Cohomology of Tiling Spaces

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3 big questions

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- How do you compute it?

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- What is it? (Much of this talk)
- How do you compute it?
  - Lots of techniques for substitution tilings,
  - Some for cut-and-project
  - Only a few for matching rules tilings
  - When in doubt, send an email to Franz Gähler.
- What is it good for? (Rest of this talk)

Outline

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What is Čech cohomology?

- Topological invariant of **continuous** spaces.
  - Doesn't work for subshifts, which are Cantor sets.
  - Works for suspensions of subshifts, in other words tilings.
- $\check{H}^k$ counts k-dimensional features. H^0 counts vertices, H^1 counts particular edges, etc.
- Plays well with inverse limits.
  - All tilings are inverse limit.
  - Cohomology pick out finite-range information about a tiling.
  - PE cohomology makes this explicit.

What is an inverse limit?

- Sequence Γ^n of topological spaces and maps $\rho_n : \Gamma^n \to \Gamma^{n-1}$.
- $$\varprojlim(\Gamma^n, \rho_n) := \{x \in \prod \Gamma^n | \forall n, x_n = \rho_{n+1}(x_{n+1})\}$$
- Note that x_n determines x_{n-1}, x_{n-2}, etc.
- Product topology. Basic open sets only control first n coordinates.
- Γ^n is called n-th *approximant* to $\varprojlim(\Gamma^n, \rho_n)$.

Examples of inverse limits

- 2-odometer $\mathbb{O}_2$. $\Gamma^n = \mathbb{Z}_{2^n}$, ρ_n is forgetful map. $\varprojlim(\Gamma, \rho)$ is not just $\mathbb{Z}$! $(1, 1, 5, 5, 21, 21, 85, 85 \ldots)$

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- 2-solenoid $\mathbb{S}_2$. $\Gamma^n = \mathbb{R}_{2^n}$, ρ_n is forgetful map. $\varprojlim(\Gamma, \rho)$ is not just $\mathbb{R}$! $(1.1, 1.1, 5.1, 5.1, 21.1, 21.1, \ldots)$

Tiling spaces are inverse limits

- Γ^n parametrizes tiling out to a distance $r_n \to \infty$.
- ρ_n is the forgetful map.
-

$$\begin{aligned} \varprojlim(\Gamma^n, \rho_n) &= \{\text{Consistent instructions for tiling}\} \\ &= \{\text{Space of tilings}\} = \Omega. \end{aligned}$$

The Anderson-Putnam (AP) complex

How can you represent Γ^0 as a geometric object? Want to place one tile at the origin.

- Pick a tile to place.
- Pick a point in that tile for location at the origin.
- Origin might be on boundary. Identify edges!
- Γ_{AP} is built from 1 copy of each tile type, with edge identifications. CW complex!

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- For Γ^n_{AP}, do same thing with collared tiles (Gähler) or supertiles (Anderson-Putnam) or other constructions.

Cohomology and inverse limits

Definition of Čech cohomology is complicated, but you only need two facts:

- On CW complexes, all cohomology theories are the same. $\check{H}^*(\Gamma^n) = H^*(\Gamma^n)$.
- $\check{H}^*(\varprojlim(\Gamma^n, \rho_n)) = \lim(\check{H}^*(\Gamma^n), \rho_n^*)$

Some simple examples

- $\mathbb{S}_2$: $\Gamma^n = \mathbb{R}_{2^n} \sim S^1$.
  $H^1(S^1) = \mathbb{Z}$, $\rho_n^* = \times 2$
  $\check{H}^1(\mathbb{S}_2) = \lim(\mathbb{Z}, \times 2) = \mathbb{Z}[1/2]$. (Identify m at stage n with $m/2^n$.
- Fibonacci tiling. Γ^n is a figure 8. $H^1(\Gamma^n) = \mathbb{Z}^2$.
  $\rho_n^* = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, which is invertible.
  $\check{H}^1(\Omega_{Fib}) = \lim(Z^2, \rho_n^*) = \mathbb{Z}^2$.

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Definitions

Decorate $\mathbb{R}^d$ with a tiling T.

- A function f on $\mathbb{R}^2$ is PE with radius R if $f(x)$ depends only on pattern of T in ball of radius R around x.
- PE $= \cup_R \{$ PE with radius R $\}$.
- Can also do PE k-forms. $d(PE)$ is PE. Look at de-Rham cohomology of complex of PE k-forms.

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- Can also do PE k-forms. $d(PE)$ is PE. Look at de-Rham cohomology of complex of PE k-forms.
- Can also do PE k-cochains (PE functions on k-cells with values in your favorite Abelian group). δ(PE) is PE. Look at cohomology of this complex.

Theorem (Kellendonk-Putnam, S-)

$H^*_{PE}(T) \simeq \check{H}^*(\Omega_T)$

Fibonacci cohomology

Substitution tiling $\sigma(a) = ab$, $\sigma(b) = a$, tilings look like

$$\ldots abaababaabaab \ldots$$

- 1-cells are tiles. 0-cells are vertices between tiles.
- $H^0_{PE} = \mathbb{Z}$, generator is constant function 1 on vertices.
- $H^1_{PE} = \mathbb{Z}^2$, generators are indicator functions i_a, i_b. Generators count a and b tiles. (i_a(interval I) = number of a's in I)
- What about counting some other pattern P? Define cochain i_P. (e.g. if $P = aaba$, define i_P(tile t) to be 1 if t is 2nd tile of $aaba$, 0 otherwise.

Regularity for counting P's

- $[i_P] = c_1[i_a] + c_2[i_b]$.
- $i_P = c_1 i_a + c_2 i_b + \delta f$, where f is PE function with radius R.
- If I is interval connecting two identical patches of size $> R$,

$$\begin{aligned} \#(P \in I) &= i_P(I) \\ &= c_1 i_a(I) + c_2 i_b(I) + f(\partial I) \\ &= c_1 \#(a \in I) + c_2 \#(b \in I) \end{aligned}$$

- $\#(a \in I)$ and $\#(b \in I)$ determine $\#(P \in I)$ **exactly**.
- Between arbitrary points, discrepancy of P is bounded by discrepancies of a and b plus boundary terms. "Exact regularity"

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Birkhoff sums

For d dimensional tilings, look at $H^d_{PE}(T)$. If rank=k, pick k patches $P_1, \ldots, P_k$ so that $\{[i_{P_j}]\}$ span $H^d \otimes \mathbb{Q}$.

- For any other patch P,

$$i_P = \sum_j c_j i_{P_j} + \delta\beta,$$

where β is PE $(d-1)$-cochain. On any region R,

$$\#(P \in R) = \sum_j c_j \#(P_j \in R) + \beta(\partial R).$$

- Discrepancies of P_j's control **all** discrepancies up to ∂ terms.
- For substitution tilings, discrepancies of P_j's controlled by action of substitution on $\check{H}^d \simeq H^d_{PE}$.
- Up to $L^{d-1}\log(L)^m$, only need substitution matrix.
- Log terms are hard. (Coronel and others)

Shape changes

You can change shape and size of tiles while preserving combinatorics.

- Shape changes mod MLD* equivalence are parametrized by $\check{H}^1(\Omega, \mathbb{R}^d)$. (Clark-S 2006). [MLD = analogue of sliding block code, MLD* also preserves tiles]
- Unlike in subshifts, not all topological conjugacies are MLD! Shape changes mod MLD* are parametrized by subgroup $\check{H}^1_{AN}(\Omega, \mathbb{R}^d)$.
- Both H^1 and H^1_{AN} are easily computed for substitution tilings.

Pisot substitutions

- For 1D (homological) Pisot substitutions, $H^1(\Omega, \mathbb{R}) = \mathbb{R} \oplus H^1_{AN}(\Omega, \mathbb{R})$.
- All shape changes are topological conjugacies up to an overall scale.
- Dynamics of self-similar tiling (e.g. Fibonacci with $|a| = \phi$, $|b| = 1$) are essentially the same as dynamics of subshift ($|a| = |b| = 1$).

Homeomorphisms

Theorem (Julien-S, 2018)

Every homeomorphism of uniquely ergodic FLC tiling spaces is homotopic to the composition of a shape change and an MLD equivalence.

Corollary

Homeomorphisms of (uniquely ergodic) FLC tiling spaces are classified by cohomology.

Rotation theory

Theorem (Aliste-Prieto, Rand, S)

Let T be a repetitive 1D FLC tiling and let $f : \Omega_T \to \Omega_T$ be PE map that is homotopic to the identity. The dynamics of f^n as $n \to \infty$ is described by a "rotation number" that lives in $H^1_{PE}(T, \mathbb{R})$.

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Behold the Hat tiling!

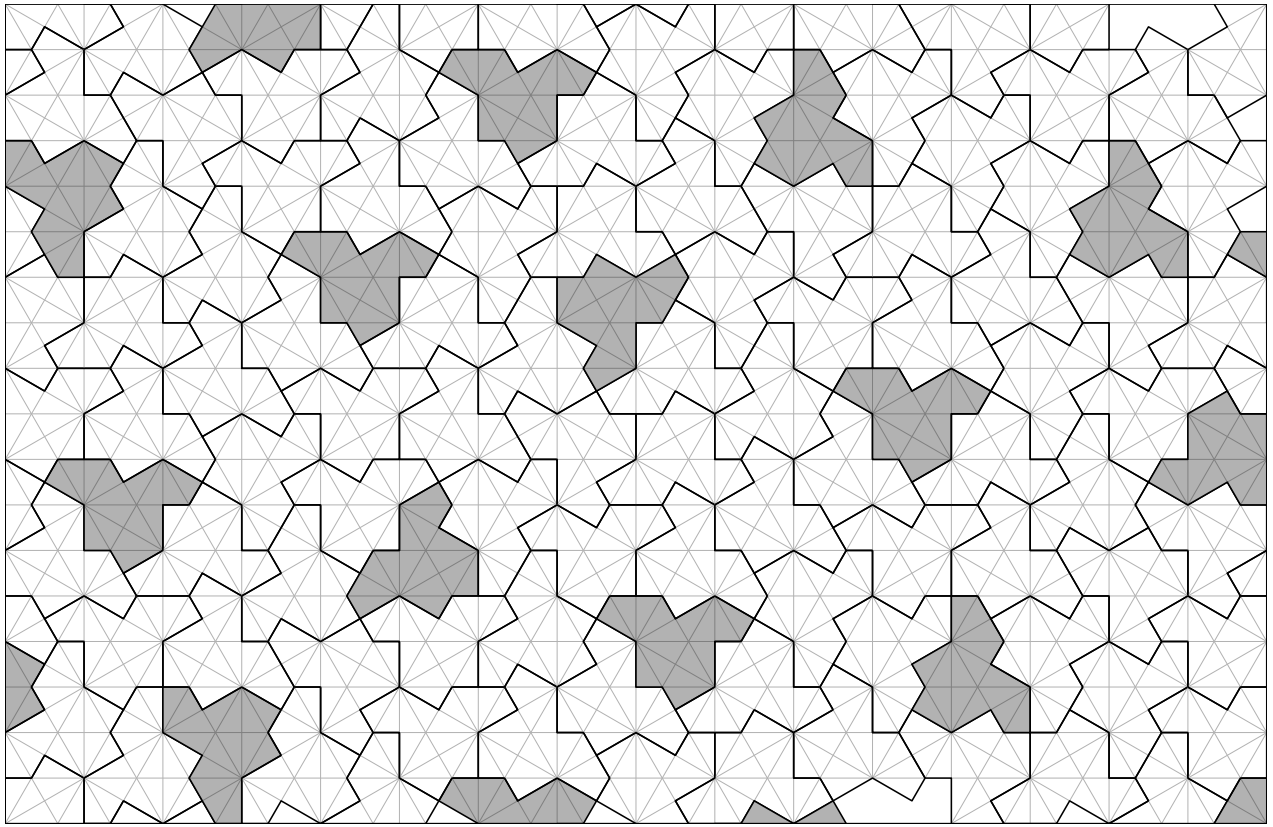
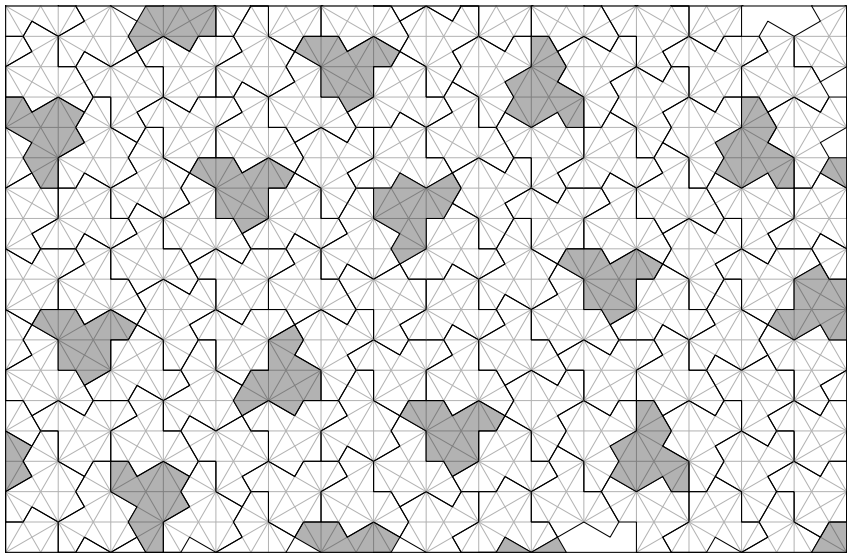


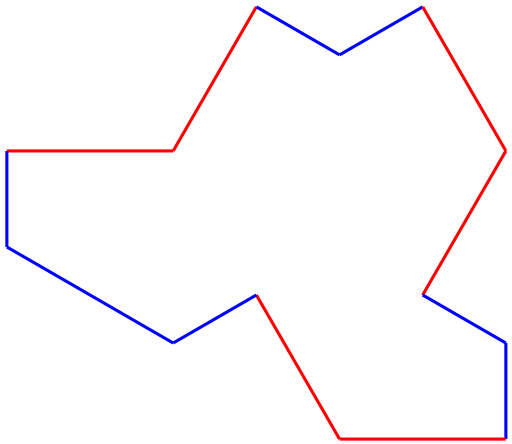
Figure: Patch of a Hat tiling. The anti-Hats are shaded.

The ingredients



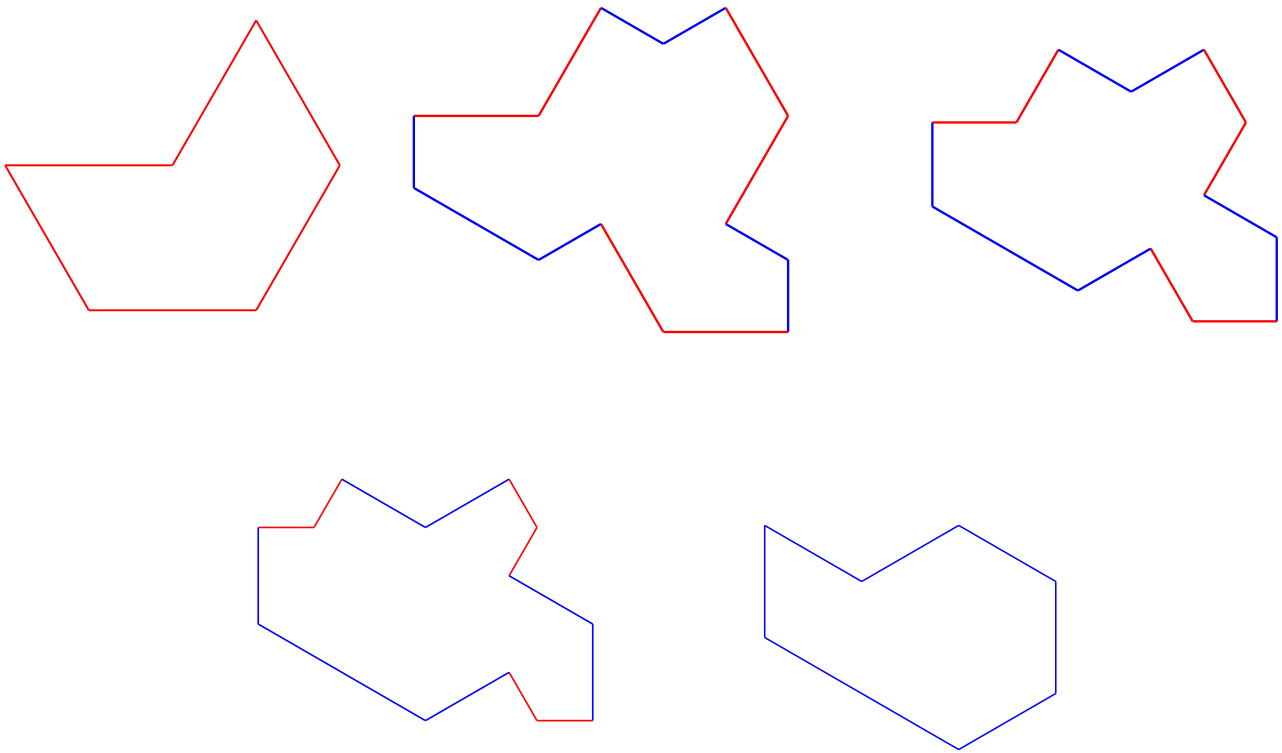
- Basic "hat" tile in 6 orientations.
- Reflected "anti-hat" in 6 orientations.
- More hats than anti-hats (or vice-versa).

Families of monotiles



- Tile(a,b) has blue edges of length a and red edges of length b.
- Need extra rules if if $a = b$, $a = 0$ or $b = 0$. (Ignore this detail)

Some family photos



Complexification

- Can extend family to include complex values of a and b.
- Tiling involves $Tile(a, b)$ and reflection of $Tile(\bar{a}, \bar{b})$.
- No longer a monotile, but still a perfectly good space of tilings.

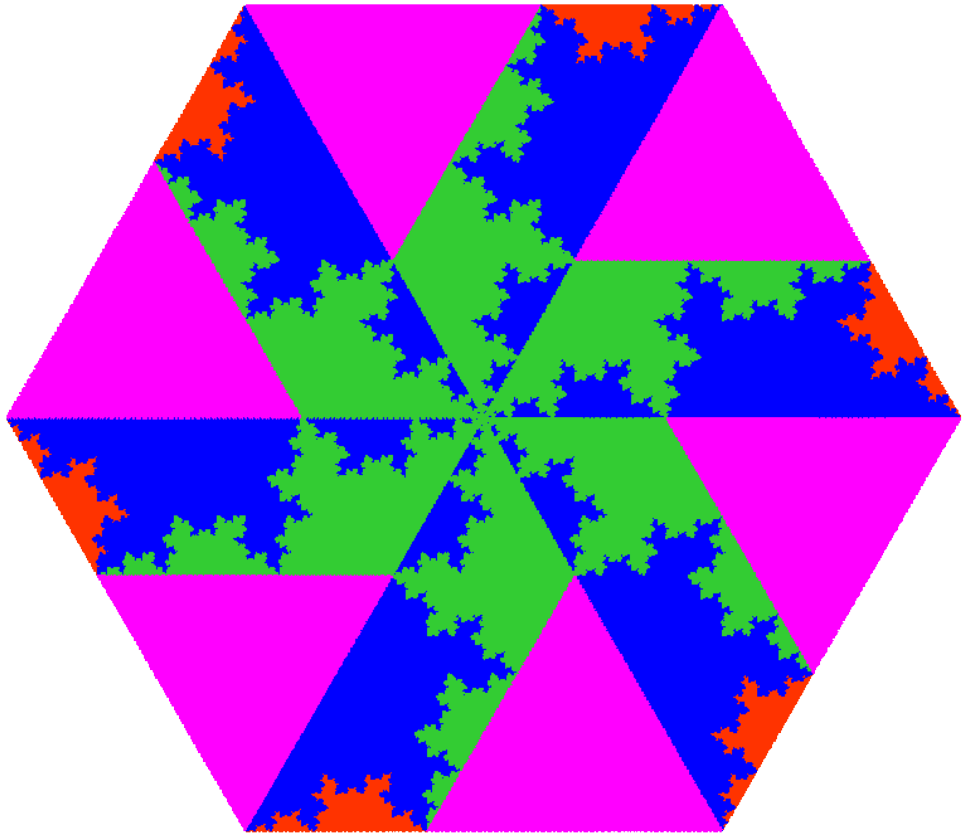
Cohomology to the rescue

- Look at $\check{H}^1(\Omega, \mathbb{R}^2) = \check{H}^1(\Omega, \mathbb{C})$.
- Calculation: $\check{H}^1(\Omega, \mathbb{C}) = \mathbb{C}^4$ and $\check{H}^1_{AN}(\Omega, \mathbb{C}) = \mathbb{C}^2$.
- All shape changes are topologically conjugacies, up to overall linear transformation. (If you preserve rotational symmetry, linear transformation is rotation plus rescaling.)
- Pick a, b so that tiling is self-similar (CAP tiling), then use toolkit for substitution tilings.

Properties of the CAP tiling

- Exactly self-similar with stretching factor ϕ^4.
- Pure point spectrum $c\mathbb{Z}[\phi, \xi]$ where $\xi = \exp(2\pi i/6)$.
- 4:2 cut and project tiling.
- Nice window.

The window



Properties of general $Tile(a, b)$ tiling

- Asymptotically self-similar with stretching factor ϕ^4.
- Pure point spectrum $c\mathbb{Z}[\phi, \xi]$ where $\xi = \exp(2\pi i/6)$.
- 4:2 cut and project tiling with **exactly the same lattice and window** as the CAP tiling.
- Only difference is projection from $\mathbb{R}^4$ to $\mathbb{R}^2$.

The Spectre tiling

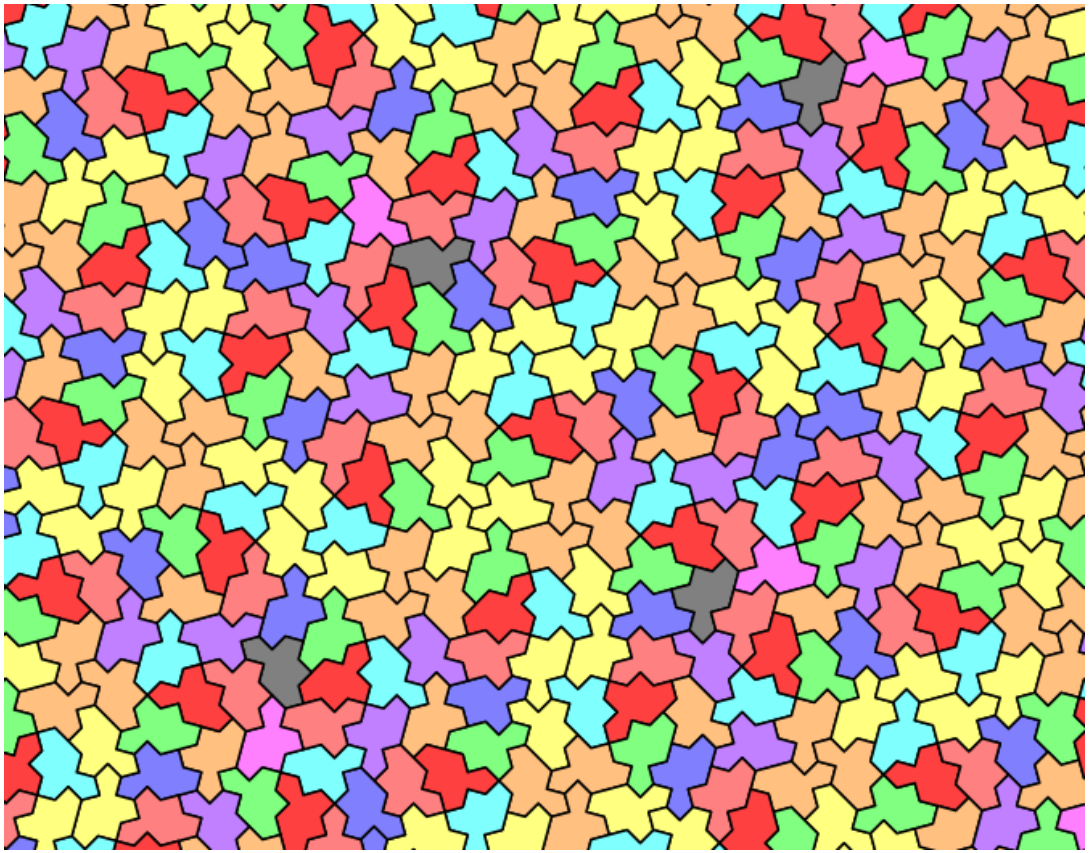


Figure: The Spectre tiling.

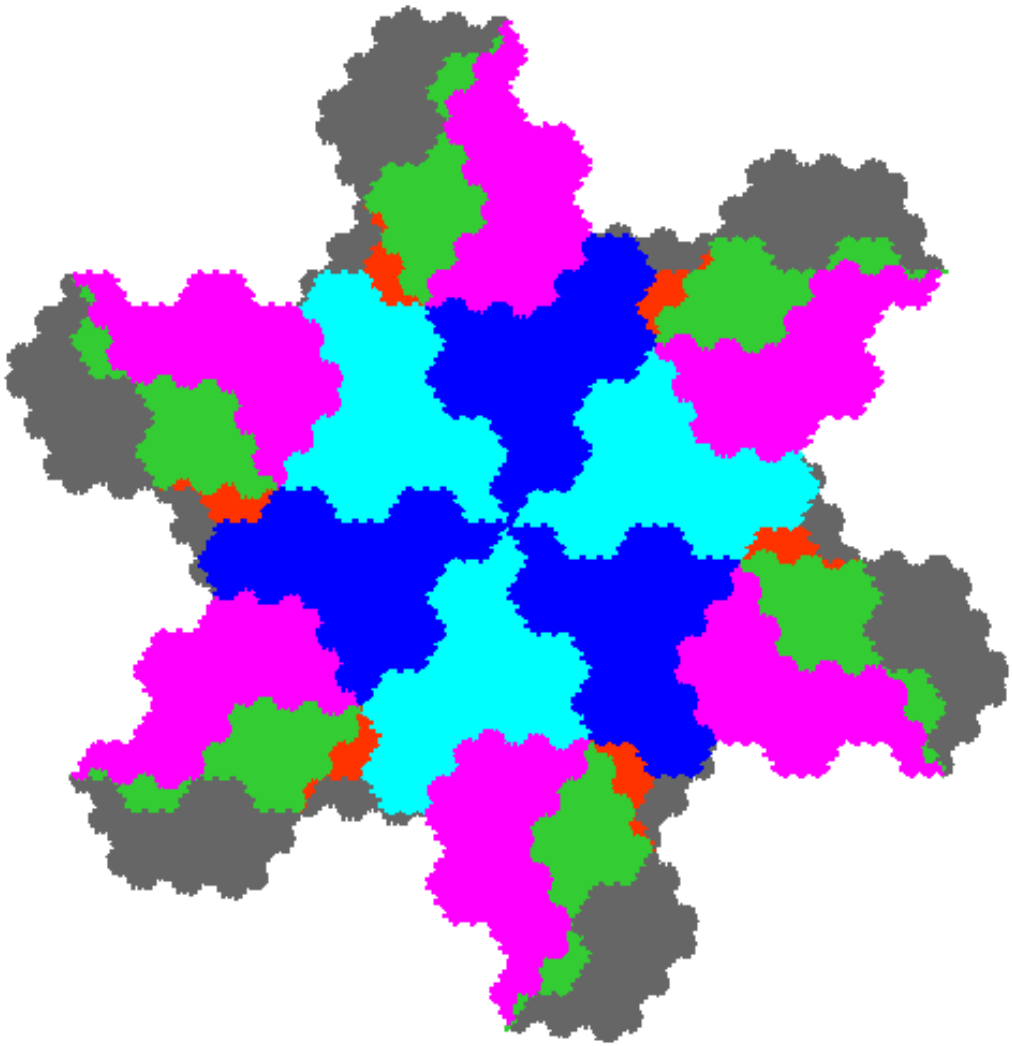
Ingredients

- Uses 12 tiles up to translation. Deformations of Tile(1,1).
- 6 orientations more common than other six.
- Can deform to tiling by Tile(a, b) in 6 orientations and Tile(b, a) in other six. Tile(b, a)'s are rotated by 30 degrees relative to Tile(a, b)'s. E.g. hats and turtles.
- Can take a, b complex. $\mathbb{C}^2$ again!

Same song, different verse

- $$\check{H}^1(\Omega, \mathbb{C}) = \mathbb{C}^2 \oplus \mathbb{C}^2.$$
- First $\mathbb{C}^2$ corresponds to linear transformations.
- Second $\mathbb{C}^2$ is asymptotically negligible. Generates topological conjugacies.
- **All** Spectre-family tilings are topologically conjugate, up to a linear transformation.
- Tiling has pure point spectrum and comes from a 4:2 cut-and-project scheme with a nice window.

The Spectre window



Morals of the story

- Subshifts don't have interesting topology, but their suspensions do!
- $\check{H}^k(\Omega)$ gives information about k-dimensional features of tilings.
- PE cohomology gives a beautiful way to visualize $\check{H}^*$.
- $\check{H}^d$ gives information about ergodic averages and fluctuations.
- $\check{H}^1$ classifies shape changes and homeomorphisms.

Application to Hat and Spectre

- Cohomology is **the** key tool for understanding Hat and Spectre tilings.
- In both families, $\check{H}^1(\Omega, \mathbb{C}) = \mathbb{C}^4$ is as small as it possibly could be. All shape changes are linear transformations plus topological conjugacies.
- In both families, self-similar version gives a convenient starting point for cut-and-project construction. Shape changes are basically just linear transformations and reprojections.

Thank you!