Periodicity of joint co-tiles in $\mathbb{Z}^d$

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Multidimensional symbolic dynamics and lattice models of quasicrystals

Joint work with Tom Meyerovitch and Shrey Sanadhya

Tilings of $\mathbb{Z}^d$

- A *tile* in $\mathbb{Z}^d$ is a finite set $F \Subset \mathbb{Z}^d$.
- For a set $A \subseteq \mathbb{Z}^d$ we denote by

$$F \oplus A = \biguplus_{a \in A} F + a,$$

the disjoint union of translations of F by the elements of A.
- We say the *F tiles $\mathbb{Z}^d$* (by translations) if there exists an $A \subseteq \mathbb{Z}^d$ that satisfies $F \oplus A = \mathbb{Z}^d$ (no rotations or reflections of F are allowed).

  Equivalently, if every $n \in \mathbb{Z}^d$ can be uniquely represented as $n = f + a$, where $f \in F$ and $a \in A$.
- Such an $A \subseteq \mathbb{Z}^d$, that satisfies $F \oplus A = \mathbb{Z}^d$, is called a *co-tile* of F.

Settings and Definitions - an example

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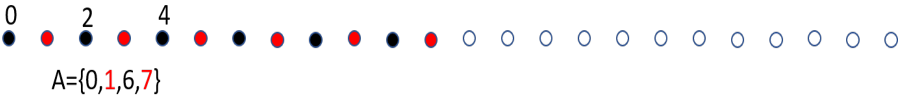

Settings and Definitions - an example

$A=\{0,1,6,7,12,13,18,19\ldots\}$

The Periodic Tiling Conjecture

- A set $A \subseteq \mathbb{Z}^d$ is called *periodic* (or d-periodic) if there exists a finite index subgroup $L \leq \mathbb{Z}^d$ such that $A + L = A$.

- The periodic tiling conjecture (PTC) in $\mathbb{Z}^d$ - Lagarias, Wang '96:
  If a finite set $F \Subset \mathbb{Z}^d$ tiles $\mathbb{Z}^d$ then it tiles periodically.

  In other words: if there exists an $A \subseteq \mathbb{Z}^d$ so that $F \oplus A = \mathbb{Z}^d$ then there exists a **periodic set** $A' \subseteq \mathbb{Z}^d$ that satisfies $F \oplus A' = \mathbb{Z}^d$.

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Greenfeld-Tao Counterexample

Theorem (Greenfeld-Tao, '22)

There exists $d \in \mathbb{N}$ and a finite set $F \Subset \mathbb{Z}^d$ that tiles $\mathbb{Z}^d$, such that F does not admit a periodic co-tile.

- In fact, Greenfeld and Tao showed that there exists a tile $F \Subset \mathbb{Z}^2 \times G$, for a certain finite abelian group G, that tiles, but cannot tile periodically. Then, they used a *lifting procedure* that shows that one can lift a counterexample in any quotient of $\mathbb{Z}^d$ to $\mathbb{Z}^d$.
- What about the PTC for small values of d?

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- What about the PTC for small values of d?

PTC in $\mathbb{Z}^d$ - $d = 1$

Theorem (Newman, ’77)

Every tiling of $\mathbb{Z}$ is periodic.

- That is: If $F \oplus A = \mathbb{Z}$, then A must be periodic.
- Thus, a tiling of $\mathbb{Z}$ by F contains a tiling of an interval.

Note that:

- Classifying the tiles $F \Subset \mathbb{Z}$ that tiles $\mathbb{Z}$ is not an easy question.
- In $'99$, Coven and Meyerowitz gave explicit sufficient conditions on a finite set $F \Subset \mathbb{Z}$ to tile $\mathbb{Z}$.
- It has been conjectured that these conditions are also necessary.
- Łaba and Londner have proved the necessity of these conditions in some specific cases.

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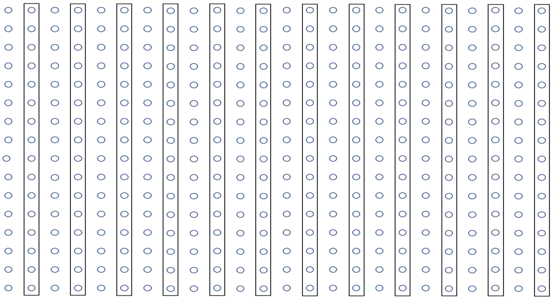
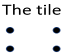
PTC in $\mathbb{Z}^d$ - $d = 2$

- In $\mathbb{Z}^2$ it is not difficult to find examples of tiles F that tiles non-periodically, and there are examples that are not even 1-periodic.





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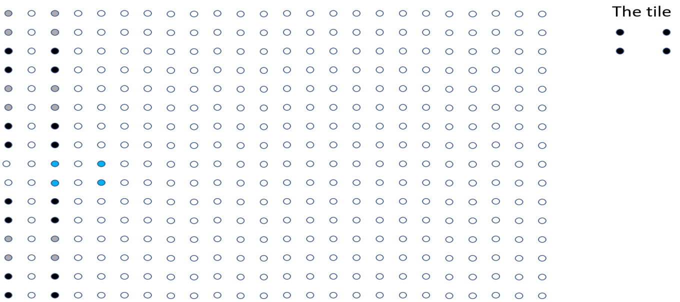

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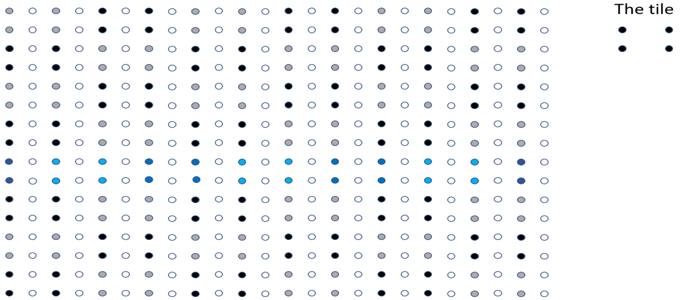

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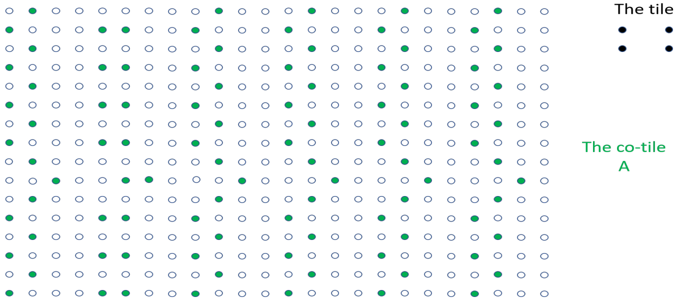

Theorem (Bhattacharya, '20)

The PTC holds in $\mathbb{Z}^2$:
If $F \in \mathbb{Z}^2$ admits a co-tile, then F admits a periodic co-tile.

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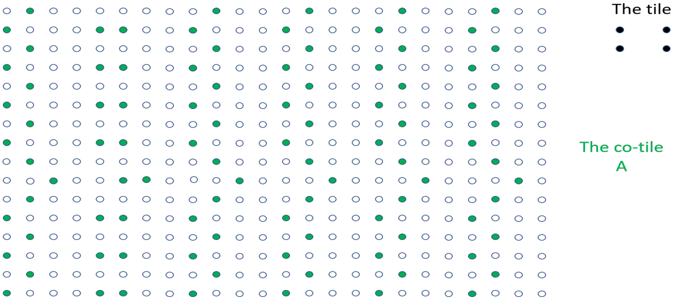

Theorem (Bhattacharya, '20)

The PTC holds in $\mathbb{Z}^2$:
If $F \Subset \mathbb{Z}^2$ admits a co-tile, then F admits a periodic co-tile.

PTC in $\mathbb{Z}^d$ - partial summary

- PTC in $\mathbb{Z}^d$:
  If a finite set $F \Subset \mathbb{Z}^d$ tiles $\mathbb{Z}^d$ then it tiles periodically.

d=1 - Newman's Theorem:

Every tiling of $\mathbb{Z}$ is periodic.

d=2 - Bhattacharyas' Theorem:

The PTC holds in $\mathbb{Z}^2$.

Huge d - Greenfeld-Tao's Theorem:

There exists $d \in \mathbb{N}$ for which the PTC fails.

- Let's see how Newman's Theorem and Bhattacharya's Theorem can still be generalized to any d.

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New setup

We propose a slightly different setup, and two new statements, such that:

1. Statement 1 coincide with Newman's theorem in $\mathbb{Z}$, and generalize it for every $d \geq 1$.
2. Statement 2 coincide with Bhattacharya's Theorem in $\mathbb{Z}^2$, and generalize it for every $d \geq 2$.
3. The conditions that we offer in the new statements are also necessary.

Our setup and results - *independent tiles*

- **Definition:** Given a k-tuple of finite sets $(F_1, \ldots, F_k)$ in $\mathbb{Z}^d$, we say that $A \subseteq \mathbb{Z}^d$ is a *joint co-tile* of $F_1, \ldots, F_k$ if

$$\text{For all} \quad 1 \leq j \leq k : \quad F_j \oplus A = \mathbb{Z}^d.$$

- **Definition:** A k-tuple of finite sets $(F_1, \ldots, F_k)$ in $\mathbb{Z}^d$ is called *independent* if $0 \in F_j$ for every $1 \leq j \leq k$, and for any choice of $v_1 \in F_1 \smallsetminus \{0\}, \ldots, v_k \in F_k \smallsetminus \{0\}$ the k-tuple $(v_1, \ldots, v_k)$ is linearly independent (say over $\mathbb{R}$).
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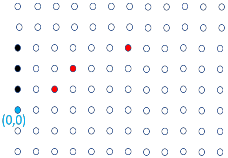

$F_1 = \{(0,0), (0,1), (0,2), (0,3)\}$

$F_2 = \{(0,0), (2,1), (3,2), (6,3)\}$

For $A = \mathbb{Z} \times 4\mathbb{Z}$ we have

$$F_1 \oplus A = \mathbb{Z}^2$$
$$F_2 \oplus A = \mathbb{Z}^2$$

and (F_1, F_2) independent

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Recall: Newman's Theorem

Every co-tile in $\mathbb{Z}$ is periodic.

Theorem 1 (Meyerovitch, Sanadhya, S., '24)

Every joint co-tile of d independent tiles in $\mathbb{Z}^d$ is periodic.

Our setup and results - *property* $(\star)$

- **Definition:** Let $(F_1, \ldots, F_{d-1})$ be a $(d-1)$-tuple of finite sets in $\mathbb{Z}^d$. We say that $(F_1, \ldots, F_{d-1})$ *has property* $(\star)$ if $(F_1, \ldots, F_{d-1})$ is an independent tuple and for every $(v_1, \ldots, v_{d-1}), (u_1, \ldots, u_{d-1}) \in (F_1 \setminus \{0\}) \times \ldots \times (F_{d-1} \setminus \{0\})$, if

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then $v_i = u_i$ for all $1 \leq i \leq d-2$.

- Note that property $(\star)$ is void for $d = 2$.

Recall: Bhattacharya's Theorem:

Suppose that $F \Subset \mathbb{Z}^2$ admits a co-tile $A \subseteq \mathbb{Z}^2$, then it admits a periodic co-tile $A' \subseteq \mathbb{Z}^2$.

Theorem 2 (Meyerovitch, Sanadhya, S., '24)

Suppose that $(F_1, \ldots, F_{d-1})$ in $\mathbb{Z}^d$ has property $(\star)$. If $(F_1, \ldots, F_{d-1})$ has a joint co-tile $A \subseteq \mathbb{Z}^d$, then it admits a periodic joint co-tile $A' \subseteq \mathbb{Z}^d$.

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- Furthermore, we show that our independence and $(\star)$ conditions are also necessary, in the following sense:

Theorem 3 (Meyerovitch, Sanadhya, S., '24)

Suppose that $F \Subset \mathbb{Z}^d$ admits a periodic co-tile $A \subset \mathbb{Z}^d$. Then there exist ("sister tiles") $F_1, \ldots, F_{d-1} \Subset \mathbb{Z}^d$, such that for every $1 \leq j \leq d-1$ we have $F_j \oplus A = \mathbb{Z}^d$, and

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