Zero temperature convergence of Gibbs measures for a locally finite potential in a 2-dimensional lattice

Philippe Thieullen
University of Bordeaux

Joint work with
S. Barbieri, R. Bissacot, and Gregorio Dalle Vedove Nosaki

Multidimensional symbolic dynamics
and lattice models of quasicrystals

1st–5 April 2024

Summary

- I. **Position of the problem**
- II. Some old results in dimension 1
- III. A new result in dimension 2
- IV. Ideas of the proof

I.1. Zero-temperature chaotic convergence

We want to understand whether some spin systems exhibit a phenomenon called *zero-temperature chaotic convergence* introduced by van Enter and Ruszel (2007).

Definition

Let $\Sigma^d(\mathscr{A}) = \mathscr{A}^{\mathbb{Z}^d}$ be a spin system over a finite alphabet $\mathscr{A}$ and $\varphi : \Sigma^d(\mathscr{A}) \to \mathbb{R}$ be a continuous function (potential). Let $\mu_{\beta\varphi}$ be any invariant Gibbs measure for the potential $\beta\varphi$.

The zero-temperature chaotic convergence is a phenomenon where there exists a sequence $(\beta_k)_{k\geqslant 0}$, $\beta_k \to +\infty$, and two disjoint invariant compact sets $K_1, K_2 \subseteq \Sigma^d(\mathscr{A})$ such that if $\mu_{\beta_k\varphi}$ is any invariant Gibbs measure,

- any weak$*$ limit of $(\mu_{\beta_{2k+1}\varphi})_{k\geqslant 0}$ is supported in K_1
- any weak$*$ limit of $(\mu_{\beta_{2k}\varphi})_{k\geqslant 0}$ is supported in K_2

I.2. Zero-temperature chaotic convergence

Remark

- By compactness argument, some subsequences $(\mu_{\beta_k})_{k\geqslant 0}$ are converging and are not chaotic. So the chaotic convergence cannot be expected for all subsequences.
- Coronel, Rivera-Letelier (2015) introduced a stronger notion of *sensitive dependence of the chaotic convergence*: for every sequence $\beta_k \to +\infty$, for every $\epsilon > 0$, there exists $\|\psi - \phi\|_\infty < \epsilon$ and a subsequence $(\beta_{\sigma(k)})_{k\geqslant 0}$ such that $(\mu_{\beta_{\sigma(k)}\psi})_{k\geqslant 0}$ has a chaotic convergence at zero temperature.
- We will not discuss that notion, but van Enter's method is robust and it is likely that our results are also true in that case.

I.3. General setting

Notation

- *A spin system*: $\mathscr{A}$ a finite alphabet, $\Sigma^d(\mathscr{A}) = \mathscr{A}^{\mathbb{Z}^d}$, the full shift
- The group of *space translations* $\sigma^k : \Sigma^d(\mathscr{A}) \to \Sigma^d(\mathscr{A})$, $k \in \mathbb{Z}^d$
- The *Hamiltonian* is given by a Lipschitz function $\varphi : \Sigma^d(\mathscr{A}) \to \mathbb{R}$

$$H_\Lambda(x) = \sum_{k \in \Lambda} \varphi \circ \sigma^k(x), \quad \Lambda \Subset \mathbb{Z}^d \tag{1}$$

Remark

We will be interested in studying *short range interactions* $\Phi = (\Phi_X)_X$ where $X = k + [\![1, D]\!]^d$ is any square of fixed size D. Our Hamiltonian is equivalent to the one defined by summing over all interactions

$$H_\Lambda^\varnothing(x) = \sum_{X \subseteq \Lambda} \Phi_X(x), \quad \varphi(x) = \frac{1}{D^2} \sum_{0 \in X} \Phi_X(x) \tag{2}$$

I.4. Equilibrium measures/Gibbs measures

Definition

An *equilibrium measure* is a shift invariant probability measure $\mu_{\beta\varphi}$ solution of the variational principle: $\mu_{\beta\varphi}$ minimizes the *free energy*

$$F_\beta(\varphi) := \inf \Big\{ \int \varphi \, d\mu - \beta^{-1}\mathrm{Ent}(\mu) : \mu \text{ shift invariant probability } \Big\}$$

- *shift invariance*: $\sigma_\sharp^k(\mu) = \mu, \quad \forall\, k \in \mathbb{Z}^d$,
- *Kolmogorov-Sinai entropy*: $\mathscr{P}$ the canonical generating partition

$$\mathrm{Ent}(\mu) = \lim_{n\to+\infty} \frac{1}{n^d}\mathrm{Ent}\Big(\mu, \bigvee_{k\in [\![1,n]\!]^d} \sigma^{-k}\mathscr{P}\Big)$$

Remark

If $\varphi : \Sigma^d(\mathscr{A}) \to \mathbb{R}$ is Lipschitz, *shift invariant Gibbs measures* defined by the DLR procedure and equilibrium measures give the same object.

I.5. Main question

Let $\mathscr{G}_\beta(\varphi)$ be the set of equilibrium measures or Gibbs measures.

Remark

- Thanks to Dobrushin's argument, $\mathscr{G}_\beta(\varphi)$ is a single element at large temperaure
- For simple systems (at least for short range φ), $\mathscr{G}_\beta(\varphi)$ may have several pure states at small temperature. For the Ising model in $\mathbb{Z}^2$

$$\forall\, \beta < \beta_c,\ \mathrm{card}(\mathscr{G}_\beta(\varphi)) = 1, \quad \forall\, \beta > \beta_c,\ \mathscr{G}_\beta(\varphi) = [\mu_\beta^+, \mu_\beta^-] \qquad (1)$$

Question

What are the limits of Gibbs measures as the temperature goes to zero? More precisely what are the limits of the whole set $\mathscr{G}_\beta(\varphi)$ as $\beta \to +\infty$? For the Ising model

$$[\mu_\beta^+, \mu_\beta^-] \to [\mu^+, \mu^-], \quad \text{there is no chaotic convergence} \qquad (2)$$

I.6. Minimizing measures

Question: What are the possible weak limits of $(\mu_\beta)_{\beta\to+\infty}$?

$$\int \varphi\, d\mu_\beta - \beta^{-1}\mathrm{Ent}(\mu_\beta) = F_\beta(\varphi) \tag{1}$$

Definition

- *The ground level energy* (by freezing the system $\beta \to +\infty$)

$$F_\infty(\varphi) := \inf\left\{ \int \varphi\, d\mu : \mu \text{ shift invariant probability} \right\}$$

- A *minimizing measure* μ_{min} is a shift invariant probability measure satisfying

$$\int \varphi\, d\mu_{min} = F_\infty(\varphi) \tag{2}$$

Theorem (Obvious)

- $F_\beta(\varphi) \to F_\infty(\varphi)$
- *Any accumulation point of $(\mu_\beta)_{\beta\geqslant 0}$ is a minimizing measure that maximizes the entropy of all minimizing measures.*

I.7. The set K of ground configurations

Observation

- Assume $X := \{\varphi = \min \varphi\}$ is shift invariant, then
  - ⋆ any weak limit of $\mu_\beta \to \mu_{min}$ satisfies $\mathrm{supp}(\mu_{min}) \subseteq X$
  - ⋆ but is it is not true in general that any invariant measure supported on X is a candidate to be a limit of a Gibbs measure.
- If in addition X has a unique invariant measure μ_{min}, then $\mathscr{G}_\beta(\varphi) \to \{\mu_{min}\}$ (no chaotic convergence)
- In general $\varphi^{-1}(\min \varphi)$ is not invariant. The existence or not of a chaotic convergence will depend strongly on the complexity of

$$\mathrm{Mather}(\varphi) = \bigcup \Big\{ \mathrm{supp}(\mu) : \ \mu \text{ is minimizing: } \int \varphi \, d\mu = F_\infty(\varphi) \ \Big\}$$

Hypothesis (a possible set of minimal hypotheses)

- $\varphi : \Sigma^d(\mathscr{A}) \to \{0, 1\}$ has finite range and depends on a finite number of coordinates: we say φ *is locally finite*
- X = Mather is a *computable subshift* (or effectively closed subshift)

I.8. Subshift of finite type

Definition

- A function $\varphi : \Sigma^d(\mathscr{A}) \to \{0,1\}$ is said to be *locally finite* if there exists $D \geqslant 1$ such that, for every $x, y \in \Sigma^d(\mathscr{A})$

$$x_{[\![1,D]\!]^d} = y_{[\![1,D]\!]^d} \quad \Rightarrow \varphi(x) = \varphi(y).$$

- A *subshift* X is a closed shift invariant subspace of $\Sigma^d(\mathscr{A})$ that is defined by a countable set $\mathscr{F}$ of *forbidden patterns (words)*
  - ⋆ $\mathscr{F} \subseteq \bigsqcup_{n \geqslant 1} \mathscr{A}^{[\![1,n]\!]^d}$
  - ⋆ $X = \Sigma^d(\mathscr{A}, \mathscr{F}) := \{x \in \Sigma^d(\mathscr{A}) : \forall\, w \in \mathscr{F},\ w \not\subset x\}$
- A subshift X is *computable* if $\mathscr{F}$ is enumerated by a *Turing machine* $\mathbb{M}$ by increasing size
- X is of *finite type* if there exists $D \geqslant 1$ such that $\mathscr{F} \subseteq \mathscr{A}^{[\![1,D]\!]^d}$

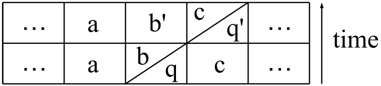
Definition

A *Turing machine* is given by $(\mathscr{A}, \{\sharp\}, \mathscr{Q}, \delta)$ where

- $\mathscr{A}$ is a finite alphabet
- $\{\sharp\}$ is an extra symbol
- $\mathscr{Q} = \{q_1, \ldots, q_n\} \sqcup \{q_{\mathtt{ini}}, q_{\mathtt{fin}}\}$
- $\delta : (\mathscr{A} \sqcup \{\sharp\}) \times \mathscr{Q} \to (\mathscr{A} \sqcup \{\sharp\}) \times \mathscr{Q} \times \{+, -\}$ (a transition function)
- an infinite ribbon where finite words of the form

$$(\cdots, \sharp, \sharp, w_1, \ldots, w_n, \sharp, \sharp, \cdots)$$

In a schematic way:

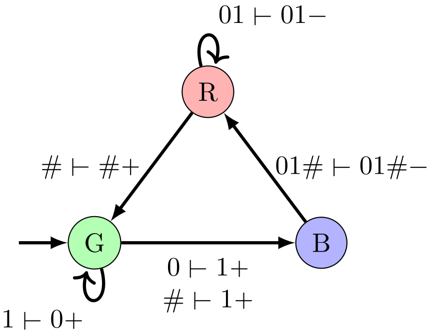


$$\delta(b, q) = (b', q', +)$$

I.10. Turing machine

#	0	0	1	#
#	0	0	#	#
#	0	1	#	#
#	1	1	#	#
#	1	1	#	#
#	1	1	#	#
#	1	1	#	#
#	0	1	#	#
#	0	1	#	#
#	0	1	#	#
#	0	1	#	#
#	0	1	#	#
#	0	#	#	#
#	1	#	#	#
#	1	#	#	#
#	1	#	#	#
#	1	#	#	#
#	0	#	#	#

Example: odometer



The notations $\boxed{a \vdash a'+}$ means the head reads a, then writes a', and moves forward.

$\Longrightarrow$ **A Turing machine is equivalent to a tiling defined by a finite number of local constraints**

I.11. Conclusion

- We want to understand whether a system is chaotic at zero temperature: do there exists a subsequence $\beta_k \to +\infty$ and $K_1, K_2 \subseteq \Sigma^d(\mathscr{A})$, compact and disjoint, such that
  - ⋆ $K_1 \bigsqcup K_2 \subseteq X$
  - ⋆ $\mathscr{G}_{\beta_{2k+1}}(\varphi) \to$ measures supported on K_1
  - ⋆ $\mathscr{G}_{\beta_{2k}}(\varphi) \to$ measures supported on K_2
- we want an example of potential φ as simple as possible:

$$\varphi = \mathbb{1}_{[\mathscr{F}]}, \quad \mathscr{F} \subseteq \mathscr{A}^{[\![1,D]\!]^d}, \quad [\mathscr{F}] \text{ denotes a cylinder}$$

⋆ we verify that ground configurations do exist

$$X = \Sigma^d(\mathscr{A}, \mathscr{F}) = \{x \in \Sigma^d(\mathscr{A} : \varphi \circ \sigma^k(x) = 0, \ \forall k \in \mathbb{Z}^d\} \boxed{\neq \emptyset}$$

obviously the ground energy and ground measures satisfy

$$F_\infty(\varphi) = 0 \ \text{ and } \ \mathrm{supp}(\mu_{min}) \subseteq X = \mathrm{Mather}(\varphi)$$

⋆ we want to work in dimension $d = 2$: our main result is an extension of Chazottes-Hochman (2010) in $d \geqslant 3$

Summary

II.1. Transfer matrix method

Assumption $d = 1$, $\mathscr{A} = \{1, 2, \cdots, n\}$, $\varphi : \mathscr{A}^{\mathbb{Z}} \to \mathbb{R}$ is locally finite

$$\varphi(x) = \varphi(i_0, i_1), \quad \forall\, x = (\ldots, i_{-1} \mid i_0, i_1, \ldots) \in \mathscr{A}^{\mathbb{Z}} \tag{1}$$

Lemma (Transfer method)

Gibbs measures are built using the following procedure

- $M_\beta(i,j) = e^{-\beta\varphi(i,j)}$
- $\lambda_\beta =$ *the largest eigenvalue of* $M_\beta = (M_\beta(i,j))_{1 \leqslant i,j \leqslant n}$
- L_β, R_β *are the left and right eigenvectors*
- *normalization:* $\sum_{i=1}^{n} L_\beta(i) R_\beta(i) = 1$
- *the unique Gibbs measure at temperature* β^{-1} *is*

$$\mu_\beta([i_0, \ldots, i_n]) = \frac{L_\beta(i_0) R_\beta(i_n)}{\lambda_\beta^n} \exp\Big(-\beta \sum_{k=1}^{n} \varphi(i_{k-1}, i_k) \Big) \tag{2}$$

II.2. Minimizing cycles

Notations $\varphi : \mathscr{A}^{\mathbb{Z}} \to \mathbb{R}$ has the form: $\varphi(x) = \varphi(i_0, i_1)$.

- The ground level energy

$$F_\infty(\varphi) = \inf \Big\{ \int \varphi \, d\mu : \mu \text{ shift invariant probability } \Big\} \quad (1)$$

- A minimizing cycle is a τ-periodic path $(i_0, i_1, \ldots, i_{\tau-1})$ such that

$$\frac{1}{\tau} \sum_{k=0}^{\tau-1} \varphi(i_k, i_{k+1}) = F_\infty(\varphi) \quad (2)$$

- $\mathscr{A}^{[\![1,2]\!]} \backslash \mathscr{F} = \{$ accepted transitions $\}$

$$\mathscr{A}^{[\![1,2]\!]} \backslash \mathscr{F} = \{ i \to j : \ i \to j \text{ belongs to a minimizing cycle} \}$$

Theorem[Obvious]

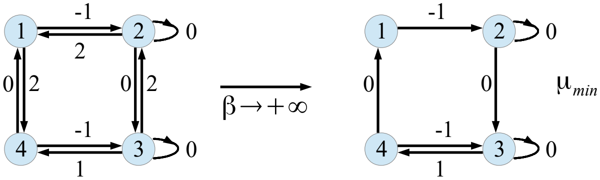
$$F_\infty(\varphi) = \lim_{n \to +\infty} \inf_{x \in \mathscr{A}^{\mathbb{Z}}} \frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ \sigma^k(x) \quad (3)$$

$$\text{Mather}(\varphi) = \bigcup \Big\{ \text{supp}(\mu) : \ \mu \text{ is minimizing } \Big\} = \Sigma^1(\mathscr{A}, \mathscr{F}) \quad (4)$$

II.3. Example of convergence

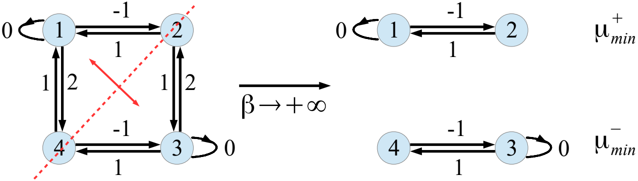
Example 1: $\mathscr{A} = \{1, 2, 3, 4\}$, $\varphi(1, 2) = -1$, $\varphi(2, 2) = 0$

$\mu_\beta \to \mu_{min}$ (1 measure of maximal entropy)



Example 2:

$\mu_\beta \to \frac{1}{2}(\mu^+_{min} + \mu^-_{min})$ (2 measures of maximal entropy)



II.4. A exact result of convergence

Theorem (Brémont (2003))

Let $d = 1$, $\varphi : \Sigma^1(\mathscr{A}) \to \mathbb{R}$ be a locally finite potential. Then

(1) $\lim_{\beta \to +\infty} \mu_\beta = \mu^*_{min}$ *(without taking a subsequence)*

(2) μ^*_{min} *is a minimizing measure (possibly non ergodic)*

(3) $\mathrm{Ent}(\mu^*_{min}) = \sup\{\mathrm{Ent}(\mu) : \ \mu$ *is minimizing* $\}$

(4) μ^*_{min} *is a barycenter of minimizing measures of maximal entropy supported on disjoint SFTs. (The coefficients of the barycenter are algebraic numbers).*

Remark

(1) The proof uses tools in semi-algebraic theory.

(2) The set Mather $:= \bigcup\{\mathrm{supp}(\mu) : \ \mu$ is minimizing $\}$ that supports all minimizing measures has a simple description: a subshift of finite type.

II.5. Examples of chaotic convergence

If the set Mather has a large complexity, one may expect a chaotic convergence at zero temperature.

Theorem (Chazottes, Hochman (2010))

There exists an invariant compact set $K \subset \Sigma^1(\mathscr{A})$ such that the potential $\varphi(x) = d(x, K)$ is chaotic at zero temperature.

But the set of minimizing measures can be as simple as possible.

Theorem (Garibaldi, Bissacot, T. (2018))

There exists a Lipschitz potential $\varphi : \{0,1\}^{\mathbb{Z}} \to [0, +\infty)$ such that

- δ_{0^∞} *and* δ_{1^∞} *are the only two ergodic minimizing measures*
- φ *is chaotic at zero temperature*
- *one defines an energy barrier $h : \Sigma^1 \times \Sigma^1 \to [0, +\infty)$ and in order to have chaotic convergence, we must have*

$$h(0^\infty, 1^\infty) = h(1^\infty, 0^\infty) = 0$$

(See also Coronel, Rivera-Letelier (2015) for the same results.)

Summary

III.1. Main result

Theorem (Barbieri, Bissacot, Dalle Vedove, T. (2022))

There exists a locally finite potential $\varphi : \Sigma^2(\mathscr{A}) \to \mathbb{R}$ that is chaotic at zero temperature.

Remark

- Chazottes-Hochman (2010) proved the previous result for $d \geqslant 3$
- Simultaneously to us, Chazottes-Shinoda (2021), extended their previous result to $d = 2$. Their proof is different: improved Kleene fixed-point theorem by Durand-Romaschenko-Shen (2012).

Theorem (Gayral, Sablik, Taati (2023))

If $\mathscr{K}$ is a finite simplex of periodic measures (or Π_2-computable simplex), there exists $\varphi : \Sigma^2(\mathscr{A}) \to \mathbb{R}$ locally finite such that

- $\mathrm{diam}(\mathscr{G}_\beta(\varphi)) \to 0$
- *any $\mu \in \mathscr{K}$ is an accumulation point of a choice of $(\mu_{\beta\varphi})_{\beta \to +\infty}$*

III.2. General strategy

- Find a set of forbidden patterns $\mathscr{F} \subset \mathscr{A}^{[\![1,D]\!]^2}$ of size $D \geqslant 1$,
- Define the potential $\varphi = \mathbb{1}_{[\mathscr{F}]}$
- Make sure that

$$X = \Sigma^2(\mathscr{A},\mathscr{F}) = \{x \in \Sigma^2(\mathscr{A}) : \varphi \circ \sigma^k(x) = 0, \ \forall\, k \in \mathbb{Z}^2\} \neq \varnothing$$

- Find a special cooling sequence $\beta_k \to +\infty$
- Take any Gibbs measure

$$\int \beta_k \varphi \, d\mu_{\beta_k} - \mathrm{Ent}(\mu_{\beta_k}) = \inf \Big\{ \int \beta_k \varphi \, d\mu - \mathrm{Ent}(\mu) : \ \mu \text{ is invariant } \Big\}$$

- Show that

$$\mu_{\beta_{2k+1}} \to \mu^1_{min}, \quad \mu_{\beta_{2k}} \to \mu^2_{min}, \quad \mu^1_{min} \neq \mu^2_{min}$$

- Notice that

$$\mathrm{supp}(\mu^1_{min}),\ \mathrm{supp}(\mu^2_{min}) \subset X$$

II.3. Step 1/6

We construct a *1d set of forbidden words* with alternating complexity

- $\tilde{\mathscr{A}} = \{0,1,2\}$, 0 is a marker,
- $\mathscr{A}^{(1)} = \{0,1\}, \quad \mathscr{A}^{(2)} = \{0,2\}$
- construct inductively two languages of words of length ℓ_k

$$\mathscr{A}_k^{(1)} = \{1^{\ell_k}, a_k^{(1)}\}, \quad \mathscr{A}_k^{(2)} = \{2^{\ell_k}, a_k^{(2)}\}$$

- choose a finite set $\mathscr{F}_k$ such that

$$\tilde{X}_k := \Sigma(\mathscr{A}, \mathscr{F}_k) = \begin{cases} \text{bi-infinite configurations obtained as} \\ \text{concatenation of words in } \mathscr{A}_k^{(1)} \text{ and } \mathscr{A}_k^{(2)} \end{cases}$$

- construct similarly

$$\tilde{X}_k^{(1)} := \Sigma^1(\mathscr{A}, \mathscr{F}_k^{(1)}) \text{ and } \tilde{X}_k^{(2)} := \Sigma^1(\mathscr{A}, \mathscr{F}_k^{(2)})$$

- $\tilde{\mathscr{F}} = \bigsqcup \mathscr{F}_k$, the *first subshift* we construct

$$\tilde{X} = \Sigma^1(\tilde{\mathscr{A}}, \tilde{\mathscr{F}}) = \bigcap_{k \geqslant 0} \tilde{X}_k$$

II.4. Step 2/6

- we summarize $\tilde X = \bigcap_{k\geqslant 0} \downarrow \tilde X_k$

$$\tilde X_k \supset \tilde X_k^{(1)} \sqcup \tilde X_k^{(2)}, \quad \tilde X_k^{(1)} \subset \{0,1\}^{\mathbb{Z}},\ \tilde X_k^{(2)} \subset \{0,2\}^{\mathbb{Z}}$$

- impose an alternating complexity

$$\begin{cases} \text{for } k \text{ even,} & \text{Complexity}(\tilde X_k^{(1)}) \ll \text{Complexity}(\tilde X_k^{(2)}) \\ \text{for } k \text{ odd,} & \text{Complexity}(\tilde X_k^{(2)}) \ll \text{Complexity}(\tilde X_k^{(1)}) \end{cases}$$

- start with $a_0^{(1)} = 01$, $a_0^{(2)} = 02$, build

$$\mathscr{A}_0^{(1)} = \{11, a_0^{(1)}\}, \quad \mathscr{A}_0^{(2)} = \{22, a_0^{(2)}\},$$

- build by induction $\mathscr{A}_{k+1}^{(1)} = \{1^{\ell_k}, a_{k+1}^{(1)}\}$, assume k even

$$a_{k+1}^{(1)} = \underbrace{a_k^{(1)} a_k^{(1)} \cdots a_k^{(1)}}_{N_k \text{ times}}, \qquad a_{k+1}^{(2)} = a_k^{(2)} \underbrace{2^{\ell_k} \cdots\cdots 2^{\ell_k}}_{(N_k-2) \text{ times}} a_k^{(2)}$$

- assume k is odd, permute (1) and (2)

II.5. Step 3/6

- assume k even

$$a_{k+1}^{(1)} = \underbrace{a_k^{(1)} a_k^{(1)} \cdots a_k^{(1)}}_{N_k \text{ times}}, \qquad a_{k+1}^{(2)} = a_k^{(2)} \underbrace{2^{\ell_k} \cdots\cdots 2^{\ell_k}}_{(N_k-2)\text{ times}} a_k^{(2)}$$

- define $f_k^{(i)}$ to be the frequency of 0 in the word $a_k^{(i)}$ (recall: $a_k^{(1)} \in \{0,1\}^*$, $a_k^{(2)} \in \{0,2\}^*$)

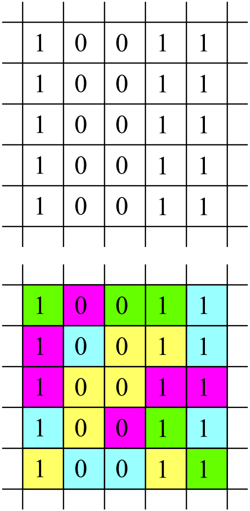
$$\begin{cases} \text{for } k \text{ even}, & f_k^{(1)} \ll f_k^{(2)} \\ \text{for } k \text{ odd}, & f_k^{(2)} \ll f_k^{(1)} \end{cases}$$

- the complexity could have been measured by the entropy (but as we will see, entropy is not the right notion)

Observation

$\mathscr{F}$ can be constructed *recursively* (provided $(N_k)_{k\geqslant 0}$ is also recursive). That is, there exists a Turing machine $\mathbb{M}$ that enumerates the words of $\mathscr{F}$ by increasing size and polynomial time enumeration $T_{\mathbb{M}}$ function;

II.6. Step 4/6



- Embed a 1d subshift into a 2d subshit by vertically repeating the symbols

$$\tilde{\tilde{X}} = \Sigma^2(\tilde{\mathscr{A}}, \tilde{\tilde{\mathscr{F}}})$$

- notice

$$\text{Ent}(\tilde{\tilde{X}}) = 0$$

- Extend the 2d subshift by adding additional colors

$$\Pi : \hat{X} \to \tilde{\tilde{X}}$$

- Find a finite set of local constraints between the colors and the digits so that the vertically aligned subshift $\tilde{\tilde{X}}$ is revealed by erasing the colors

II.7. Step 5/6

Theorem (Aubrun, Sablik (2013))

Let $\tilde{\mathscr{F}}$ be a 1d computable set of forbidden words on the alphabet $\tilde{\mathscr{A}}$. Let $\tilde{X}$ the corresponding subshift and $\tilde{\tilde{X}}$ the vertically aligned extended subshift. Then $\tilde{\tilde{X}}$ is sofic.

- *One can decorate the original symbols:* $\hat{\mathscr{A}} = \tilde{\mathscr{A}} \times \mathscr{B}$
- *There exists $D \geqslant 1$ and $\hat{\mathscr{F}} \subset \hat{\mathscr{A}}^{[\![1,D]\!]^2}$ such that $\hat{X} := \Sigma^2(\hat{\mathscr{A}}, \hat{\mathscr{F}})$ is a shift equivariant extension of $\tilde{\tilde{X}}$:*
  - ⋆ *There exists a commuting diagram*

$$\begin{array}{ccc} \hat{X} & \xrightarrow{\ \sigma\ } & \hat{X} \\ \downarrow \Pi & & \downarrow \Pi \\ \tilde{\tilde{X}} & \xrightarrow{\ \sigma\ } & \tilde{\tilde{X}} \end{array}$$

  - ⋆ Π *is surjective and is defined by erasing the decorations, by using a one-bloc factor map $\pi : \hat{\mathscr{A}} \to \tilde{\mathscr{A}}$, (the first projection)*

II.8. Step 6/6

The entropy of Aubrun-Sablik has also zero entropy. For the purpose of the rest of the proof we need alternating subshifts of large and small entropies.

- Duplicate the symbol 0

$$\hat{\mathscr{A}} = \{0,1,2\} \times \mathscr{B} \;\rightarrow\; \mathscr{A} = \{0',0'',1,2\} \times \mathscr{B}$$

- Duplicate the forbidden words

$$\hat{\mathscr{F}} \;\rightarrow\; \mathscr{F} \subset \mathscr{A}^{[\![1,D]\!]^2}$$

- We constructed successively

$$\tilde{X} \leftarrow \tilde{\tilde{X}} \leftarrow \hat{X} \leftarrow X$$

$$\tilde{X} = \bigcap_{k \geqslant 0} \tilde{X}_k, \;\; \tilde{X}_k \supset \tilde{X}_k^{(1)} \bigsqcup \tilde{X}_k^{(2)}$$

1	0''	0''	1	1
1	0'	0''	1	1
1	0'	0'	1	1
1	0'	0''	1	1
1	0''	0'	1	1

Entropy estimate: $\quad \mathrm{Ent}(X_k^{(1)}) = \log(2) f_k^{(1)}$

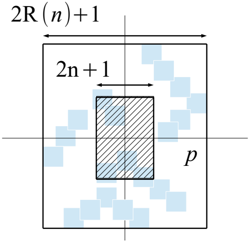
II.9. Our contribution

Our contribution in that problem is to put in light two estimates that where needed in Chazottes-Hochman proof.

Definition (Shift reconstruction function)

Let $\mathscr{F}$ be a set of forbidden patterns and $X = \Sigma^2(\mathscr{A}, \mathscr{F})$ be the corresponding subshift.

- A finite pattern $p \in \mathscr{A}^{[\![1,n]\!]^2}$ is *locally admissible* if no forbidden pattern appears in p.
- A finite pattern is *globally admissible* if it is a sub-pattern of an (infinite) configuration $x \in X$



- The reconstruction function is the function that associates for every $n \geqslant 1$, the smallest size $R \geqslant n$ such that if $p_{[\![-R(n),R(n)]\!]^2}$ is any locally admissible pattern, then $p_{[\![-n,n]\!]^2}$ is globally admissible.

II.10. Reconstruction function

Proposition

Let $\tilde{\mathscr{F}}$ be a 1d computable set of forbidden patterns. Assume

- The time enumeration function $T_{\tilde{\mathscr{F}}} = \mathscr{O}(P(n)|\mathscr{A}|^n)$
- The reconstruction function $R_{\tilde{\mathscr{F}}}(n) = \mathscr{O}(n)$

Then the Aubrun-Sablik extension $\hat{X} = \Sigma^2(\hat{\mathscr{A}}, \hat{\mathscr{F}})$ satisfies

$$\limsup_{n\to+\infty} \frac{1}{n} \log R_{\hat{\mathscr{F}}}(n) < +\infty$$

Notice that we don't say that the growth of the reconstruction function is computable. Actually any a priori growth would be enough for the rest of the proof.

Remark

The set $\tilde{\mathscr{F}}$ constructed before satisfies the hypothesis of the lemma

II.11. Relative complexity function

Definition (Relative complexity)

Let $\Pi : \hat{X} = \Sigma^2(\hat{\mathscr{A}}, \hat{\mathscr{F}}) \to X = \Sigma^2(\mathscr{A}, \mathscr{F})$ be an extension with a one-bloc factor map $\pi : \hat{\mathscr{A}} \to \mathscr{A}$. For every $n \geqslant 1$, for every globally admissible pattern $p \in \mathscr{A}^{[\![1,n]\!]^2}$, let $\mathscr{L}(n,p)$ be the set of globally admissible patterns $\hat{p} \in \hat{\mathscr{A}}^{[\![1,n]\!]^2}$ that project onto p.

The relative complexity function is

$$C_{\hat{\mathscr{F}}}(n) := \sup_p \operatorname{card}(\mathscr{L}(n,p))$$

The Aubrun-Sablik extension is more than a zero-entropy extension.

Proposition

The Aubrun-Sablik extension satisfies

$$\limsup_{n\to+\infty} \frac{1}{n} \log C_{\hat{\mathscr{F}}}(n) < +\infty.$$

Summary

- I. Position of the problem
- II. Some old results in dimension 1
- III. A new result in dimension 2
- IV. **Ideas of the proof**

II.1. Construction of a 1d subshift

We construct a 1d subshift containing two subshifts with alternating complexity. We start by construction a decreasing sequence of SFTs $(\tilde{X}_k)$ obtained by concatenating 4 words of equal length ℓ_k in two different alphabets $\tilde{\mathscr{A}}_k^{(1)}$ and $\tilde{\mathscr{A}}_k^{(2)}$

$$\tilde{\mathscr{A}}_k^{(1)} = \{a_k^{(1)}, 1^{\ell_k}\}, \quad \tilde{\mathscr{A}}_k^{(2)} = \{a_k^{(2)}, 2^{\ell_k}\},$$
$$a_k^{(1)} \in \{0,1\}^{[\![1,\ell_k]\!]}, \quad a_k^{(2)} \in \{0,2\}^{[\![1,\ell_k]\!]}.$$

The final 1d subshift is obtained by

$$\tilde{X} = \bigcap_{k \geqslant 0} \tilde{X}_k, \quad \tilde{X}_k = \langle \tilde{\mathscr{A}}_k^{(1)}, \tilde{\mathscr{A}}_k^{(2)} \rangle.$$

At each scale ℓ_k it contains two SFTs

$$\tilde{X}_k^{(1)} \bigsqcup \tilde{X}_k^{(2)} \subseteq \tilde{X}_k, \quad \tilde{X}_k^{(1)} = \langle \tilde{\mathscr{A}}_k^{(1)} \rangle, \quad \tilde{X}_k^{(2)} = \langle \tilde{\mathscr{A}}_k^{(2)} \rangle.$$

II.2. Alternating complexity

We will assume that the complexity alternates between $\tilde{X}_k^{(1)}$ and $\tilde{X}_k^{(2)}$:

- for k odd: complexity$(\tilde{X}_k^{(1)}) \gg$ complexity$(\tilde{X}_k^{(2)})$,
- for k even: complexity$(\tilde{X}_k^{(1)}) \ll$ complexity$(\tilde{X}_k^{(2)})$.

The complexity of each SFT is measured by counting the frequency of the symbol 0 in the words $a_k^{(1)}$ or $a_k^{(2)}$. Let $f_k^{(1)}$ be the frequency in the word $a_k^{(1)}$.

As seen later entropy is not the correct notion at the 1d level.

II.3. Explicite construction

We construct by induction $\tilde{\mathscr{A}}_k^{(1)}$ and $\tilde{\mathscr{A}}_k^{(2)}$.

- for k odd ($a_k^{(1)}$ contains many zeros)

$$a_{k+1}^{(1)} = a_k^{(1)} \underbrace{1^{\ell_k} \cdots\cdots 1^{\ell_k}}_{(N_k-2) \text{ times}} a_k^{(1)}, \quad a_{k+1}^{(2)} = \underbrace{a_k^{(2)} \cdots\cdots a_k^{(2)}}_{N_k \text{ times}}$$

- for k even ($a_k^{(2)}$ contains many zeros)

$$a_{k+1}^{(1)} = \underbrace{a_k^{(1)} \cdots\cdots a_k^{(1)}}_{N_k \text{ times}}, \quad a_{k+1}^{(2)} = a_k^{(2)} \underbrace{2^{\ell_k} \cdots\cdots 2^{\ell_k}}_{(N_k-2) \text{ times}} a_k^{(2)}.$$

II.4. Recursiveness of the construction

Assume k even. Denote $\rho_k^{(1)} = \ell_k f_k^{(1)}$, $\rho_k^{(2)} = \ell_k f_k^{(2)}$.

- $N'_k := \left\lceil \dfrac{2k\rho_{k-1}^{(1)}}{\rho_{k-1}^{(2)}} \right\rceil, \; \ell'_k = N'_k \ell_{k-1},$
- $\beta_k := \left\lceil \dfrac{\ell_{k-1}^2 2^{k\ell'_k}}{(\rho_{k-1}^{(2)})^2} \right\rceil,$
- $N_k := N'_k \left\lceil \dfrac{k\beta_k}{N'_k \rho_{k-1}^{(2)}} \right\rceil, \; \ell_k = N_k \ell_{k-1},$
- $\rho_k^{(1)} = 2\rho_{k-1}^{(1)}, \; \rho_k^{(2)} = N_k \rho_{k-1}^{(2)},$

II.5. Aubrun-Sablik extension

Let $\tilde{\mathscr{F}}$ be the 1d set of forbidden words and $\tilde{X} = \Sigma^1(\tilde{\mathscr{A}}, \tilde{\mathscr{F}})$, $\tilde{\mathscr{A}} = \{0, 1, 2\}$. $\tilde{X} \in \tilde{\mathscr{A}}^{\mathbb{Z}}$ is an effective subshift.
Let $\tilde{\tilde{X}} \in \tilde{\mathscr{A}}^{\mathbb{Z}^2}$ be the vertically align subshift replicating $\tilde{X}$

Theorem (Aubrun Sablik)

There exist a finite alphabet $\widehat{\mathscr{A}} = \tilde{\mathscr{A}} \times \mathscr{B}$, a finite size $D \geqslant 1$, a finite set of forbidden patterns $\widehat{\mathscr{F}} \subseteq \widehat{\mathscr{A}}^{[\![1,D]\!]^2}$, and a SFT $\widehat{X} = \Sigma^2(\widehat{\mathscr{A}}, \widehat{\mathscr{F}})$ such that $(\widehat{X}, \hat{\sigma})$ is an extension of $(\tilde{\tilde{X}}, \tilde{\tilde{\sigma}})$. More precisely

$$\begin{array}{ccc} \hat{X} & \xrightarrow{\hat{\sigma}} & \hat{X} \\ \Big\downarrow \hat{\Pi} & & \Big\downarrow \hat{\Pi} \\ \tilde{\tilde{X}} & \xrightarrow{\tilde{\tilde{\sigma}}} & \tilde{\tilde{X}} \end{array}$$

where Π is surjective and a one-bloc factor that erases the symbols in $\mathscr{B}$

II.6. Duplicated SFT

Let $\mathscr{A} = \{0', 0'', 1, 2\} \times \mathscr{B}$ and $\mathscr{F} = \{p \in \mathscr{A}^{[\![1,D]\!]^2} : p_{0',0'' \to 0} \in \widehat{\mathscr{F}}\}$, and t $X = \Sigma^2(\mathscr{A}, \mathscr{F})$ and

$$\begin{array}{ccc} X & \xrightarrow{\ \sigma\ } & X \\ \downarrow{\scriptstyle \Pi} & & \downarrow{\scriptstyle \Pi} \\ \tilde{\tilde{X}} & \xrightarrow{\ \tilde{\tilde{\sigma}}\ } & \tilde{\tilde{X}} \end{array}$$

X has the same structure

$$X = \bigcap X_k, \quad X_{k+1} \subseteq X_k$$

$$X_k = \langle L_k \rangle, \quad L_k = \mathscr{L}(X, \ell_k)$$

$$L_k^{(1)} = \{p \in L_k : \Pi(p) \in \langle \tilde{X}_k^{(1)} \rangle\}, \quad L_k^{(2)} = \{p \in L_k : \Pi(p) \in \langle \tilde{X}_k^{(2)} \rangle\}$$

$$X \supseteq X_k^{(1)} \bigsqcup X_k^{(2)}, \quad X_k^{(1)} = \langle L_k^{(1)} \rangle$$

II.7. Gibbs measures

The short range potential

$$\varphi = \mathbb{1}_{[\mathscr{F}]}, \quad X = \left\{x \in \mathscr{A}^{\mathbb{Z}^2} : \varphi \circ \sigma^k(x) = 0, \ \forall\, k \in \mathbb{Z}^2\right\}$$

Contrary to the 1d case, $\varphi(x) = d(x, X)$ is not the correct function.

A gibbs measure at temperature β^{-1} is

$$\text{Press}(\beta\varphi) := \sup_{\mu} \left\{ \text{Ent}(\mu) - \beta \int \varphi d\mu \right\} = \text{Ent}(\mu_\beta) - \beta \int \varphi d\mu_\beta$$

II.8. General strategy

Assume k even

- bound from below using a measure $\mu_k^{(2)}$ of large entropy of $X_k^{(2)}$

$$\text{Press}(\beta_k\varphi) \geqslant \ln(2) f_k^{(2)} - 2D\frac{\beta_k}{\ell_k} = \ln(2) f_k^{(2)} + o(f_k^{(2)})$$

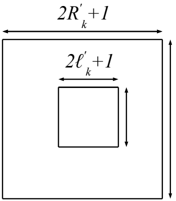
- bound from above using the Gibbs measure

$$\text{Press}(\beta_k\varphi) \leqslant \mu_{\beta_k}(\Sigma^2 \backslash [1]) \ln(2) f_k^{(2)} + o(f_k^{(2)})$$

Conclusion: $\mu_{\beta_k}([1]) \to 0$

II.9. Equivalent notion of a distance to X

We fix an intermediate scale $\ell_{k-1} < \ell'_k < \ell_k$, $\ell'_k = N'_k \ell_{k-1}$. Let R'_k be the reconstruction length:



Let $M'_k \subseteq \mathscr{A}^{2[\![-R'_k, R'_k]\!]}$ be the set of locally admissible patterns. Then

$$\mu_{\beta_k}(\Sigma^2 \backslash [M'_k]) \leqslant \frac{{R'_k}^2}{\beta_k} \ln \mathrm{card}(\mathscr{A})$$

II.10. The final two bounds

Let C'_k be the cardinal of patterns $p \in \tilde{\mathscr{A}}^{[\![1,\ell'_k]\!]^2}$ in the Aubrun-Sablik extension that projects to a fixed word at scale ℓ'_k

A lower bound:

- $\text{Press}(\beta_k \varphi) \geqslant \ln(2) f_k^{(2)} - 2D \dfrac{\beta_k}{\ell_k}$

An upper bound:

- $$\text{Press}(\beta_k \varphi) \leqslant \text{cte}\Big[\frac{{R'_k}^2}{\beta_k} + \frac{1}{{\ell'_k}^2}\ln(C'_k) + \frac{1}{N'_k} f_{k-1}^{(1)}\Big] \\ + \mu_{\beta_k}(\Sigma^2 \backslash [1]) \ln(2) f_k^{(2)}$$

We need two estimates

$$\limsup_{k\to+\infty} \frac{1}{\ell'_k}\ln(C'_k) < +\infty, \quad \limsup_{k\to+\infty} \frac{1}{\ell'_k}\ln(R'_k) < +\infty$$

Thank you

Bibliography

- S. Barbieri, R. Bissacot, G. Dalle Vedove, Ph. Thieullen. Chaos in bidimensional models with short-range. Preprint (2022).
- J.-R. Chazottes, M. Hochman On the Zero-Temperature Limit of Gibbs States, *Commun. Math. Phys.*, Vol. 297, No. 1 (2010), 265–281.
- J.-R. Chazottes, and M. Shinoda. On the absence of zero-temperature limit of equilibrium states for finite-range interactions on the lattice $\mathbb{Z}^2$. Preprint (2020).
- L. Gayral, M. Sablik, S. Taati. Characterisation of the Set of Ground States of Uniformly Chaotic Finite-Range Lattice Models. Preprint (2023).