Universality in dynamical systems

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Goal of the talk

- To introduce the notion of universal dynamical systems.
- Some examples of universal $\mathbb{Z}^d$-systems, $d \geq 1$.
- Specification "type" property.
- Universality for $\mathbb{R}^d$-flows, $d \geq 1$.
- Sketch of a proof in $\mathbb{R}^d$ setting.

Embedding

- (Y, S) - a topological dynamical system and (X, μ, T) - probability, ergodic, measure preserving system.

- An **embedding** of (X, μ, T) into (Y, S) is a measurable map

$$\phi : X \to Y$$

s.t. $\exists\ X_0 \subseteq X$, $\mu(X_0) = 1$, ϕ restricted to X_0 is an injection and $\phi(T(x)) = S(\phi(x))$ for $x \in X_0$.

Universality

- We say a topological dynamical system (Y, S) is **universal** if every probability, ergodic, measure preserving system (X, μ, T) with $h_\mu(T) < h(S)$ can be embedded into (Y, S).

- Krieger's generator theorem implies that full shift on finite alphabet is **universal**.

Why care for universal dynamical systems

- Extension of Krieger's generator theorem.
  - Krieger showed that mixing $\mathbb{Z}$-subshifts of finite type are universal.
- Measure theoretic isomorphism.
  - (Y, S) a topological dynamical system and $\mathcal{M}(Y, S)$ be set of all S invariant probability measures.
  - Question : For a given ergodic measure preserving (X, μ, T), when can we "model" it in measure theoretic fashion by (Y, S, ν), for some $\nu \in \mathcal{M}(Y, S)$.
  - Variation principal provides the necessary condition i.e. $h_\mu(T) \leq h(S)$.
  - Universality of (Y, S) provides the sufficient condition.

Some Results

- Lind-Thouvenot (1977) proved that hyperbolic (two-dimensional) toral automorphisms are ergodic universal.

- Quas and Soo (2016) extended Lind-Thouvenot result to non-hyperbolic case.

- Robinson and Şahin (2001) proved that any $\mathbb{Z}^d$ subshift of finite type that satisfies a specification property (called the *uniform filling property*) and having dense periodic points is ergodic universal.

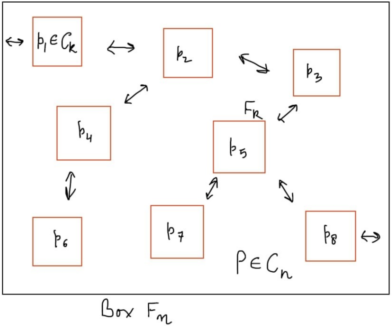
Flexible sequence of patterns

- Let $\mathcal{A}$ be a finite alphabet and $X \subset \mathcal{A}^{\mathbb{Z}^d}$ a subshift.
- $F_n = \{-n, ..., n\}^d \subset \mathbb{Z}^d$ is a box (shape) in $\mathbb{Z}^d$.
- The set of patterns of shape F_n in $\mathbb{Z}^d$ subshift $X \subset \mathcal{A}^{\mathbb{Z}^d}$,

$$\mathcal{L}(X, F_n) = \{x \mid_{F_n}: \ x \in X\} \subset \mathcal{A}^{F_n}.$$

- A sequence $\mathcal{C} = (C_n)_{n=1}^{\infty}$ where $C_n \subset \mathcal{L}(X, F_n)$ is a **flexible sequence of patterns** if given $k, n \in \mathbb{N}$ with $k << n$
    - sufficiently spaced smaller Boxes $F_k \subset F_n$ with patterns from C_k can be extended to a pattern in C_n.

Flexible sequence of patterns



- Examples of flexible $\mathbb{Z}^d$-systems: Domino tilings of $\mathbb{Z}^d$, coprime box tilings of $\mathbb{Z}^d$.
- Entropy of flexible sequence $h(\mathcal{C}) = \limsup_{n\to\infty} \frac{1}{|F_n|} \log |C_n|$.
- $h(\mathcal{C})$ is bounded above by topological entropy of the subshift.

Universality for $\mathbb{Z}^d$-systems

- Theorem (Chandgotia, Meyerovitch 2021): Let (X, S) be a topological $\mathbb{Z}^d$ dynamical system that admits a flexible sequence of patterns $\mathcal{C}$ with entropy $h(\mathcal{C})$. Then any free ergodic measure preserving $\mathbb{Z}^d$ dynamical system having entropy less than $h(\mathcal{C})$ measurably embeds in (X, S).

- As a consequence C-M showed
  - Domino tilings in $\mathbb{Z}^2$ is universal.
  - Coprime box tilings of $\mathbb{Z}^d$ is h-universal for some $h > 0$

Universality of flows

- Theorem (Ambrose 1942): Every free, ergodic measure preserving $\mathbb{R}$-flow of a probability space is isomorphic (measurably) to a *flow built under a function*.

- Theorem (Rudolph's two step coding, 1976): Given any irrational $\alpha > 0$, we can choose the ceiling function to take only two values 1 and $1 + \alpha$.

Universality of flows

- In other words every free, ergodic measure preserving $\mathbb{R}$-flow of a probability space is isomorphic to translation action of $\mathbb{R}$ on the space of tilings of $\mathbb{R}$ by two tiles.

- Theorem (Rudolph 1988): Every free, ergodic measure preserving $\mathbb{R}^d$-flow of a probability space can be **factored** on a translation action of $\mathbb{R}^d$ on the space of tilings of $\mathbb{R}^d$ by 2^d rectangular tiles.

Tiling of $\mathbb{R}^d$

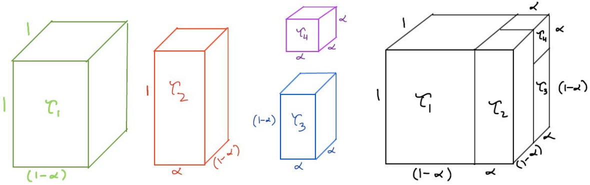
- Let $\mathcal{T} = \{\tau_1, \tau_2, \ldots, \tau_k\}$ a finite collection of d-dimensional tiles.
- A $\mathcal{T}$-*tiling* of $\mathbb{R}^d$ is a covering of $\mathbb{R}^d$ by translates of tiles from $\mathcal{T}$ such that they only overlap on boundaries of the tiles.
- $Y_{\mathcal{T}}$ - set of all tilings of $\mathbb{R}^d$ by $\mathcal{T}$. Let $S = \{S_{\vec{v}}\}_{\vec{v} \in \mathbb{R}^d}$ - translation action of $\mathbb{R}^d$ on $Y_{\mathcal{T}}$.
- Thus $(Y_{\mathcal{T}}, S)$ forms a dynamical system.
- Theorem (Kra, Quas, Şahin 2015): Any free, ergodic and measure preserving $\mathbb{R}^d$ action can be factored on an $\mathbb{R}^d$ tiling dynamical system with $d+1$ rectangular tiles.

The $d+1$ rectangular tiling in KQS 2015

- Let $d \in \mathbb{N}$ and $\alpha \in (0,1)$ irrational.
- Set of basic tiles $\mathcal{T}_\alpha = \{\tau_1, \dots, \tau_{d+1}\}$ where,

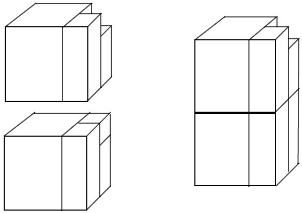
$$\begin{aligned} \tau_i &= [0,\alpha)^{i-1} \times [0,1-\alpha) \times [0,1)^{d-i} \qquad \text{for } i < d \\ \tau_d &= [0,\alpha)^{d-1} \times [0,1-\alpha) \qquad \text{and} \\ \tau_{d+1} &= [0,\alpha)^d. \end{aligned}$$

- $(Y_{\mathcal{T}_\alpha}, S)$-the corresponding $\mathbb{R}^d$ dynamical system. We call it the $\boldsymbol{\alpha}$-**tiling dynamical system**. Figure below shows basic tiles for $d = 3$.



Periodic tiling by the supertile

- Supertile $\tau^* \subset \mathbb{R}^d$ - putting two copies of tiling of unit cube on top of each other and removing the corner tile.



- There exists a periodic tiling of $\mathbb{R}^d$ by the supertile (KQS-'15, see also Stein 1990). That is, there exists an invertible $d \times d$ matrix A such that $\tau^* \oplus A\mathbb{Z}^d = \mathbb{R}^d$.
- Following KQS-'15 we will call the periodic tiling, **notched cube tiling** and denote it by $\mathcal{N}$.

Ergodic Universality of $\mathbb{R}^d$ flows

- Theorem (Meyerovitch-S '24): Let (X, S) be a topological $\mathbb{R}^d$ dynamical system that admits a flexible sequence of patterns $\mathcal{C}$ with entropy $h(\mathcal{C})$. Then any free ergodic measure preserving $\mathbb{R}^d$ dynamical system having entropy less than $h(\mathcal{C})$ measurably embeds in (X, S).

- Corollary : The α-tiling dynamical system admits a flexible sequence of patterns. Hence it is h-universal for some $h > 0$.

Brief discussion of r-entropy (Feldman 1980, Ornstein-Weiss 1987)

- (X, μ, T) is a probability measure preserving $\mathbb{R}^d$-flow. $\mathcal{P}$ a finite measurable partition of X and $F_n = [-n, n]^d \subset \mathbb{R}^d$, $n \in \mathbb{R}^+$.

- For $r \in (0, 1)$, a family $\mathcal{B}$ of disjoint sets in X is called a $(\boldsymbol{\mathcal{P}, n, r})$**-family** if : $x, y \in B \in \mathcal{B}$ implies

$$d_n^{\mathcal{P}}(x, y) = \frac{1}{|F_n|} \int_{F_n} 1_{\{\vec{t} \in F_n : \mathcal{P}(T^{\vec{t}}(x)) \neq \mathcal{P}(T^{\vec{t}}(y))\}} d\vec{t} \leq r$$

r-Entropy cont.

- Define $h_r(T, \mathcal{P})$, the ***r*-entropy** as follows :

$$= \sup_{\epsilon>0} \overline{\lim_{n\to\infty}} \inf\left\{ \frac{H_\mu(\mathcal{B})}{|F_n|} : \mathcal{B} \text{ a } (\mathcal{P}, n, r) \text{ family with } \mu(\bigcup \mathcal{B}) > 1-\epsilon \right\}$$

where $H_\mu(\mathcal{B}) := -\sum_{B\in\mathcal{B}} \mu(B)\log(\mu(B))$.

- Theorem (Feldman 1980): $h_r(T, \mathcal{P})$ is monotonic non-increasing function of r and

$$\lim_{r\to 0} h_r(T, \mathcal{P}) = h_\mu(T, \mathcal{P}).$$

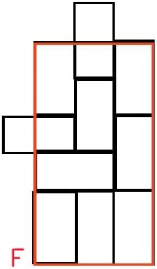
Proof Sketch: α-tilings admit flexible sequence of patterns

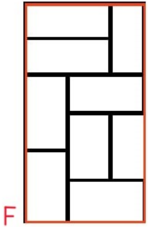
Perfect tiling of a shape by the basic tiles.

- $\mathcal{T}$ - set of basic tiles and $Y_{\mathcal{T}}$ - space of all tilings of $\mathbb{R}^d$ by $\mathcal{T}$.
- Let $F \subset \mathbb{R}^d$ be a shape and $\mathcal{L}(Y_{\mathcal{T}},, F)$ be the set of F-patterns

$$\mathcal{L}(Y_{\mathcal{T}},, F) = \{y\,|_F\colon\ y \in Y_{\mathcal{T}}\}$$

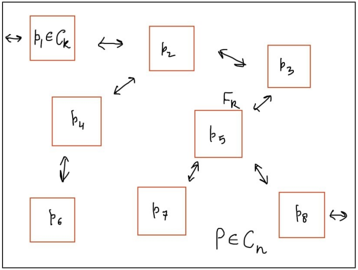
- A pattern $p \in \mathcal{L}(Y_{\mathcal{T}},, F)$ perfectly tiles F if it consists of full basic tiles.

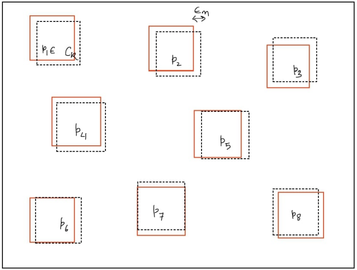




Proof Sketch

Flexible sequence of patterns revisited.





Steps in the proof sketch

The α-tiling dynamical system admits a flexible sequence of patterns. Steps 1-3 appears in KQS 2015.

- Step 1: There exists a periodic tiling by the supertile.
- Step 2: Moving down a shape tiled by supertile by a unit does not change the pattern in the shape.
- Step 3: Utilize compactness of supertile to approximate any vector in $\mathbb{R}^d$ in finitely many down shifts.
- Step 4: Define flexible sequence of patterns appropriately and utilize step 2 and 3.

Proof Sketch: Step 1. (KQS 2015)

Supertile τ^* tiles $\mathbb{R}^d$ periodically.

- There exists $d \times d$-invertible matrix A such that τ^* is the fundamental domain for translation action of $A\mathbb{Z}^d$ on $\mathbb{R}^d$ ($\tau^* \oplus A\mathbb{Z}^d = \mathbb{R}^d$).
  - Vol $(\tau^*) = 2 - \alpha^d =$ det $(A) =$ Vol of $A(0,1]^d$.
  - $(\tau^* + A\vec{v}) \cap \tau^* = \emptyset$ for all $v \in \mathbb{Z}^d \setminus \{0\}$.
- Denote the periodic tiling by $\mathcal{N}$

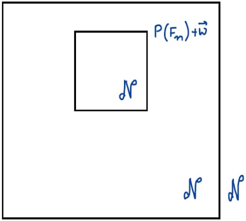
The matrix A for $\alpha \in (0,1)$.

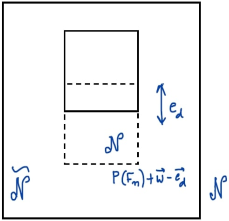
$$A = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 & 1-\alpha \\ -\alpha & 1 & 0 & \cdots & 0 & 1-\alpha \\ 0 & -\alpha & 1 & 0 & \cdots & 1-\alpha \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & \cdots & 0 & -\alpha & 1 & 1-\alpha \\ 0 & \cdots & & 0 & -\alpha & 2-\alpha \end{bmatrix}$$

Proof Sketch: Step 2. (KQS 2015)

Moving down a shape tiled by supertile by a unit does not change the pattern in the shape.

- For $F_n \subset \mathbb{Z}^d$, define **Grid shape** $P(F_n) = \bigcup_{\vec{v} \in F_n} (\tau^* + A\vec{v}) \subset \mathbb{R}^d$.
- Grid shapes $P(F_n)$ are perfectly tiled by τ^*.
- Moving down grid shapes located at some $\vec{w} \in A\mathbb{Z}^d$ maintains the pattern.





Proof Sketch: Step 3. (KQS 2015)

Utilize compactness of supertile to approximate any vector in $\mathbb{R}^d$ in finitely many steps.

- Choose appropriate α such that
  - Multiples of $A^{-1}\vec{e_d}$ are dense in $\mathbb{R}^d/\mathbb{Z}^d$.
  - Thus multiples of $\vec{e_d}$ are dense in $\mathbb{R}^d/A\mathbb{Z}^d$ (hence in τ^*).
- Since τ^* is compact, for every $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that any $\vec{v} \in \mathbb{R}^d$ (mod $A\mathbb{Z}^d$) can be approximated by $n.\vec{e_d}$ for $0 \leq n \leq K$.
- In other words every $\vec{v} \in \mathbb{R}^d$ there exists $\vec{w} \in A\mathbb{Z}^d$ and $0 \leq n \leq K$ such that $|\vec{v} - (\vec{w} - n\,.\vec{e_d})| < \epsilon$.

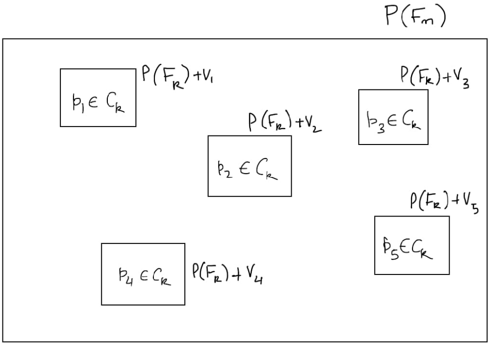
Proof Sketch: Step 4.

Define flexible sequence of patterns appropriately and utilize step 2 and 3.

- Define flexible sequence

$$C_n = \text{Set of all perfect tilings of the grid shape } P(F_n).$$

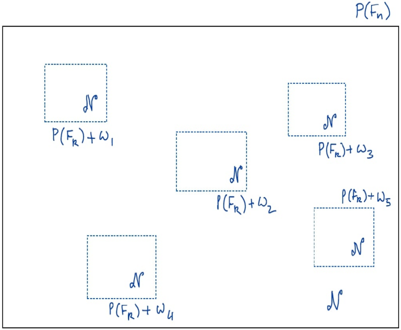
- Goal : Extending smaller patterns to larger patterns

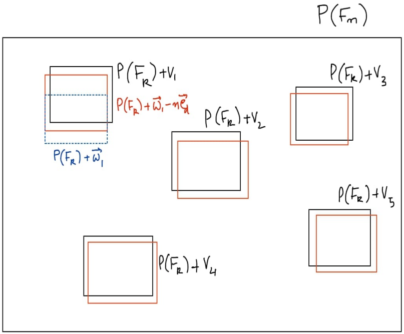


For each $\vec{v_i} \in \mathbb{R}^d$, there exists $\vec{w_i} \in A\mathbb{Z}^d$ such that $|\vec{v_i} - (\vec{w_i} - n.\vec{e_d})| < \epsilon_n$ for a uniformly bounded n.

Step 4.

Starting with $\mathcal{N}$ (tiling by τ^*) and sliding the smaller grid patches finitely many times.





Red shapes are perfectly tiled by τ^*. By replacing the patterns in red shapes with $p_1, .., p_5 \in C_k$, we get flexibility.

Thank You!