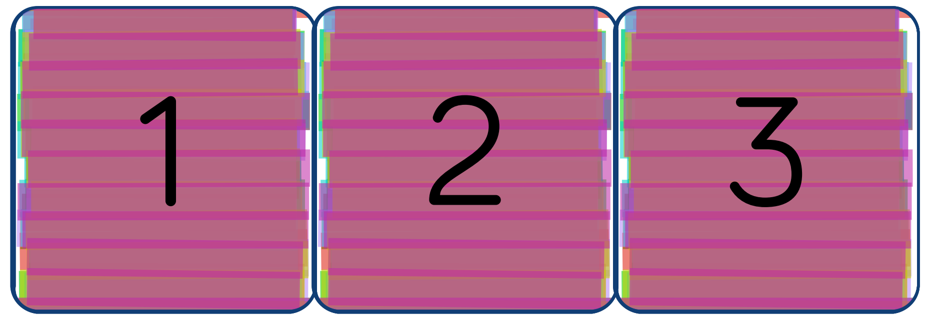
$P(0|1)$



$P(1|1)$



Facets of the Bell-LTI Polytope



$$-2E_0 - 4E_1 - 2E^{1,2}_{00} + 2E^{1,2}_{01} + 2E^{1,2}_{10} + 2E^{1,2}_{11} + E^{1,3}_{00} + E^{1,3}_{11} \geq -4$$

$$-4E_0 - 6E_1 - 3E^{1,2}_{00} + 2E^{1,2}_{01} + 3E^{1,2}_{10} + 2E^{1,2}_{11} + 2E^{1,3}_{00} + E^{1,3}_{10} + E^{1,3}_{11} \geq -6$$

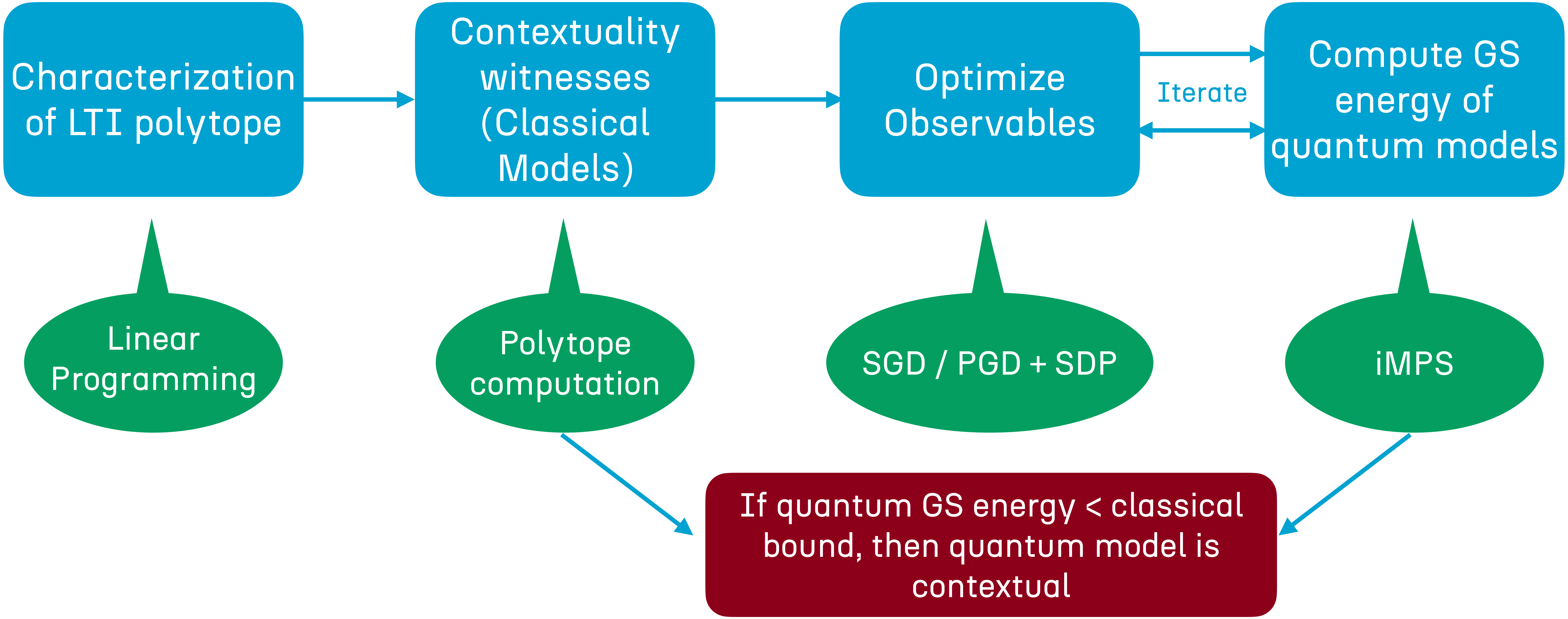
Ground State Energy of Quantum Hamiltonians

Bell Local

Contextuality of Local 1D TI Local Hamiltonians

Bounded Interaction Distance

From Contextuality Witnesses to GS Energy of Quantum Models



What Have We Learned?

- In 1D, the LTI condition is both necessary and sufficient for a distribution to have a TI extension.
- The LTI condition allows a nice polytope characterization.
- The extreme points of the LTI polytope can be visualized using Dominoes.
- Classical TI/LTI behavior can be completely characterized using the Bell-LTI polytope.


BRACE YOURSELF
ANOTHER DIMENSION IS COMING
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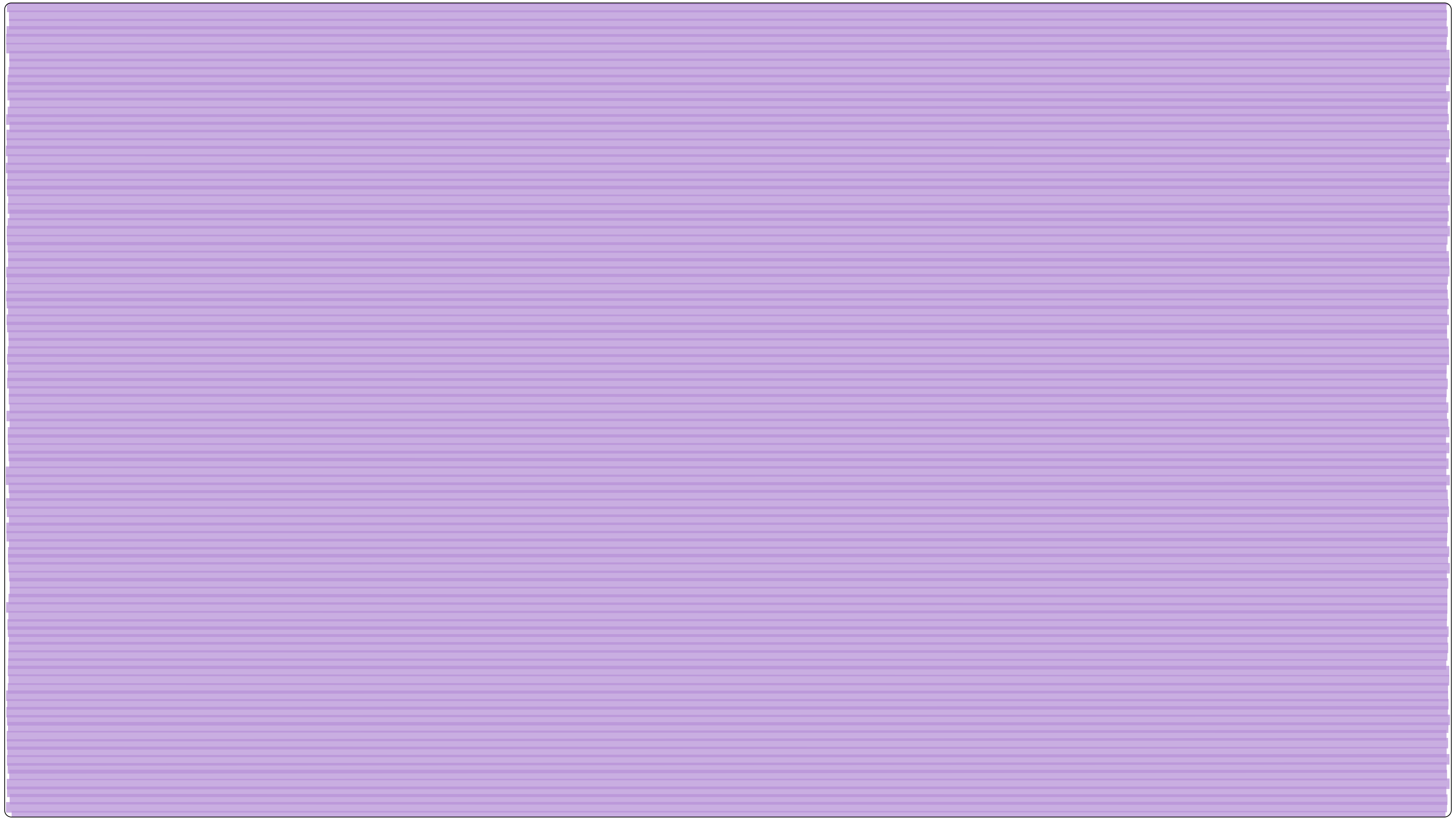
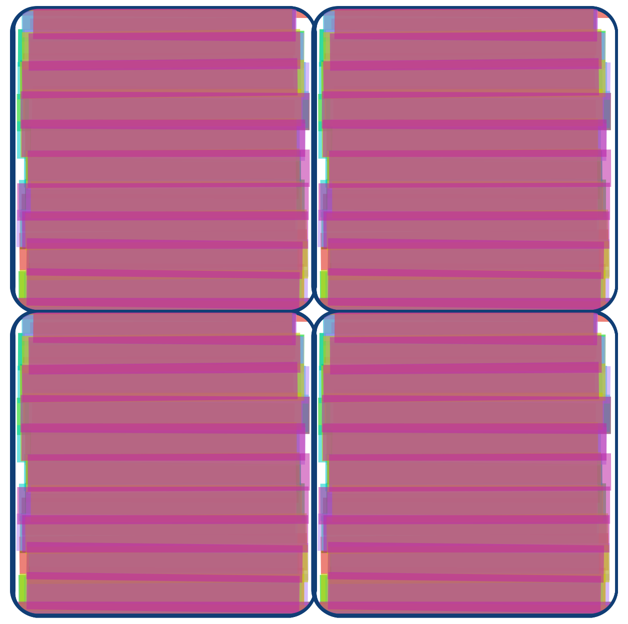
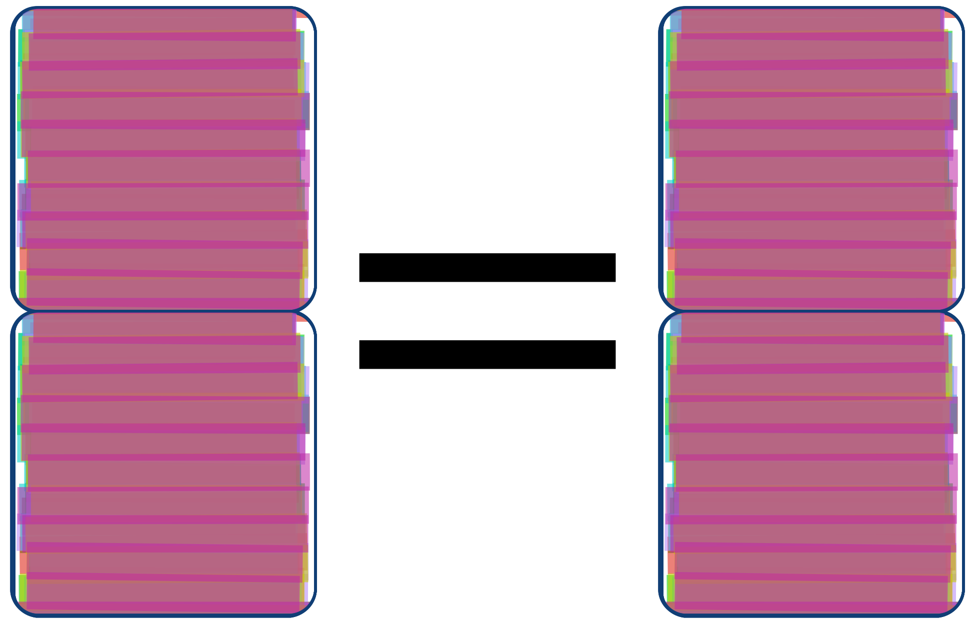
Tiles and 2D Marginals

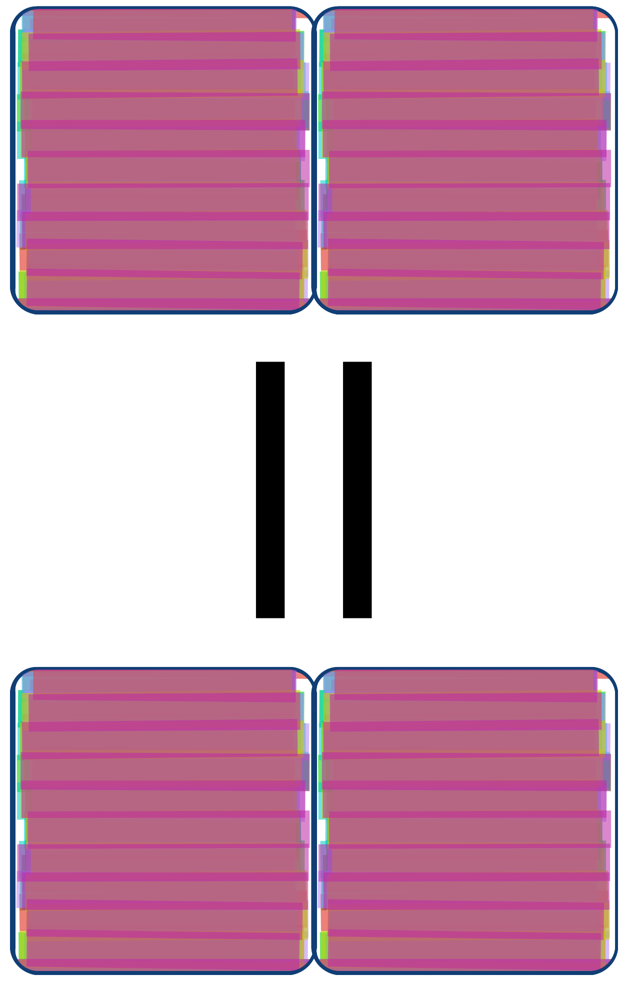




LTI Condition in 2D



LTI Condition in 2D



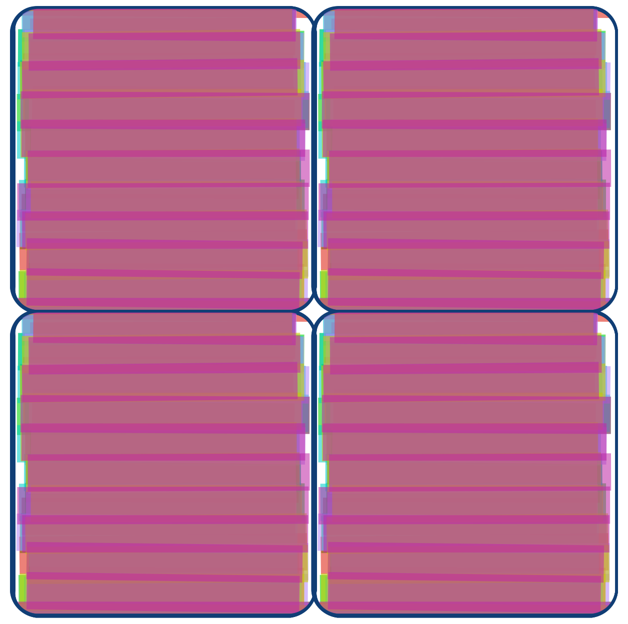
Local Translation Invariance in 2D

Necessary ✓

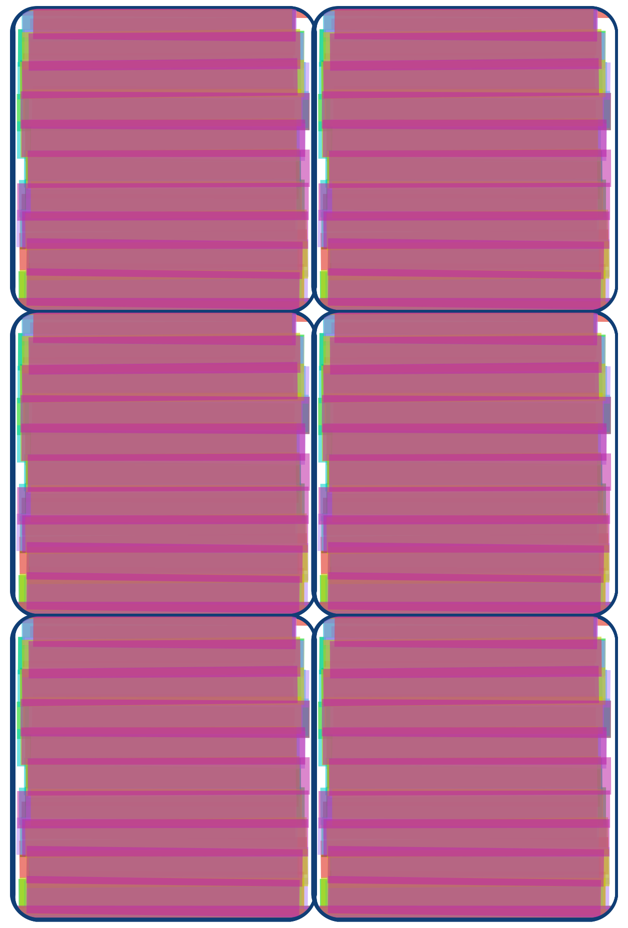
Sufficient★ ✗

★More on this later.

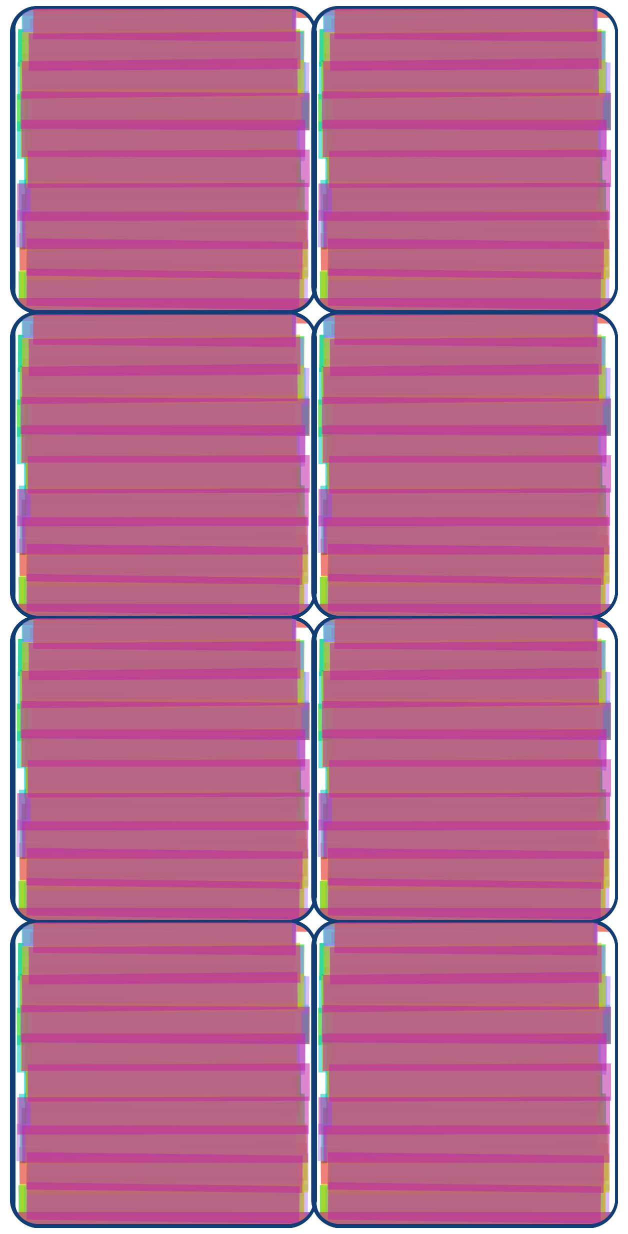
Approximations of 2D TI Marginals



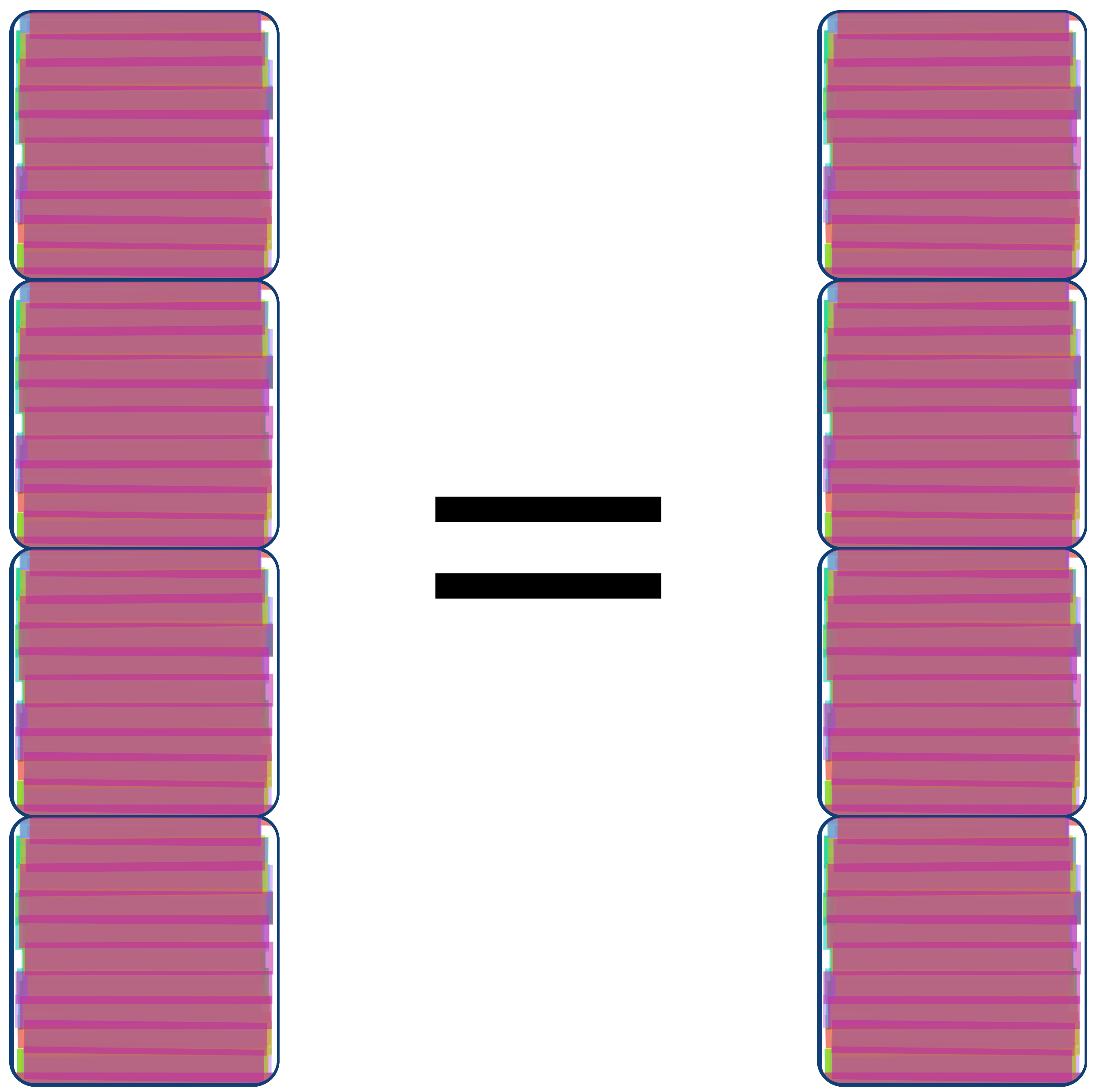
Approximations of 2D TI Marginals



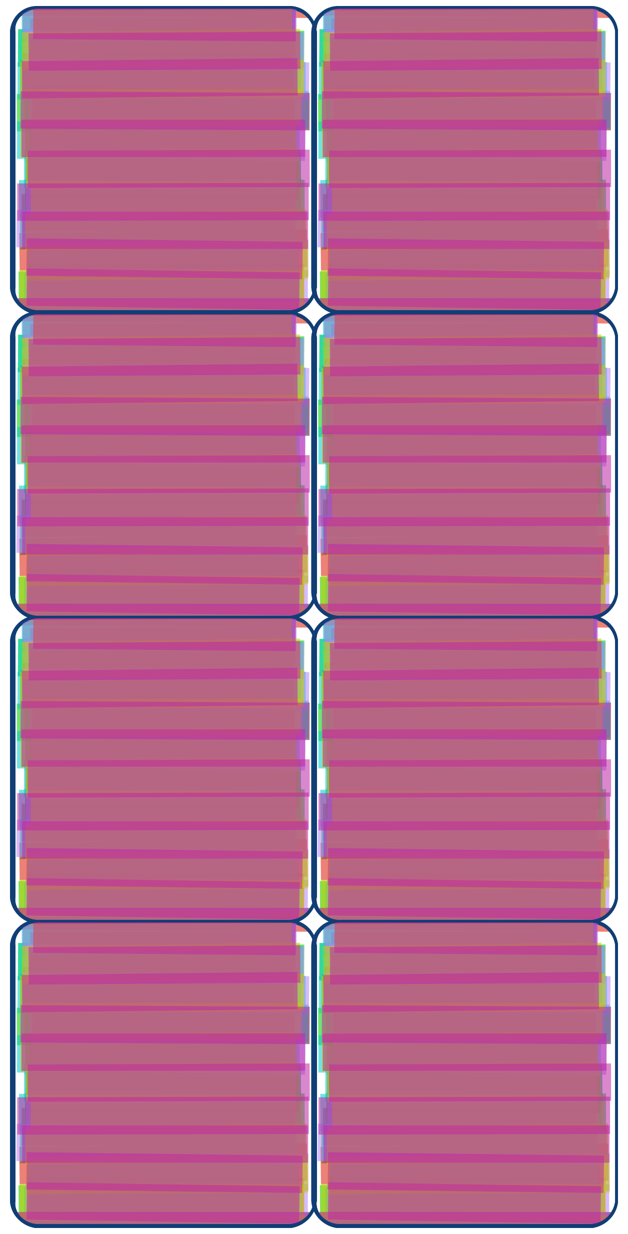
Approximations of 2D TI Marginals



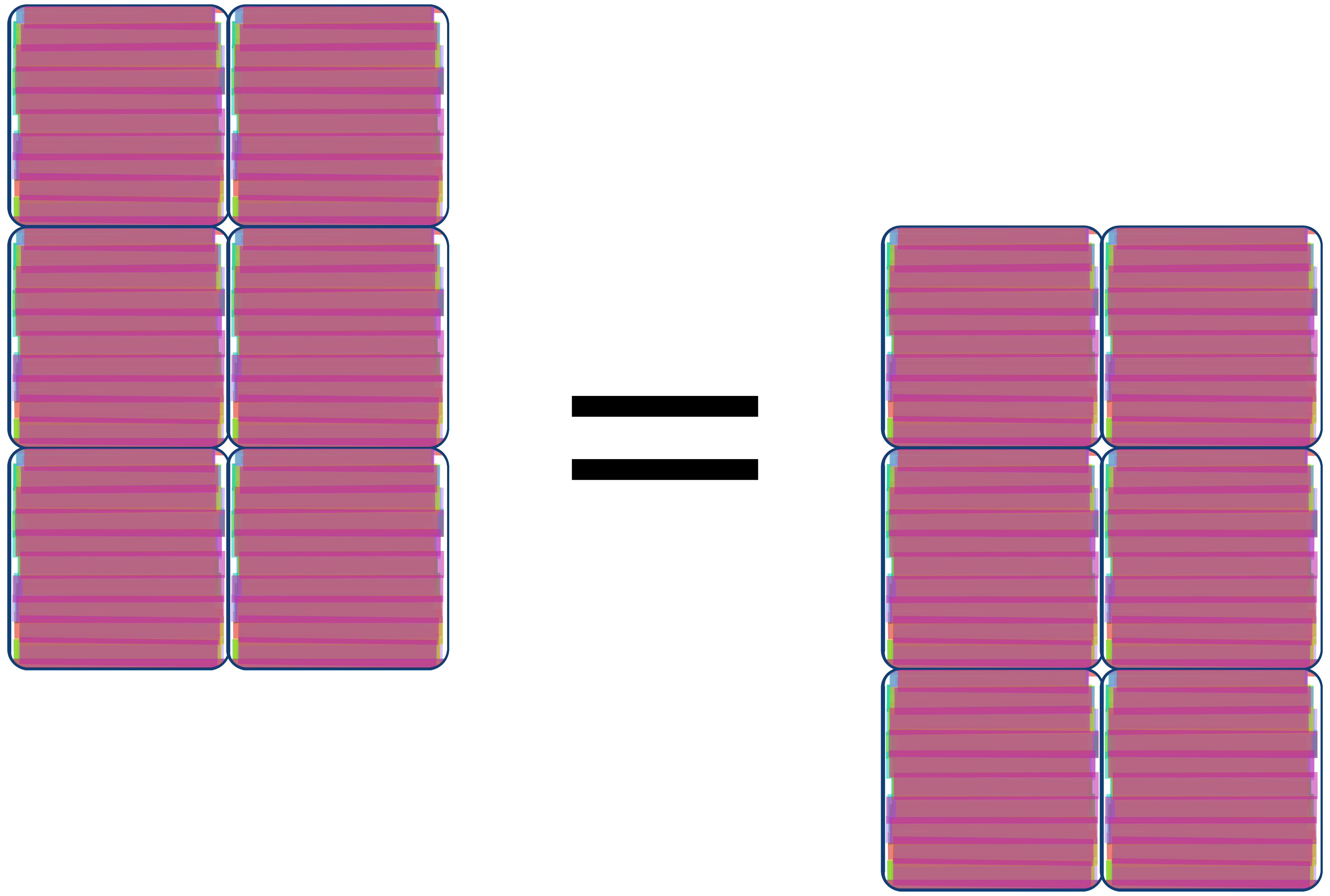
Approximations of 2D TI Marginals



Approximations of 2D TI Marginals



Approximations of 2D TI Marginals

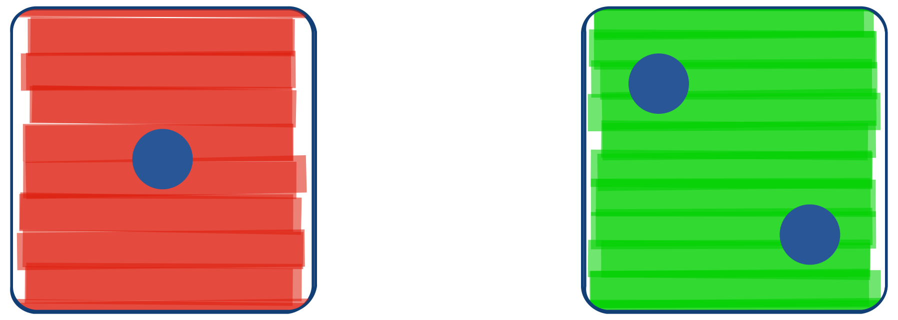


Easy Scenarios In 2D

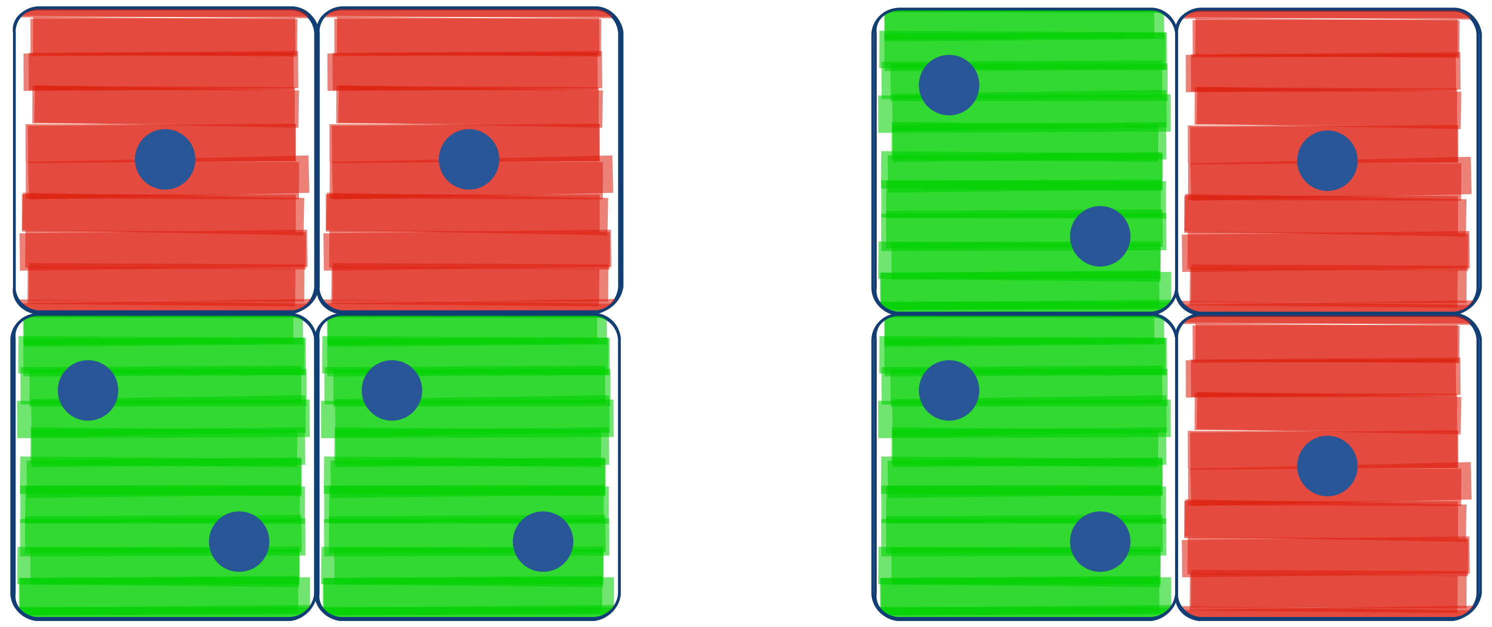
- The 2D LTI condition is sufficient only when each site takes 2 values.
- When each site takes 2 values, the marginals of the nearest and next-to-nearest neighbor distributions can be characterized by projecting from the 2D LTI polytope.
- If there is also a reflection symmetry, then the problem becomes essentially 1D.

Extreme Points of the 2D Binary LTI Polytope

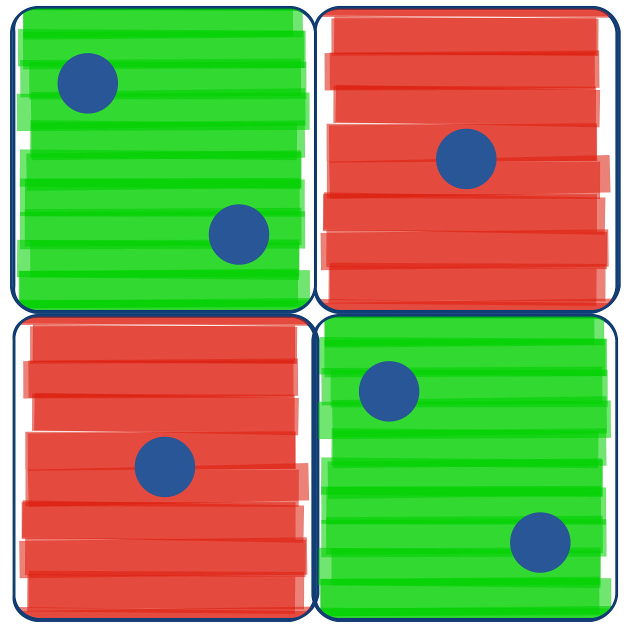
C1



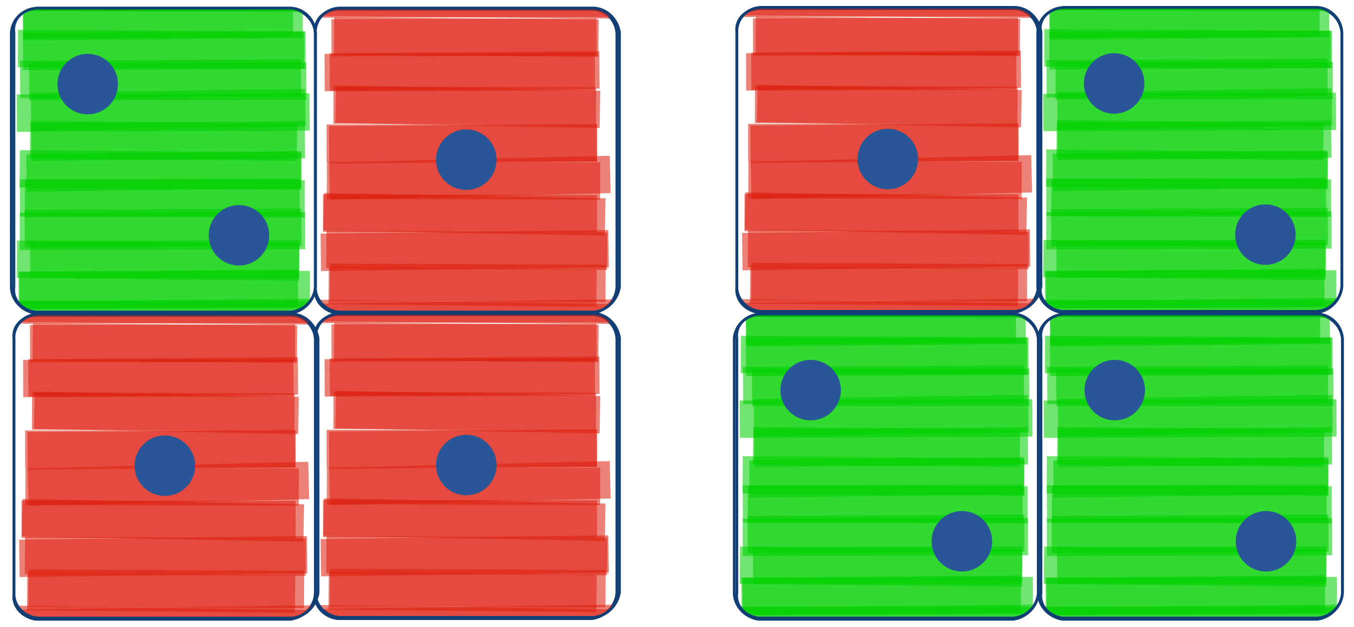
C2



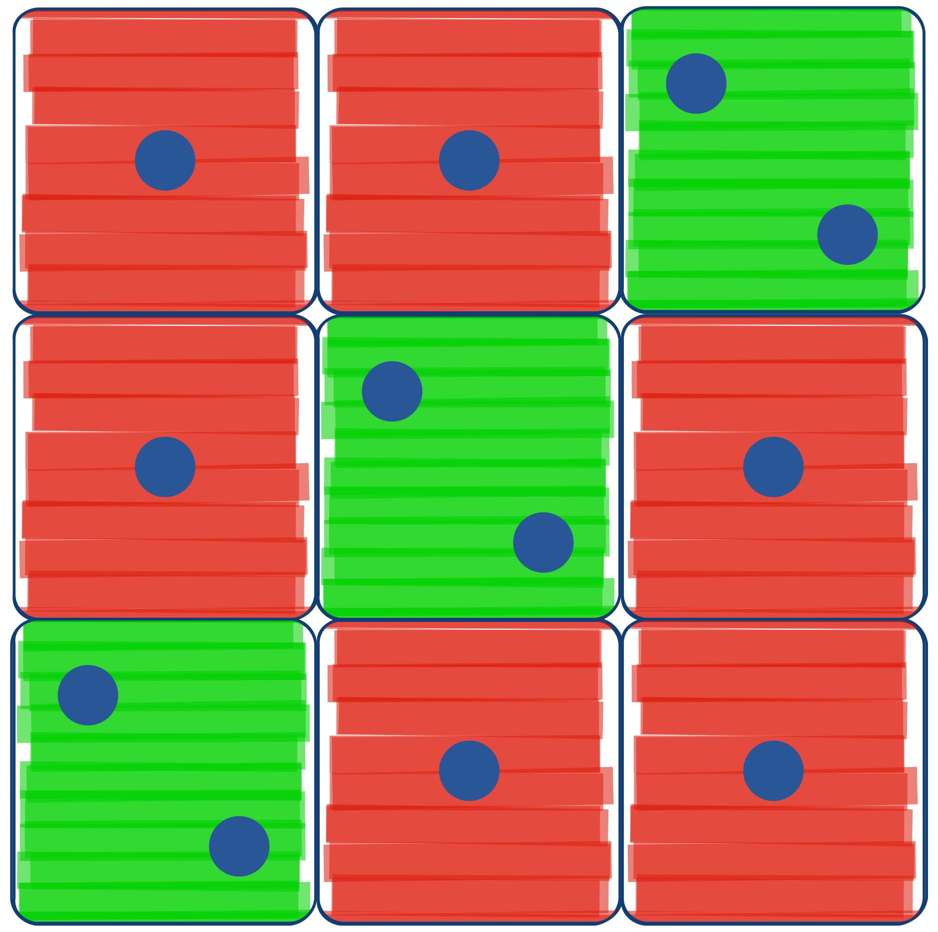
C3



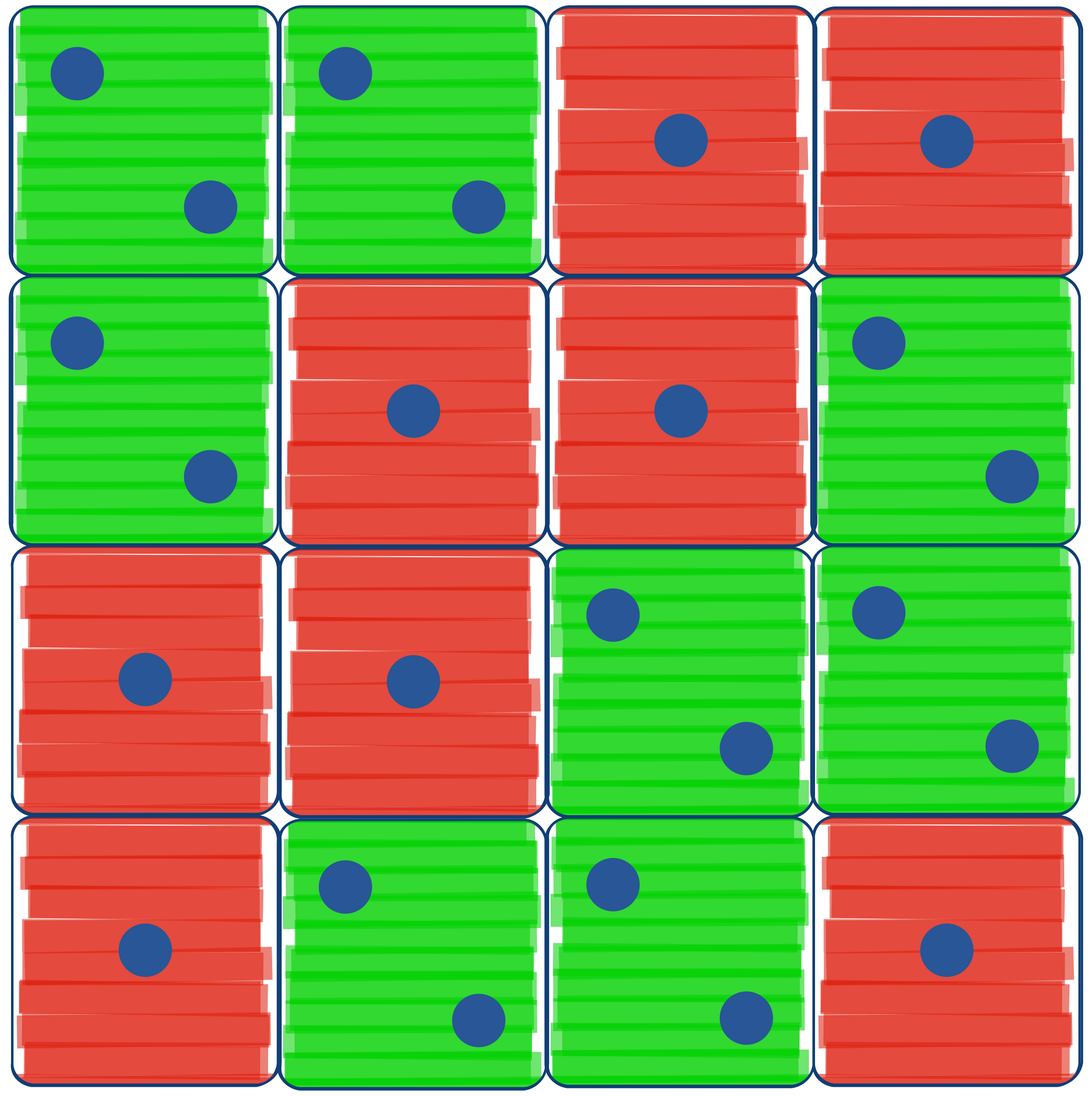
C4

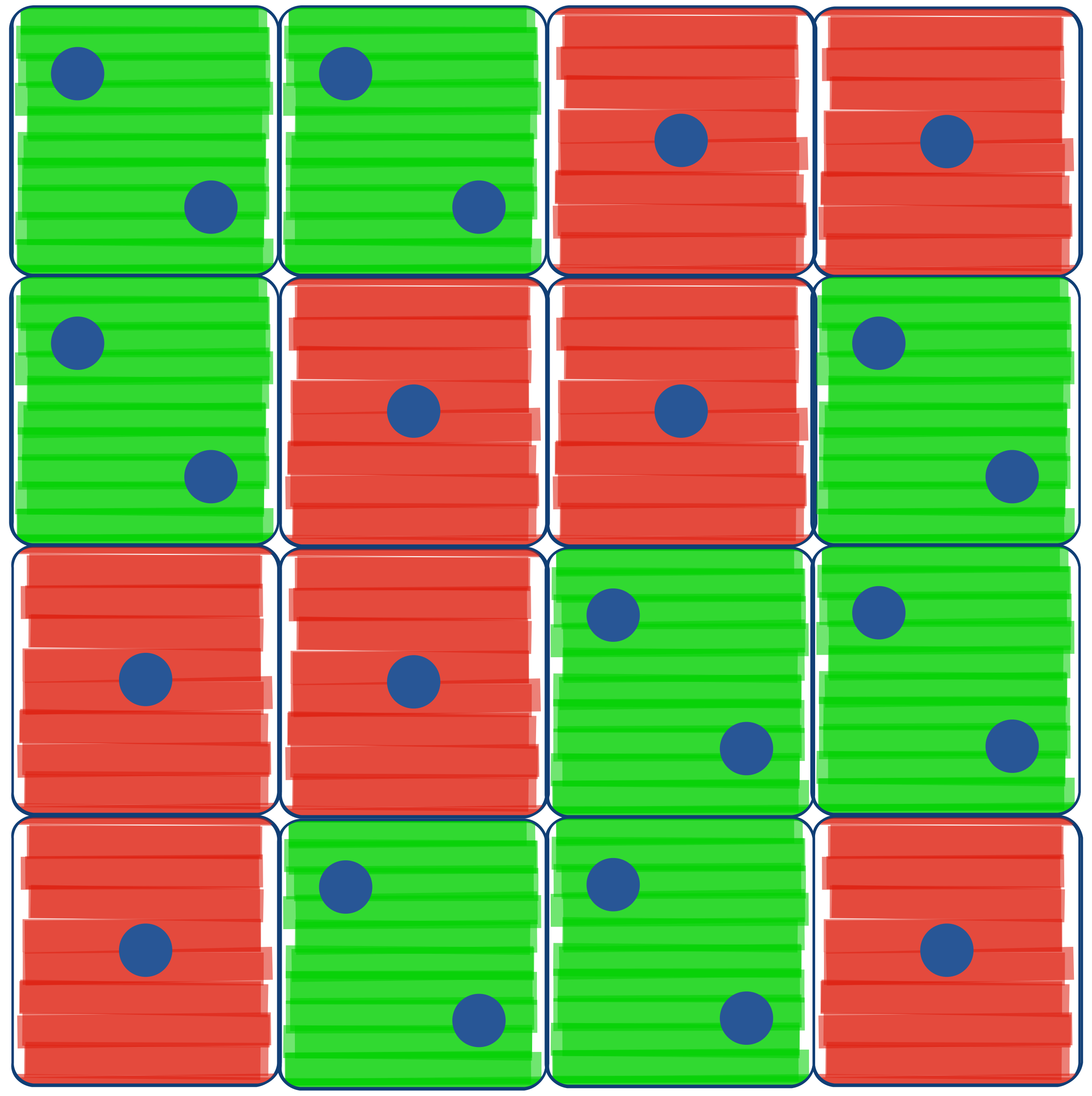


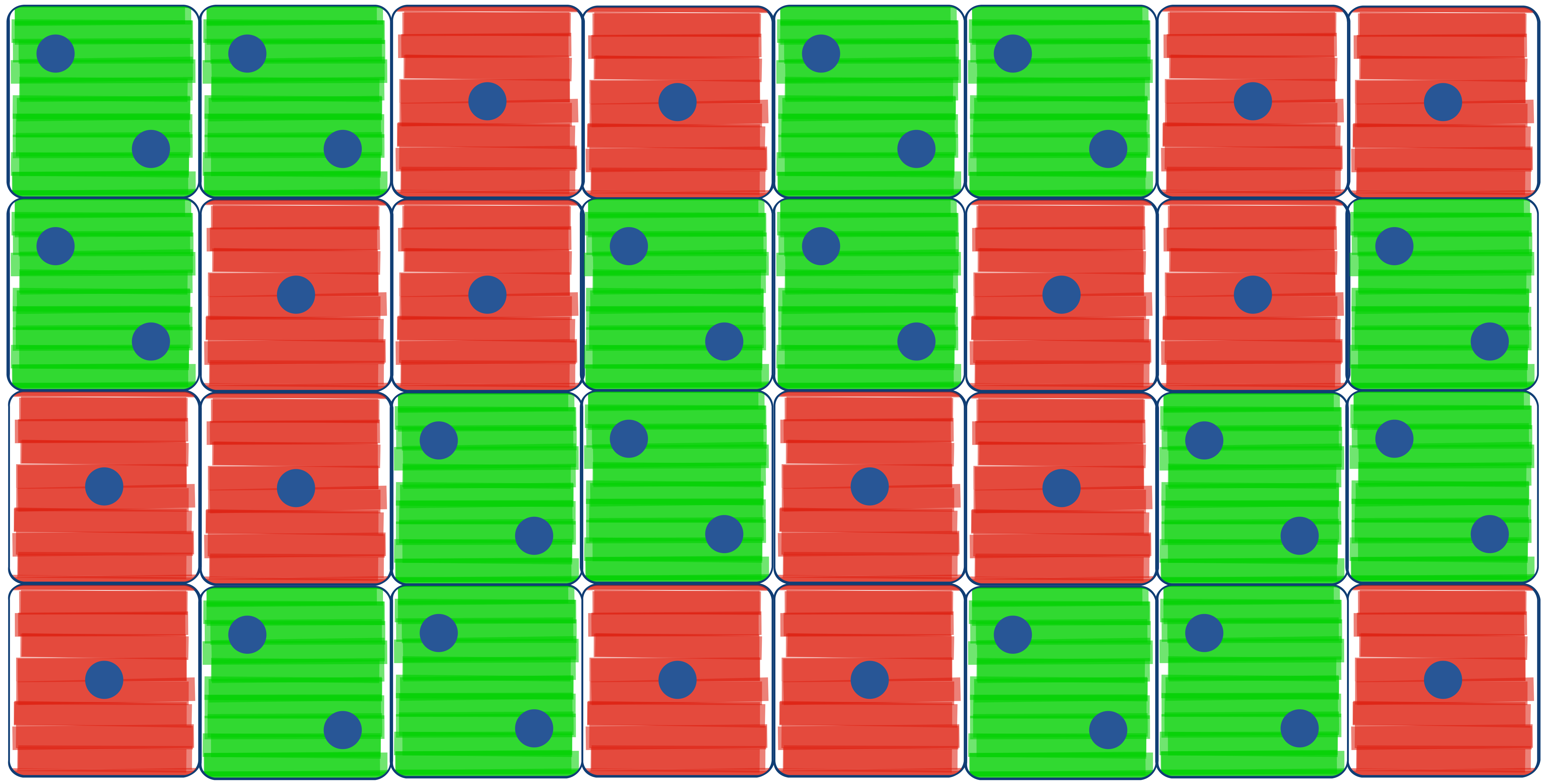
C5

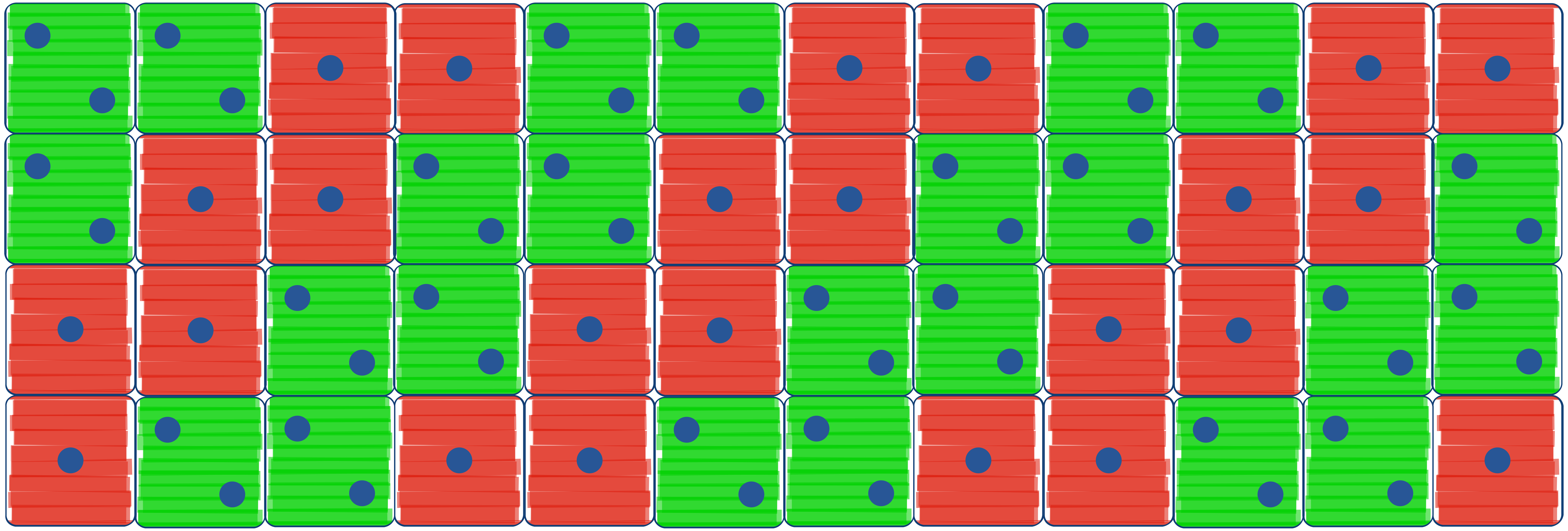


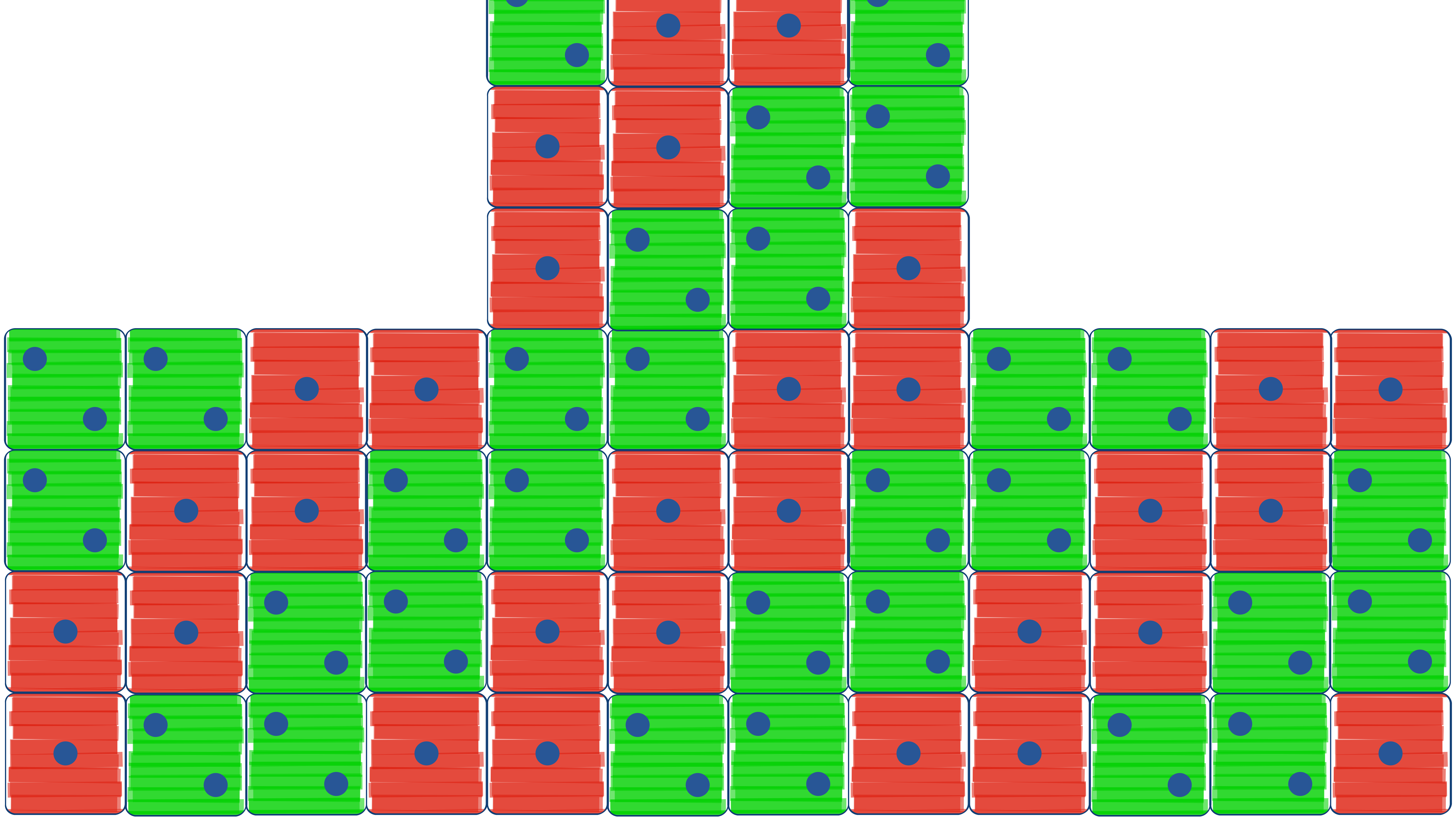
C6

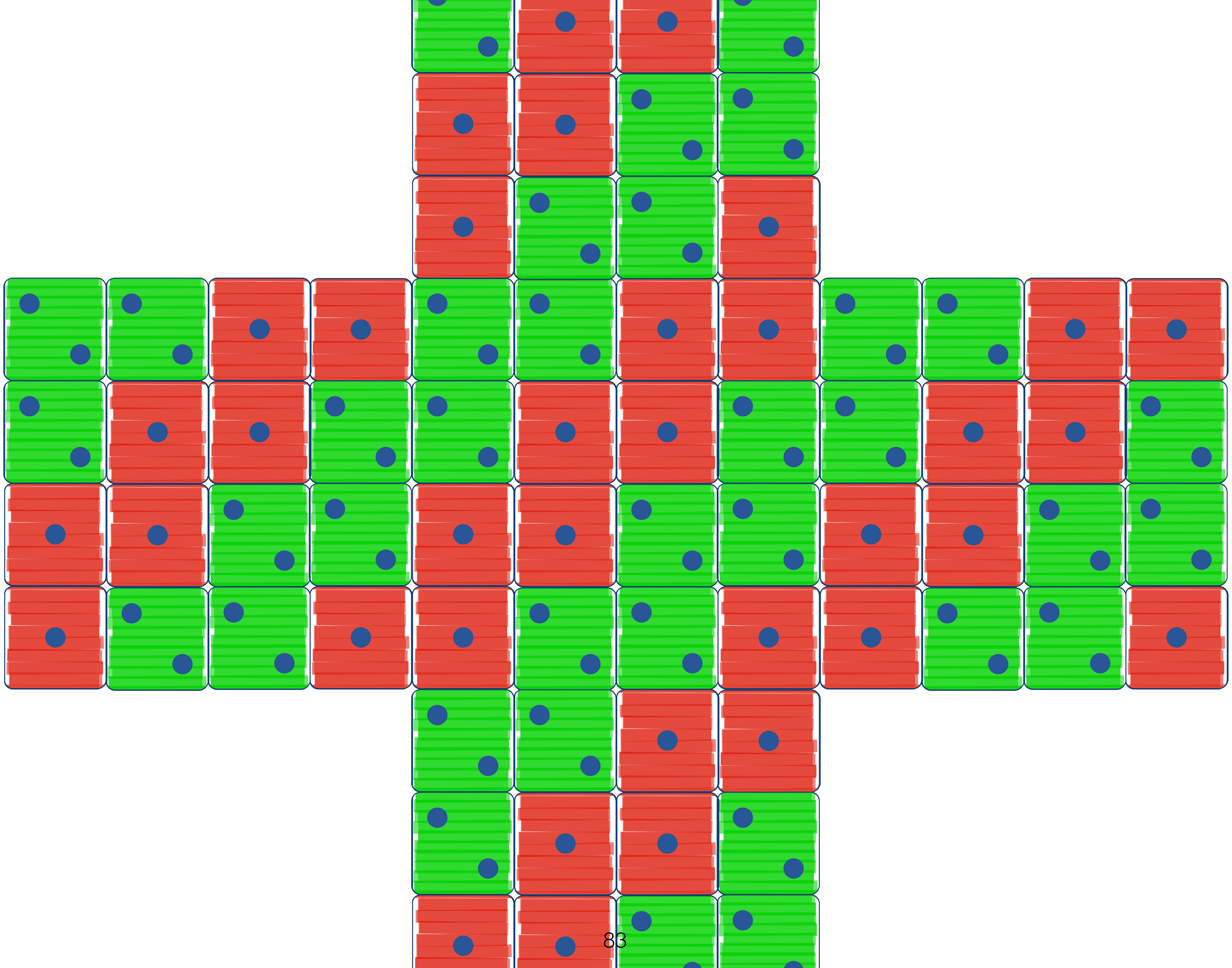








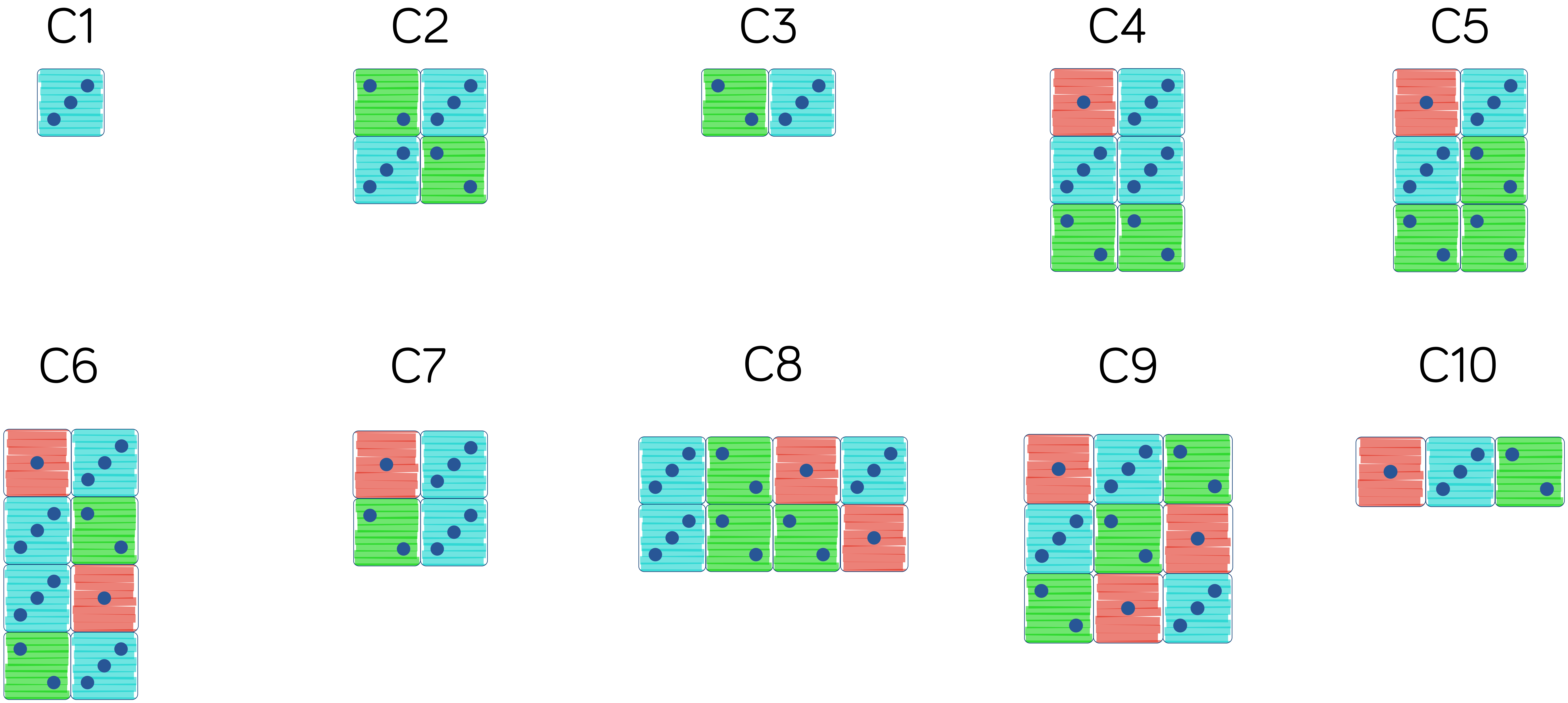




Hard Scenarios In 2D

- The 2D Binary LTI polytope only has 13 vertices, which can be grouped into 6 classes.
- If the each site takes 3 values, how many vertices will the polytope have?
- HUNDREDS OF MILLIONS!
- Can we say anything in this scenario?
- Yes! If we only look at nearest neighbors.
- This polytope only has 98 vertices, grouped into 10 classes.

Extreme Points of the 2D Ternary Nearest-Neighbor TI Marginals



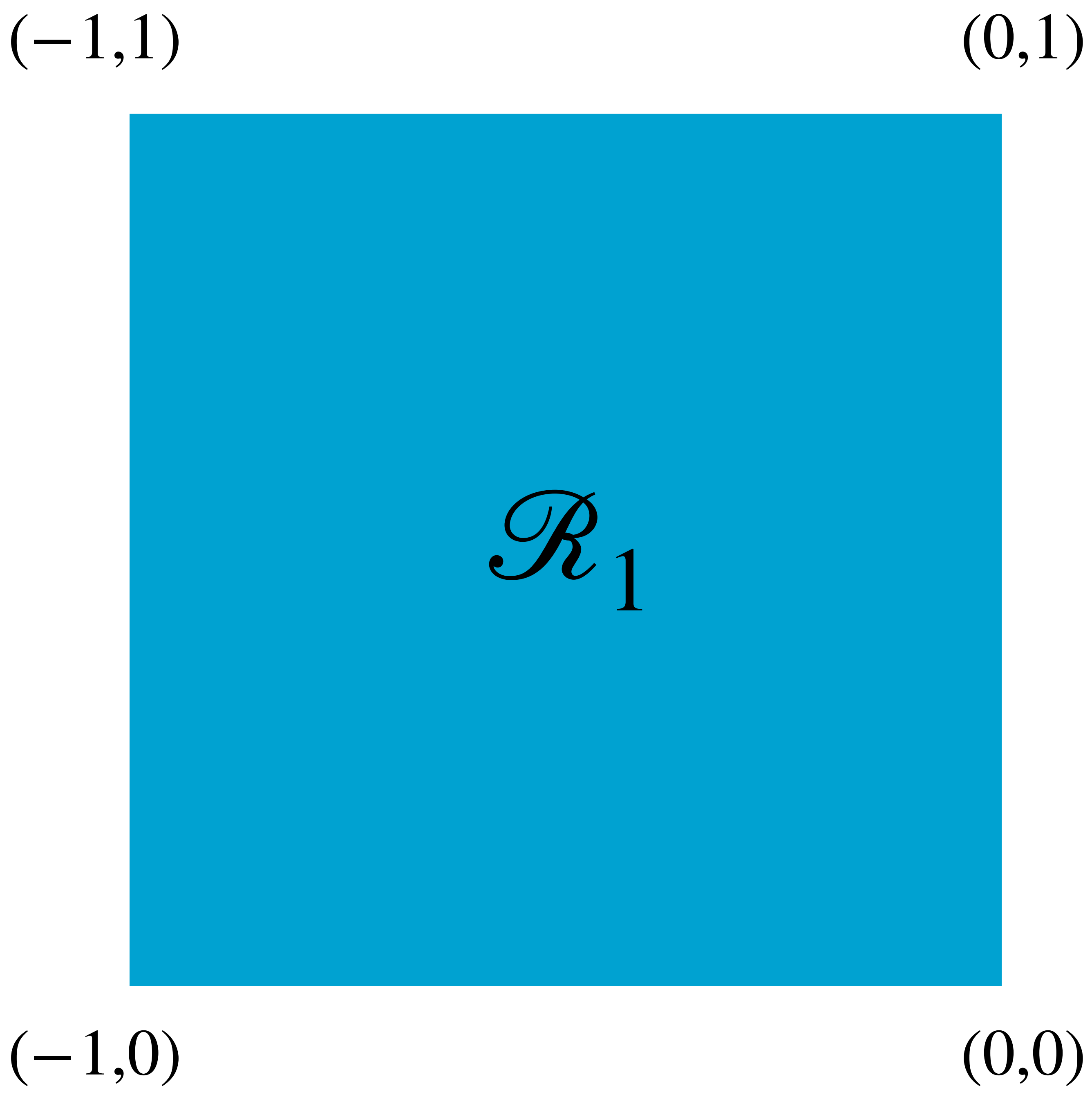
We Have Seen the Easy, the Hard.

MISSION: IMPOSSIBLE ★

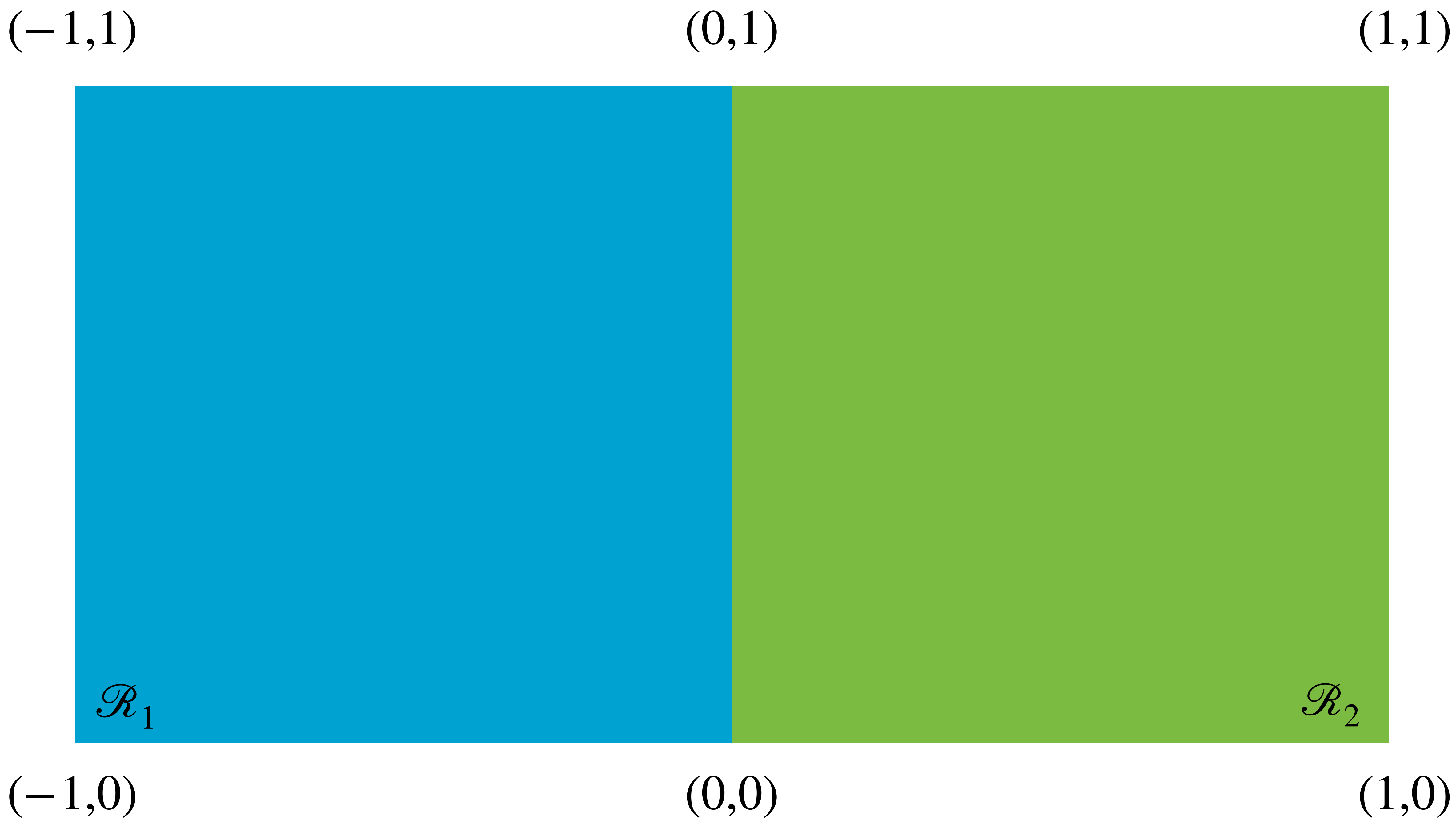
★ No Scientologist was harmed during the completion of the following proofs.

Theorem Impossible 1: Computing the Average Energy per Site of an Arbitrary TI Hamiltonian With Only Nearest-Neighbor Interaction Is Undecidable.

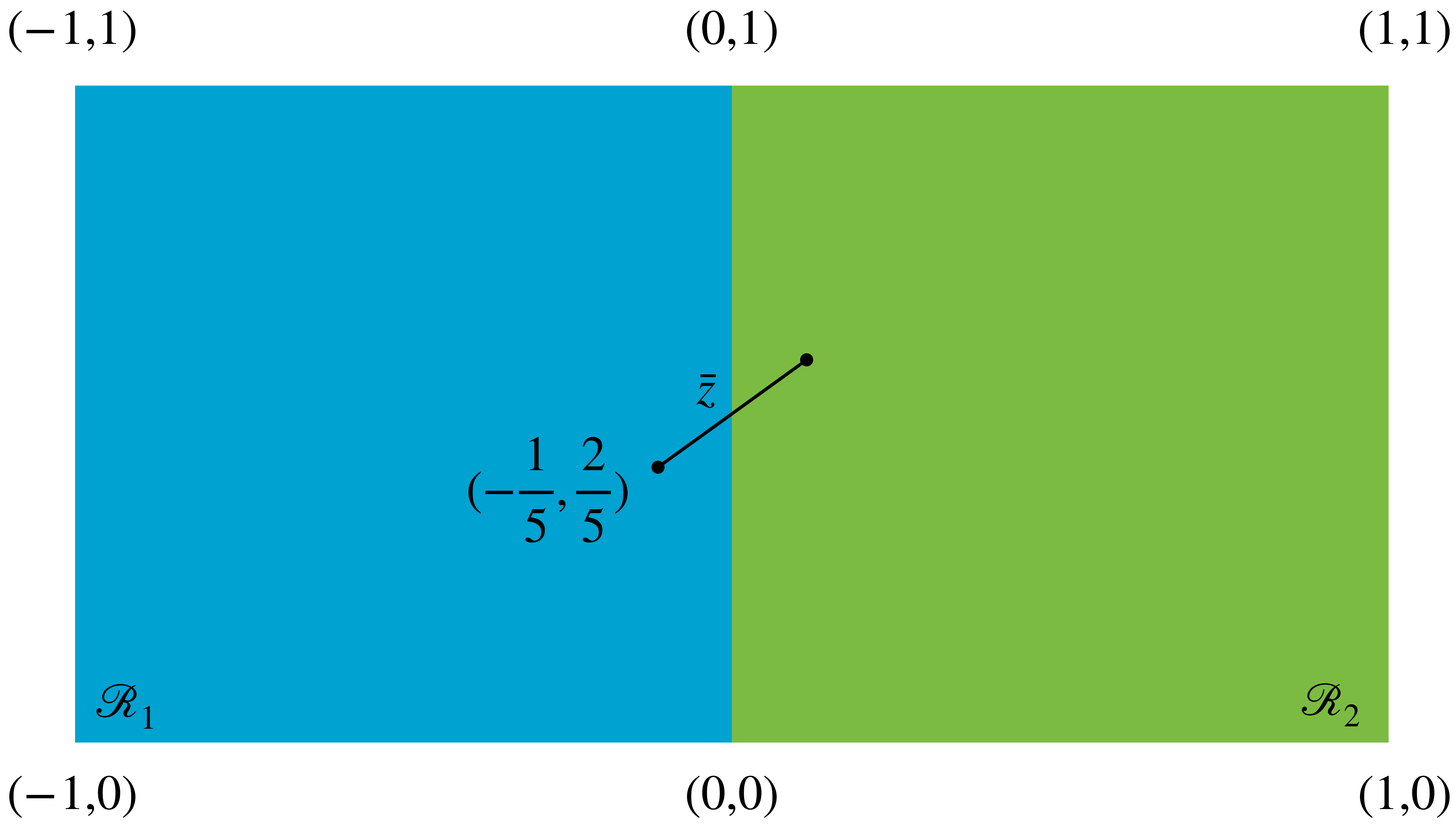
Aperiodic Tilings and Dynamical Systems



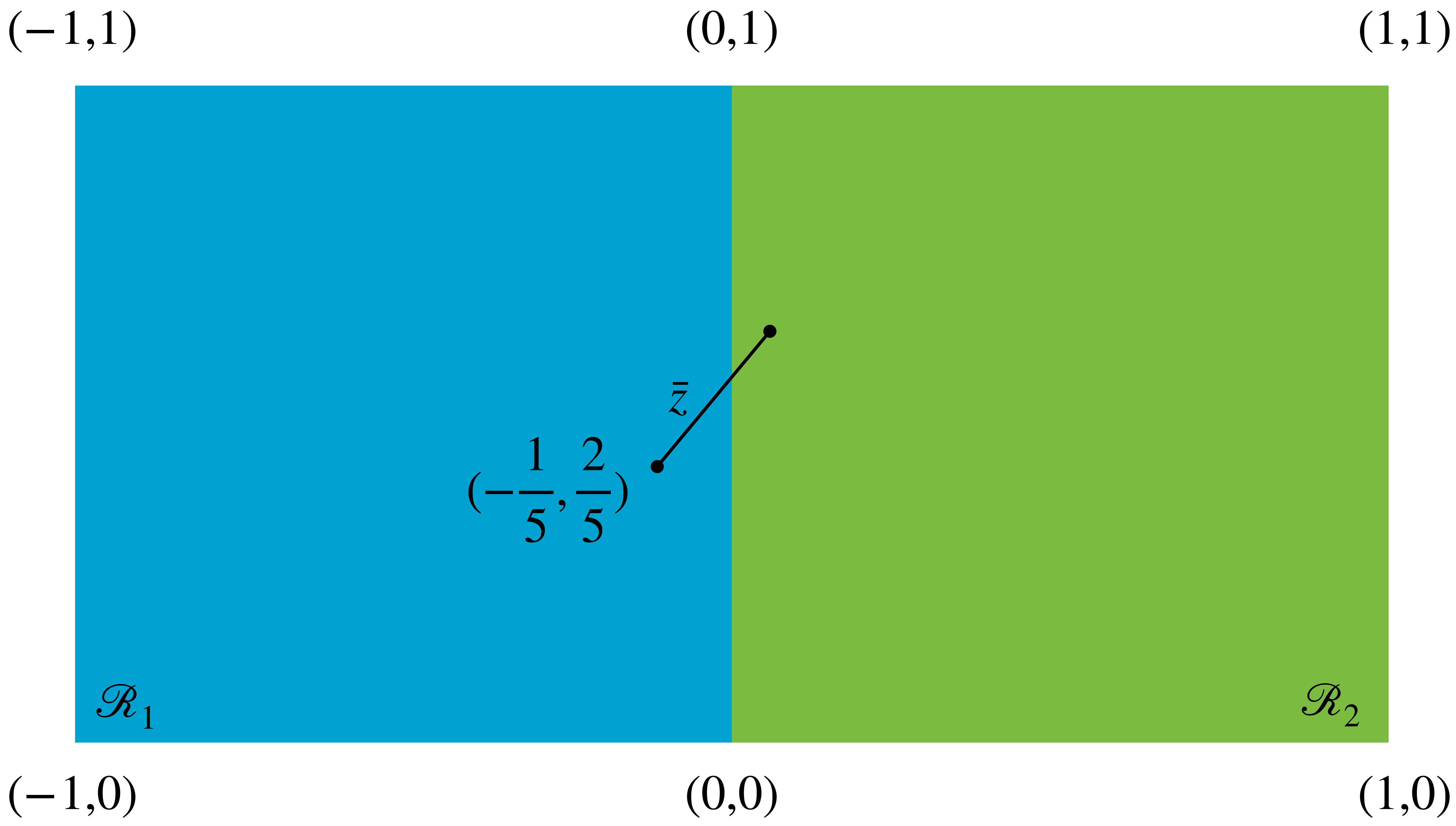
Aperiodic Tilings and Dynamical Systems



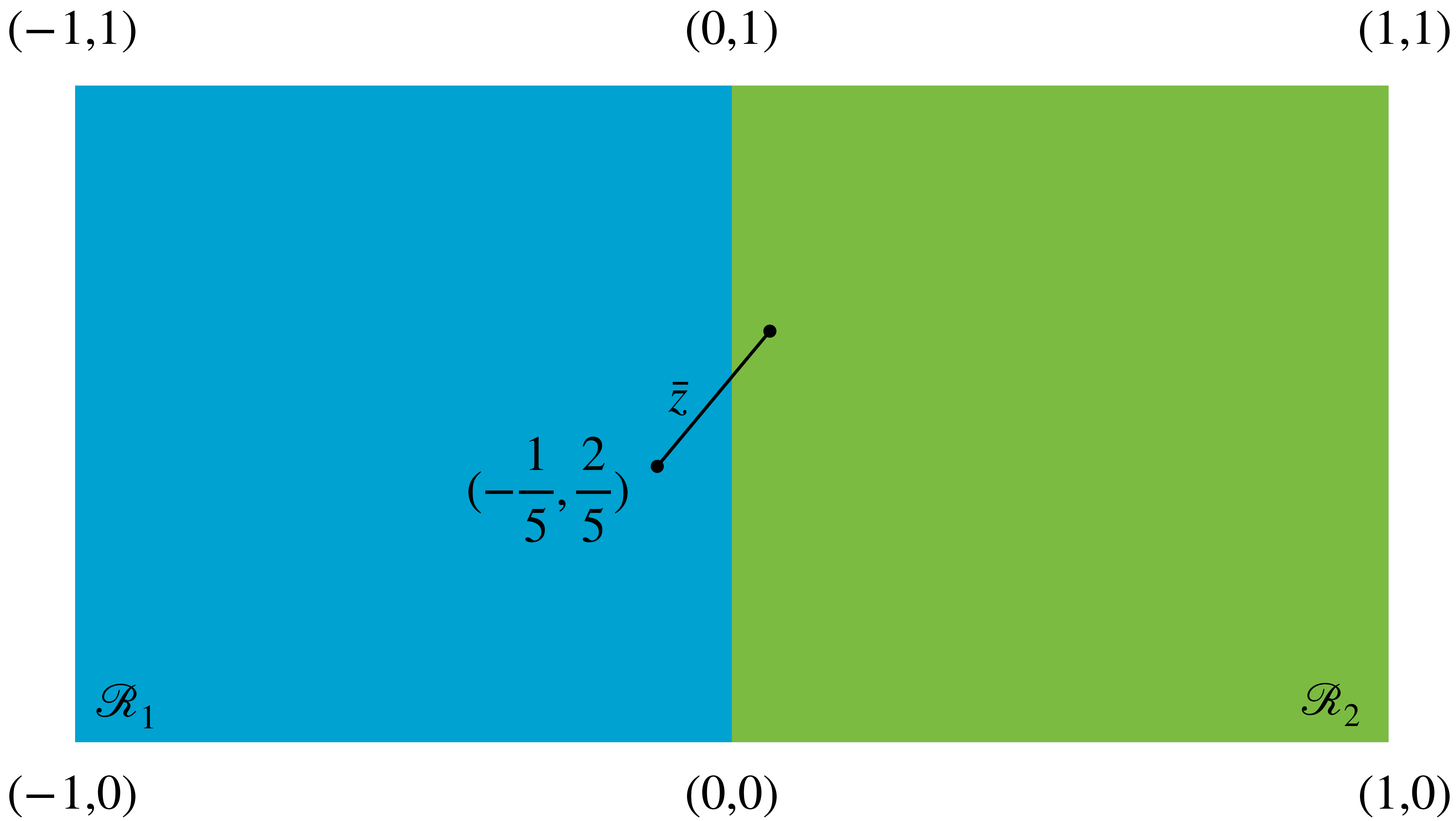
Aperiodic Tilings and Dynamical Systems



Aperiodic Tilings and Dynamical Systems



Aperiodic Tilings and Dynamical Systems



$$f(\bar{z}) = M(\bar{z} - c) + c$$

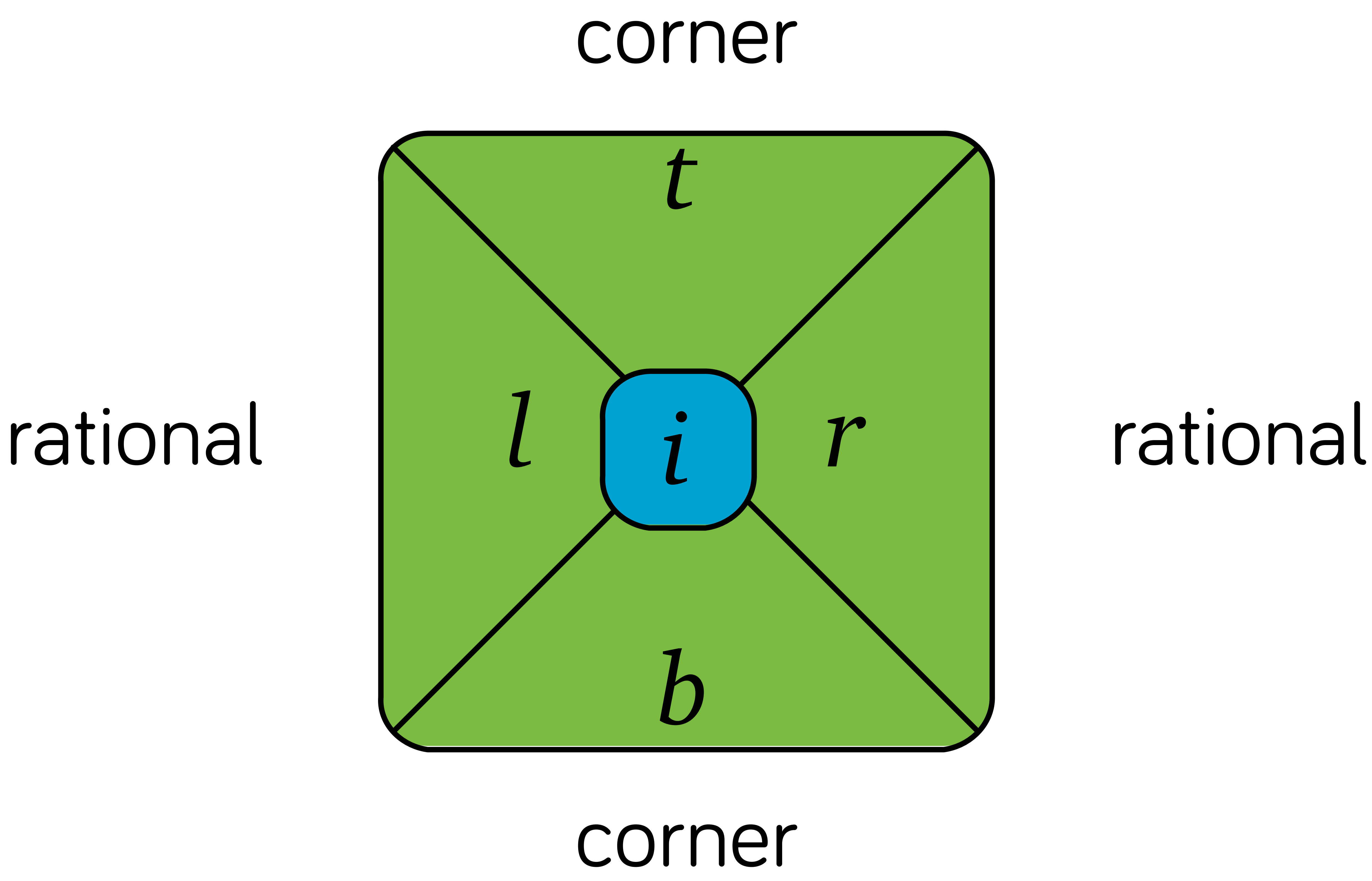
$$M = \begin{pmatrix} \frac{4}{5} & -\frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{pmatrix} \qquad c = \begin{pmatrix} \frac{1}{5} \\ \frac{1}{5} \end{pmatrix} \qquad \|\bar{z} - c\|_2 \leq \frac{2}{5}$$

Aperiodic Tilings and Dynamical Systems

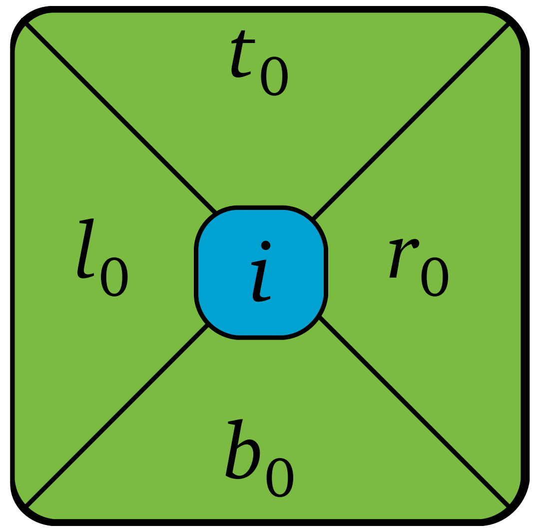
- The action of this dynamical system is to rotate a vector by an angle which is an irrational multiple of 2π.
- A vector which allows this dynamical system to continue forever is called an “immortal point”.
- This system has an (uncountably) infinite number of immortal points.
- However, all of them, together with the action of the dynamical system, can be simulated by a finite number of tiles.

Aperiodic Tilings and Dynamical Systems

$$f(t) + l = b + r$$

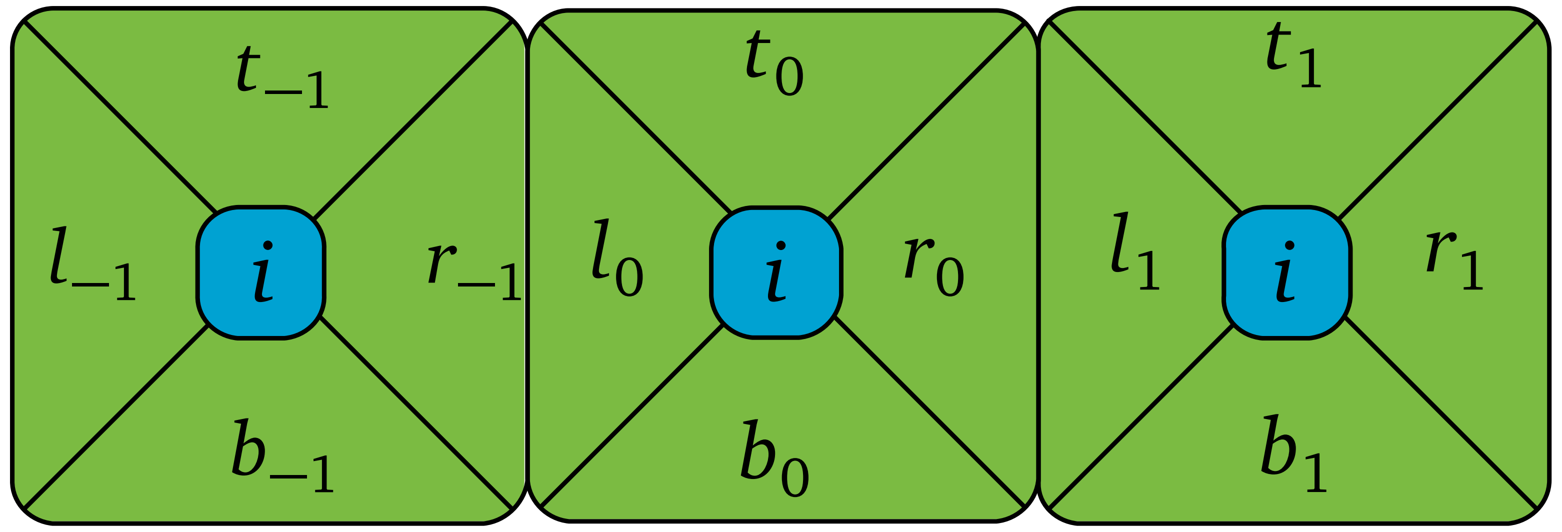


Aperiodic Tilings and Dynamical Systems



Aperiodic Tilings and Dynamical Systems

$$\langle t \rangle = \frac{1}{m} \sum_{k=1}^{m} t_k$$



$$\langle b \rangle = \frac{1}{m} \sum_{k=1}^{m} b_k$$

$$f(\langle t \rangle) \approx \langle b \rangle, m \to \infty$$

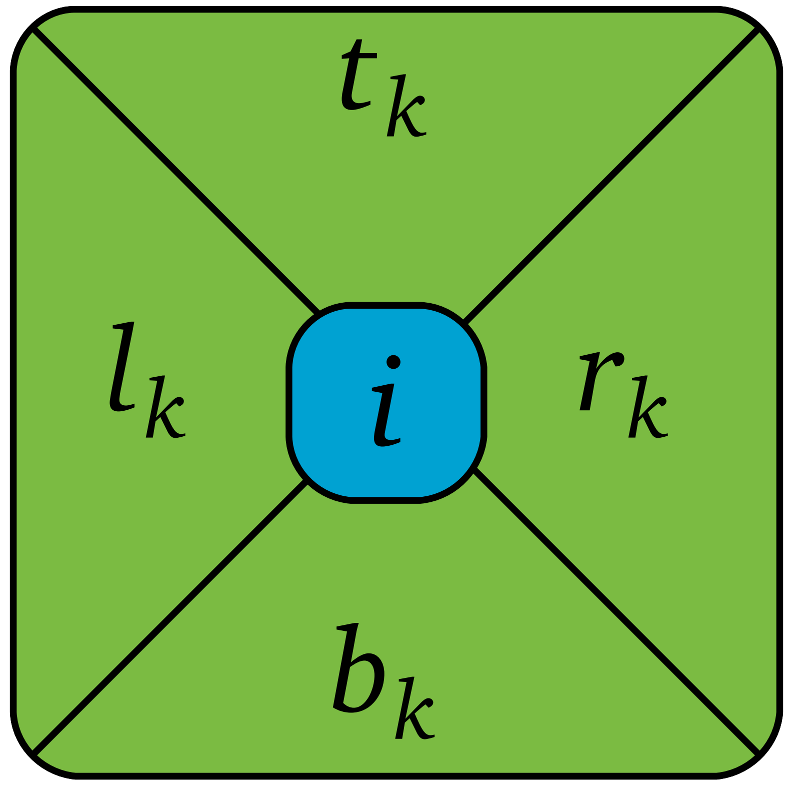
Aperiodic Tilings and Dynamical Systems

$$A_k(\bar{v}) \equiv \lfloor k\bar{v} \rfloor$$

$$B_k(\bar{v}) \equiv A_k(\bar{v}) - A_{k-1}(\bar{v})$$

$B_k(\bar{v})$

$f(A_{k-1}(\bar{v})) - A_{k-1}(f(\bar{v})) + (k-1)c$



$f(A_k(\bar{v})) - A_k(f(\bar{v})) + kc$

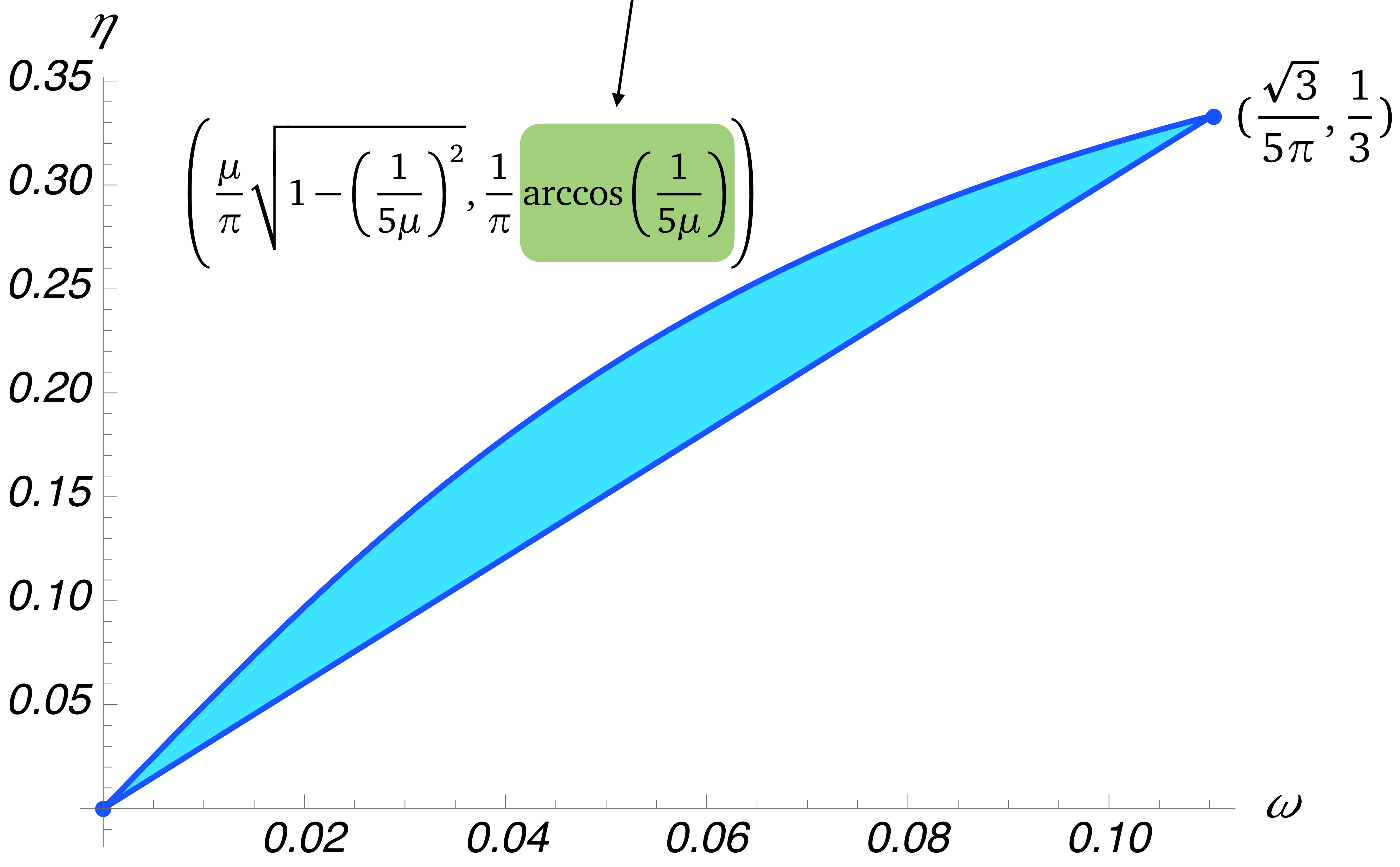
$B_k(f(\bar{v}))$

$$f(t) + l = b + r$$

Aperiodic Tilings and Dynamical Systems

- The construction is similar to Kari, SOFSEM 2008, LNCS 4910 (2008).
- The dynamical system can be simulated using 2947 tiles.
- The number 2947 is not optimal and it may go down a lot with better construction.
- Maybe it will go to 4.
- We can define a set of 2D TI marginals implementing this tiling, and look at the shape of one of its linear projections.

transcendental function



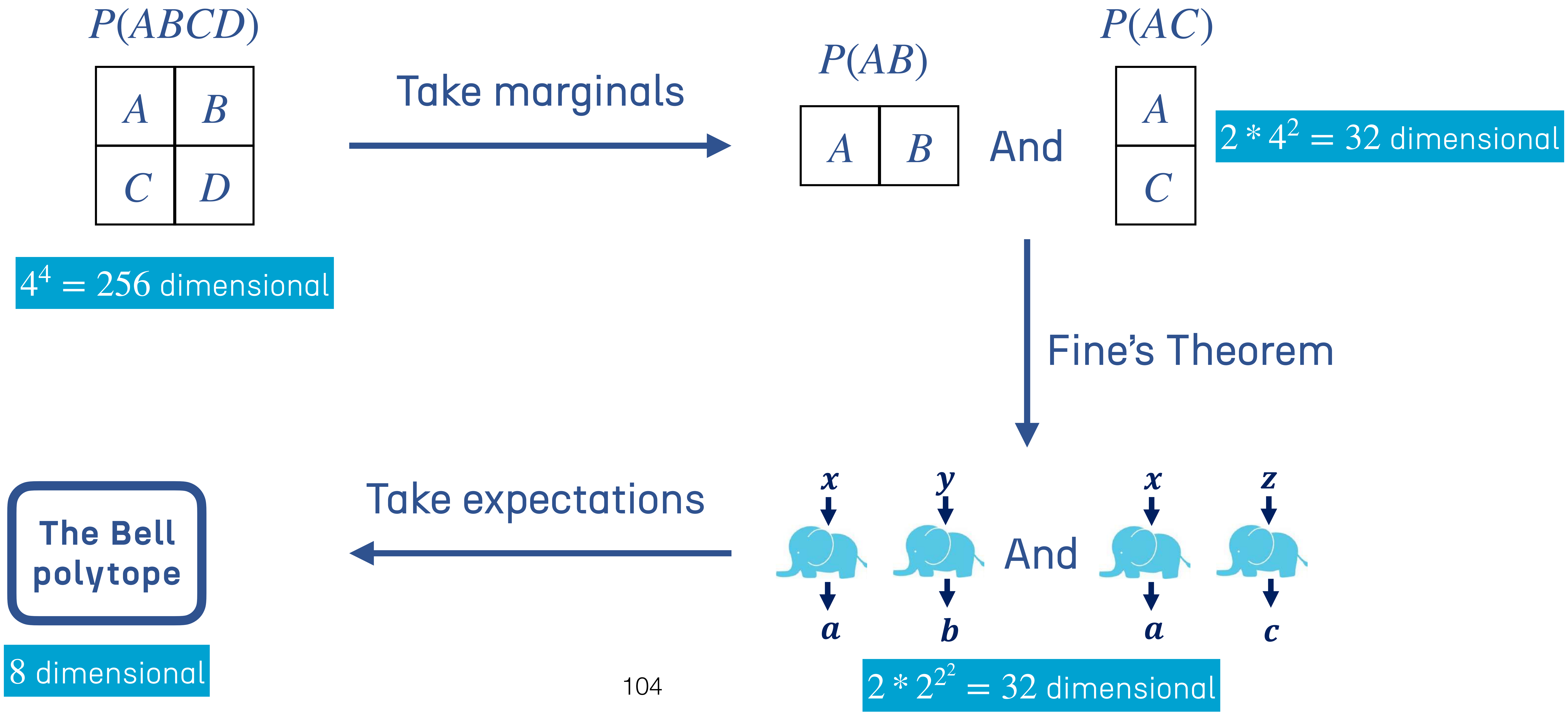
Theorem Impossible 2: When the local dimension of each site is 2947, the set of 2D TI marginals is not semi-algebraic.

Progress and Open Questions

- Similar to the 1D case, we wish to characterize the set of Bell local Hamiltonians with local interactions in 2D.
- Again, we need to use Fine's theorem, so this problem reduces to finding an inner approximation of TI marginals with 4 values per site.
- Again, we need to use tilings, but this time it's corner tilings.

The End Game: Characterization of 2D Local TI Hamiltonians

The Workflow



The End Game: Characterization of 2D Local TI Hamiltonians

Linear Programming Technique

- After the projections M, a LTI probability distribution ($d = 4$, $K = 2 \times 2$), denoted by x, is projected to a 8 -dimensional vector y in the LTI Bell polytope. Call these projections M.

1. Use random objective function c to solve the following linear program (LP)

$$\begin{aligned} \text{Maxmize} \quad & c^T y \\ \text{subject to} \quad & Mx = y \\ & x \text{ is } 2 \times 2\text{-LTI} \end{aligned}$$

2. Solve the LP many times to get many points of y and denote this set by S_y.
3. Compute the polytope from S_y to get the vertices V and facets F. Call the coefficient vector of the facets c_i and the bounds b_i.
4. Update $S_y = V$. For each facet of F, solve the same LPs by substituting c with c_i. If the optimal value exceeds b_i, add the corresponding y to S_y. If no violation appears, go to step 5. Otherwise, go to step 3.
5. We get the LTI Bell polytope V.

The End Game: Characterization of 2D Local TI Hamiltonians

LTI but not TI Vertices

- The LTI Bell polytope has 192 vertices.
- Trying to tile the plane with these 192 vertices as corner tiles, we find only 128 correspond to valid tilings, the remaining 64 vertices are essentially the same.
- Computing the polytope $\mathscr{E}^1$ from the 128 vertices, we have an internal approximation. $\mathscr{E}^1$ has 808 facets, and we call the coefficient vectors f_i and bounds b_i.
- 2×2-LTI is not sufficient to characterize the TI Bell polytope.

The End Game: Characterization of 2D Local TI Hamiltonians

From 2X2 to 3X3 LTI Patch

- A LTI probability distribution ($d = 4, K = 3 \times 3$), denoted by x (4^9-dimensional), is projected to a 8-dimensional vector in the LTI Bell polytope. Call these projections M.

- Using f_i from $\mathscr{E}^1$ as the objective function in the following linear programming:

$$\begin{aligned} \text{Maxmize} \quad & f_i^T y \\ \text{subject to} \quad & Mx = y \\ & x \text{ is } 3 \times 3 \text{ LTI} \end{aligned}$$

  If the optimal value exceeds c_i, we check y using corner tiling. If it can tile the plane, we have a new valid vertex.

- Using the above method, we find 64 new valid vertices.

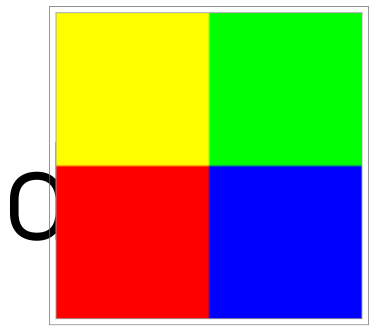
- Combined with the former 128 points, we have 192 valid vertices.

The End Game: Characterization of 2D Local TI Hamiltonians

LTI is too weak to do the job!

The Search for Tighter Relaxations in 2D

Corner Tilings and Translation Invariance

- Corner tile: a square with its four corners colored 
- For a given tile set, there are three possibilities:
  - Cannot tile a square of size n for some n
  - Can tile a square of size n for some n with the same colors on the borders——periodic tiling
  - Can tile the entire plane but cannot do it periodically——aperiodic tiling
- There exist aperiodic tilings.
- The corner tiling problem is undecidable.

Ref: Lagae A, Kari J, Dutré P. Aperiodic Sets of Square Tiles with Colored Corners, Report CW 460, KU Leuven (2006)

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- The simplest case: Each random variables at the region $K = 2 \times 2$ takes value $0(1)$ with probability 1.

$$\begin{array}{|c|c|}\hline 0 & 0 \\ \hline 0 & 0 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & 1 \\ \hline \end{array}$$

  These two probability distributions are clearly TI marginals.

- What about other deterministic probability distributions?

  For instance:

$$\begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & 1 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 0 & 1 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 0 & 1 \\ \hline 0 & 1 \\ \hline \end{array}$$

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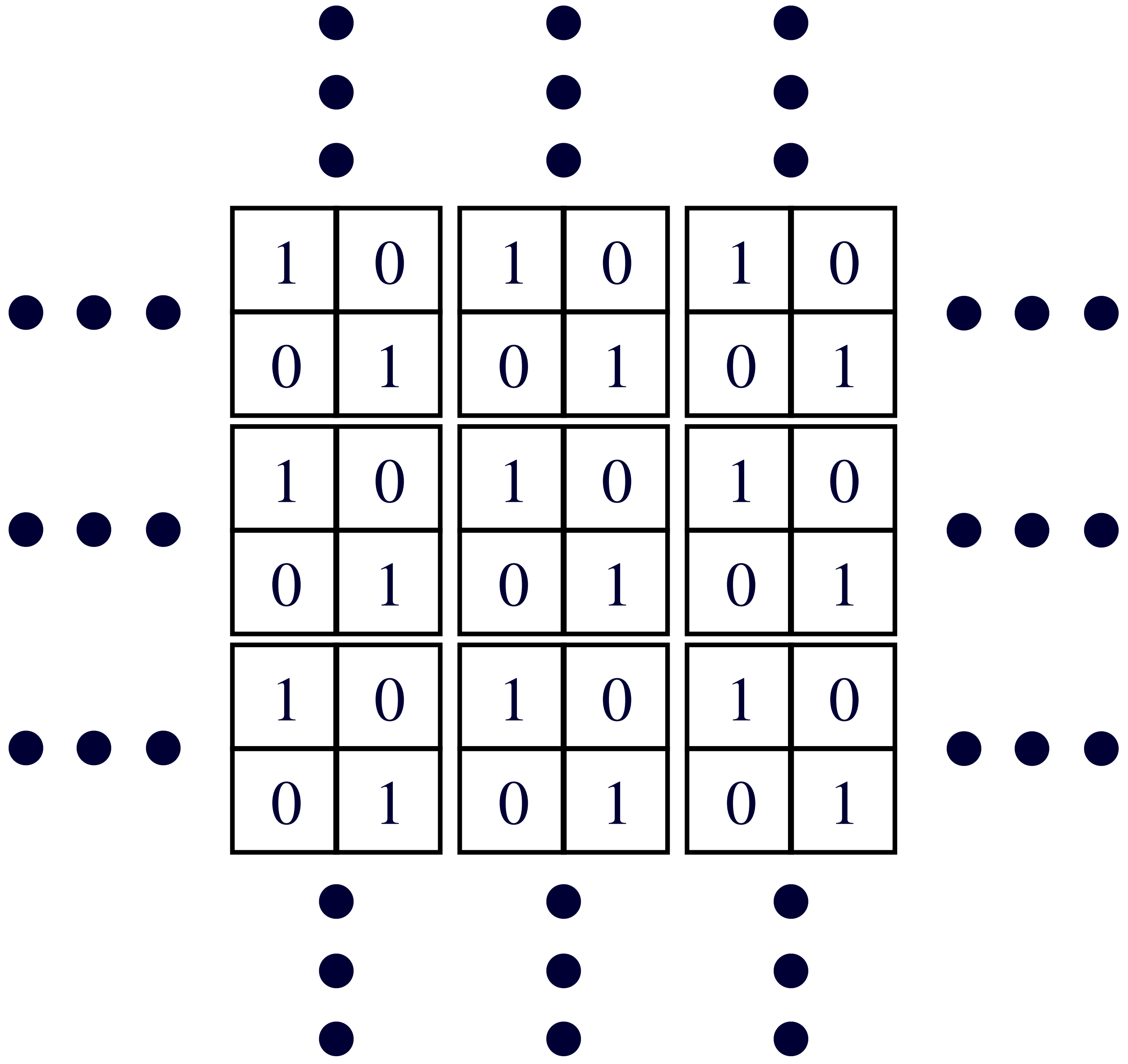
- Taking $\begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & 1 \\ \hline \end{array}$ for example, repeat this probability distribution all over the plane.



This probability distribution is 2-TI

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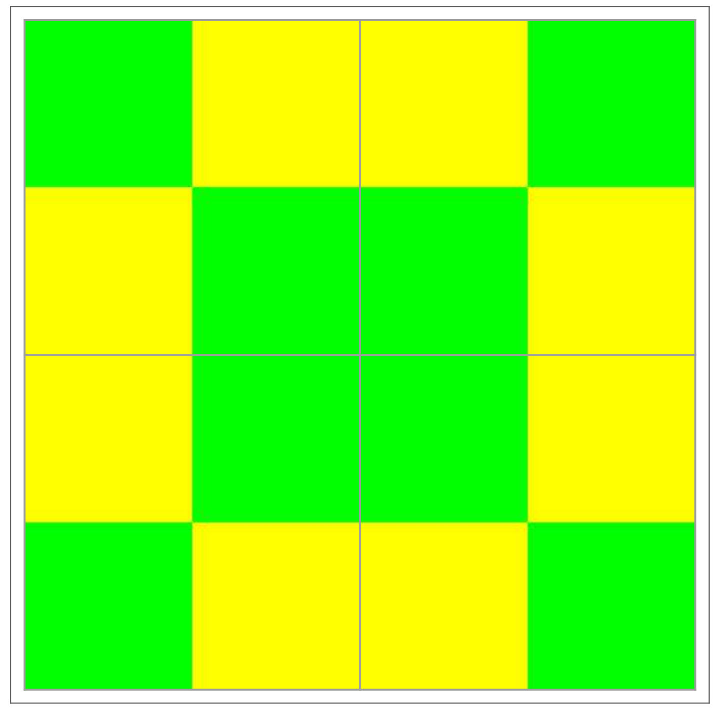
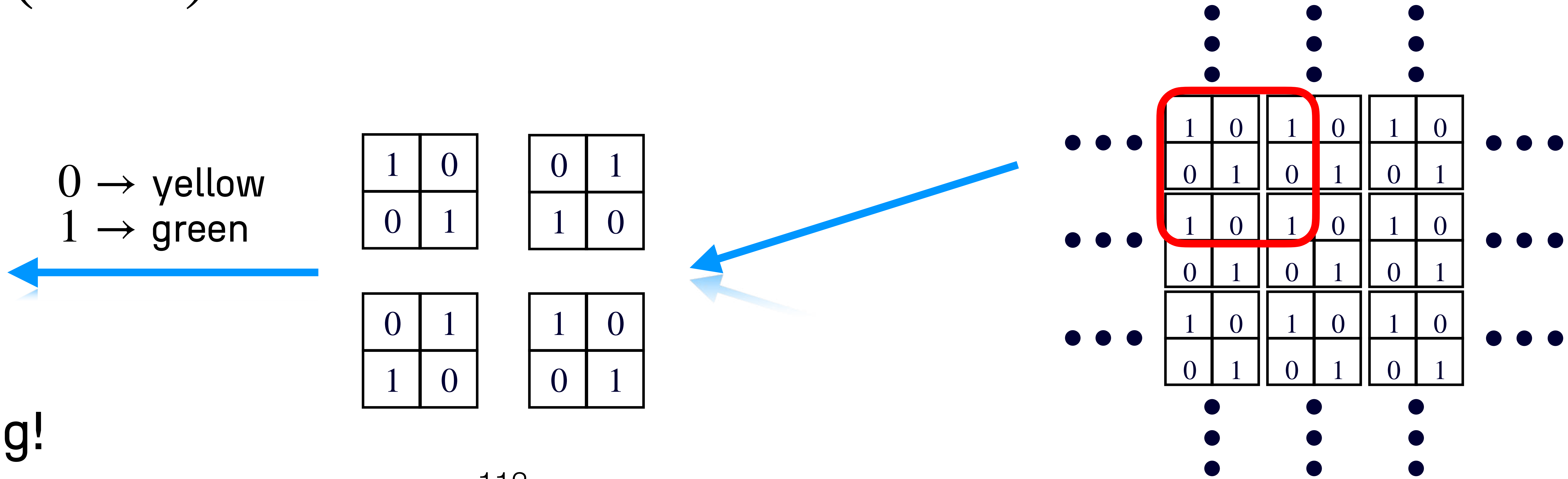
- By summing over the four 2×2 distributions inside the 3×3 red box and taking the average, we get a 2×2 TI marginal.

- Every periodic corner tiling corresponds to a TI marginal. Count the number of tiles: $P(1001) = P(0110) = 2/4 = 1/2$



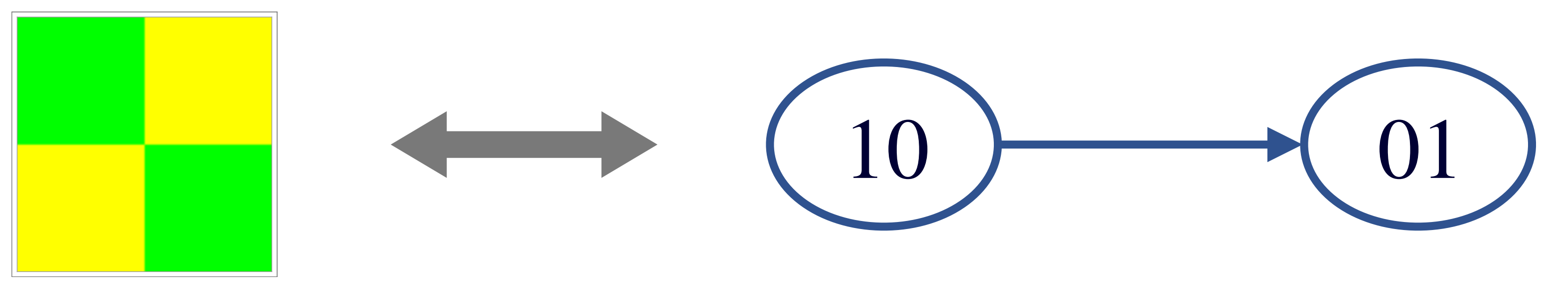
Periodic corner tiling!



The Search for Tighter Relaxations in 2D

Transducers

- How to efficiently tile a strip of height N
  - $N = 1$
    - View a corner tile as a mapping (green $\to 1$, yellow $\to 0$):



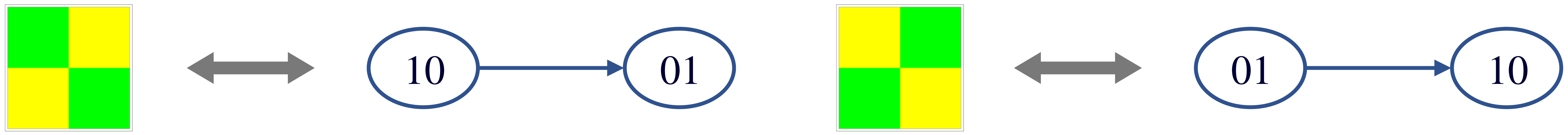
    - The mapping is to map left two colors to the right two colors.
    - The nodes are named by the colors from top to bottom. The left two colors of the tile are green and yellow, so the left node is 10. Same for the right node.

Ref: E. Jeandel and M. Rao, an Aperiodic Set of 11 Wang Tiles, AiC (2021).

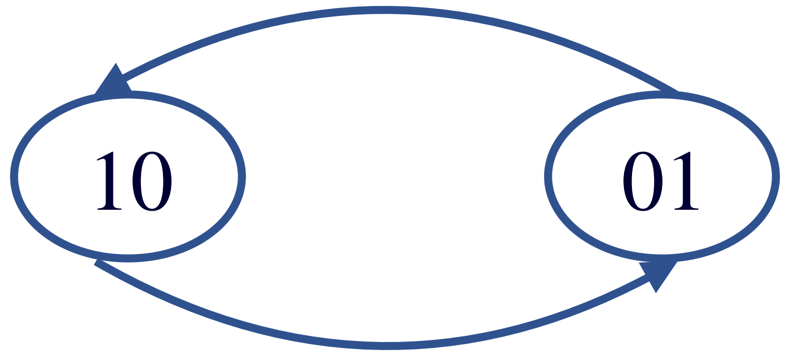
The Search for Tighter Relaxations in 2D

Transducers

- $N = 1$
  - Suppose we have two tiles



  - Construct a transducer, a directed graph using the nodes and edges

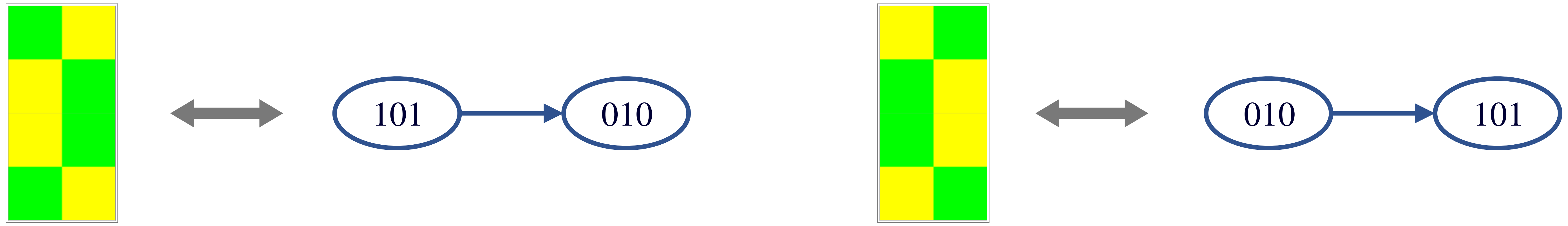


  - A tiling of height $N = 1$ is a biinfinite path on the transducer. The tiling exists if and only if there exists a cycle in the graph.
  - For this tiling to be periodic, we just need to check each node whether the first number equals the second number. For our example, it is not.

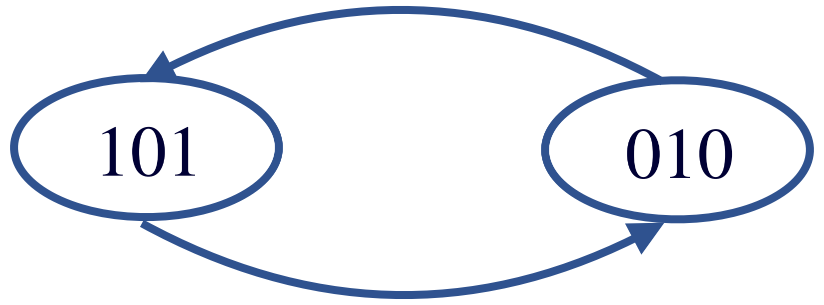
The Search for Tighter Relaxations in 2D

Transducers

- $N = 2$
  - Construct tiles of size 2×1



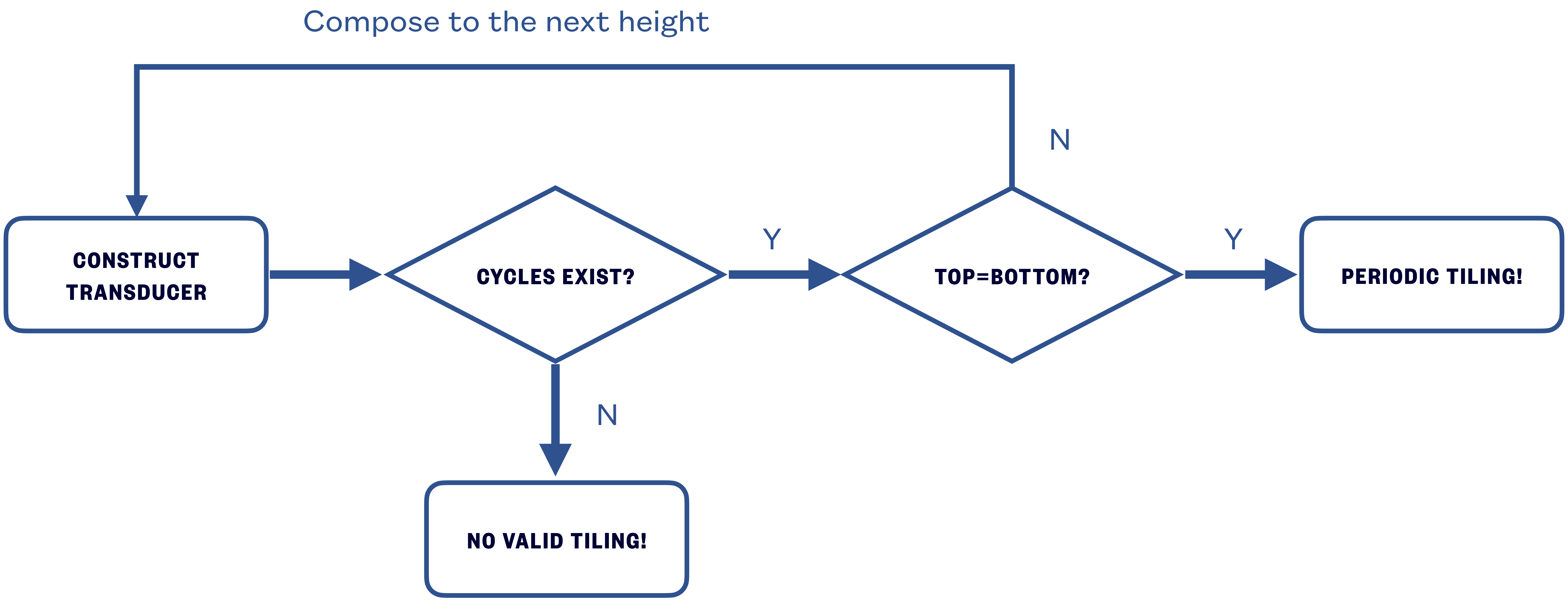
  - The bottom colors of the top tile must be identical to the top colors of the bottom tile. The nodes are named the same way before, except the overlapping colors are identified: $1001 \to 101$
  - Construct the graph:



  - Check each node: the first number equals to the last number. We get a valid periodic tiling!
  - If $N = 2$ is not enough, we go to next height until we find periodic tilings, or find that there is no cycle in the graph.

The Search for Tighter Relaxations in 2D

Transducers



The Search for Tighter Relaxations in 2D

Corner Tilings and Cycle Generators in 2 Colors

- Minimal cycle generator: a tile set that admits a valid tiling, but none of its subsets can.

- There are 17 minimal cycle generators for $d = 2$ (2 colors), corresponding to 17 TI vertices.

- We compute the convex polytope from these 17 TI vertices, denoted by $\mathscr{P}^{mcg}_{2\times 2}$.

- Comparing to the facet representation of $\mathscr{P}^{LTI}_{2\times 2}$, $\mathscr{P}^{mcg}_{2\times 2}$ has 8 more linear inequalities.

Ref: W.-G. Hu and S.-S. Lin, Nonemptiness Problems of Plane Square Tiling with Two Colors, Proc. Amer. Math. Soc. 139, 03 (2011).

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The Search for Tighter Relaxations in 2D

The Extra Facets for a Tighter Relaxation

$$P(\bar{e}_1) - P(\bar{e}_4) \leq P(e_2) + P(e_3)$$
$$P(\bar{e}_2) - P(\bar{e}_3) \leq P(e_1) + P(e_4)$$
$$P(\bar{e}_3) - P(\bar{e}_2) \leq P(e_1) + P(e_4)$$
$$P(\bar{e}_4) - P(\bar{e}_1) \leq P(e_2) + P(e_3)$$
$$P(\bar{e}_1) \leq P(\bar{e}_2) + P(\bar{e}_3) + P(\bar{e}_4)$$
$$P(\bar{e}_2) \leq P(\bar{e}_1) + P(\bar{e}_3) + P(\bar{e}_4)$$
$$P(\bar{e}_3) \leq P(\bar{e}_1) + P(\bar{e}_2) + P(\bar{e}_4)$$
$$P(\bar{e}_4) \leq P(\bar{e}_1) + P(\bar{e}_2) + P(\bar{e}_3)$$

$$e_1 \; \begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & 0 \\ \hline\end{array} \qquad e_2 \; \begin{array}{|c|c|}\hline 0 & 1 \\ \hline 0 & 0 \\ \hline\end{array} \qquad e_3 \; \begin{array}{|c|c|}\hline 0 & 0 \\ \hline 1 & 0 \\ \hline\end{array} \qquad e_4 \; \begin{array}{|c|c|}\hline 0 & 0 \\ \hline 0 & 1 \\ \hline\end{array}$$

$$\bar{e}_1 \; \begin{array}{|c|c|}\hline 0 & 1 \\ \hline 1 & 1 \\ \hline\end{array} \qquad \bar{e}_2 \; \begin{array}{|c|c|}\hline 1 & 0 \\ \hline 1 & 1 \\ \hline\end{array} \qquad \bar{e}_3 \; \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 0 & 1 \\ \hline\end{array} \qquad \bar{e}_4 \; \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & 0 \\ \hline\end{array}$$

- Can these facets be generalized to more than 2 colors?
- Can we do it for 4 colors to use Fine’s Theorem?

2D Classical

Thank You

Proc. R. Soc. A 474: 20170822 (2018)

1D Classical+Quantum

Phys. Rev. Lett.118.230401 (2017)

1D Quantum

npj Quantum Information 8, 89 (2022)