Frenkel-Kontorova model in almost-periodic environments

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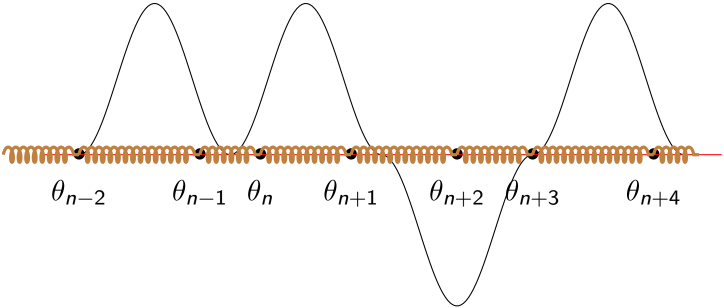
Frenkel-Kontorova Model, 1938



Used to represent dislocation in a crystal. Used to represent billard in a convex smooth domain.

$(\theta_n)_{n\in\mathbb{Z}}$: increasing sequence of reals

Total (formal) energy/Action:

$$\mathcal{E}((\theta_n)_n) = \sum_{n\in\mathbb{Z}} E(\theta_n, \theta_{n+1}),$$

for an interaction $E\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$.

Problem: Find bi-infinite configuration which "minimizes" the total action.

A configuration $(\theta_n)_{n\in\mathbb{Z}}$ is said minimizing if for any segment $(\theta_m, \theta_{m+1}, \dots, \theta_n)$

$$\sum_{j=m}^{n-1} E(\theta_i, \theta_{i+1}) \leq \sum_{j=m}^{n-1} E(\theta'_i, \theta'_{i+1})$$

for any segment $\theta'_m < \theta'_{m+1} < \cdots < \theta'_n$ with $\theta'_m = \theta_m$ and $\theta'_n = \theta_n$.

Q. What are the properties of minimizing configurations?
(Existence ?,...)

Examples of Frenkel-Kontorova model

Periodic 1D FK model:

$$\begin{aligned} E(x,y) &= \frac{1}{2}|x-y-\lambda|^2 + K(1-\cos(2\pi x)) \\ &= W(x-y) + V(x) \end{aligned}$$

- $\lambda \in \mathbb{R}$ distance when there is no external interaction
- $W(x-y)$: elastic internal interaction
- $V(x)$: periodic external interaction.

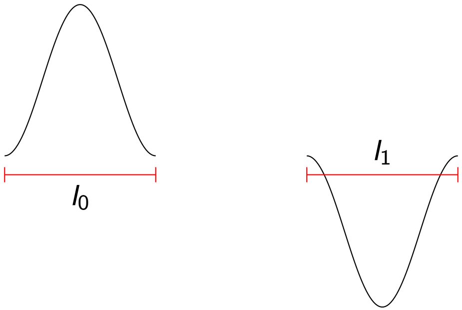
Almost-periodic 1D FK model:

$$E(x,y) = \frac{1}{2}|x-y-\lambda|^2 + K_1(1-\cos(2\pi x 1)) + K_2(1-\cos(2\pi x\sqrt{2}))$$

Quasicrystalline 1D FK model: To explain

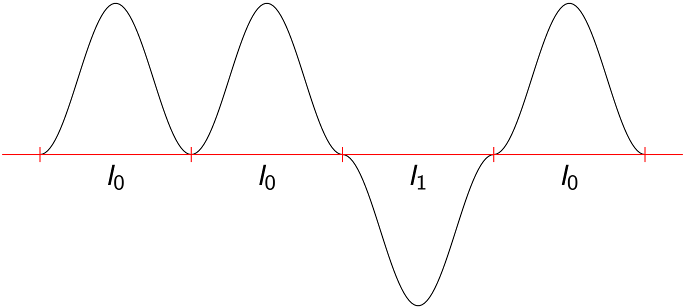
Quasicrystalline interaction 1D FK model

Let V be defined on two intervals I_0, I_1



Quasicrystalline interaction 1D FK model

A sequence $(s_n)_n \in \{0;1\}^{\mathbb{Z}}$ codes a tiling T of $\mathbb{R}$ by intervals I_0, I_1.
Ex. $s_n = \lfloor n\alpha \rfloor - \lfloor (n-1)\alpha \rfloor$, $\alpha \in (0,1)$.



This defines $V : \mathbb{R} \to \mathbb{R}$ that is strongly pattern equivariant with respect to the tiling $(I_{s_n})_{n\in\mathbb{Z}}$.

Strongly pattern equivariant potential

Setting for any interval $I \subset \mathbb{R}$

$$\mathcal{R}_I := \{x \in \mathbb{R} : V(\cdot)|_I = V(\cdot + x)|_I\},$$

- Finite complexity: There are finitely many intervals $I_1, \ldots, I_n$ s.t. $\mathcal{R}_{I_i}$ is discrete and $\mathbb{R} = \bigcup_{i=1}^n \mathcal{R}_{I_i} + I_i$.
- Repetitivity: Every set $\mathcal{R}_I$ is relatively dense.
- Uniform density: Every $\mathcal{R}_I$ admits a density: the limit $\lim_{n\to\pm\infty} \frac{\#\mathcal{R}_I \cap [-n,n]}{2n} > 0$ exists.

Common feature on interaction E

A family of interactions E_ω, indexed by an environnent $\omega \in \Omega$

1. $E_\omega \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is C^2.
2. Twist: for some $\alpha > 0$, $\frac{\partial^2 E_\omega}{\partial y \partial x} \leq -\alpha < 0$.
3. Equivariance: $\forall x, y, t \in \mathbb{R}$
$$E_\omega(x+t, y+t) = E_{\tau_t(\omega)}(x, y),$$
for some **stricly ergodic continuous** flow $\mathbb{R} \overset{\tau}{\curvearrowright} \Omega$
    - $\Omega = \mathbb{R}/\mathbb{Z}$, $\tau_t\omega = \omega + t \mod \mathbb{Z}$ Periodic model
    - $\Omega = \mathbb{R}^2/\mathbb{Z}^2$, Almost-periodic model
    $\tau_t(\omega_1, \omega_2) = (\omega_1 + t, \omega_2 + \sqrt{2}t)$
    - $\Omega =$ tiling space, $\tau_t =$ translation action Quasicrystalline model
4. Lagrangian form $E_\omega(x, y) = L(\tau_x(\omega), y - x)$, for some $L \colon \Omega \times \mathbb{R} \to \mathbb{R}$.
5. Superlinearity: $\lim_{|y-x| \to +\infty} \frac{E_\omega(x,y)}{|x-y|} = +\infty$

A minimizing configuration is critical.

Euler-Lagrange Equation

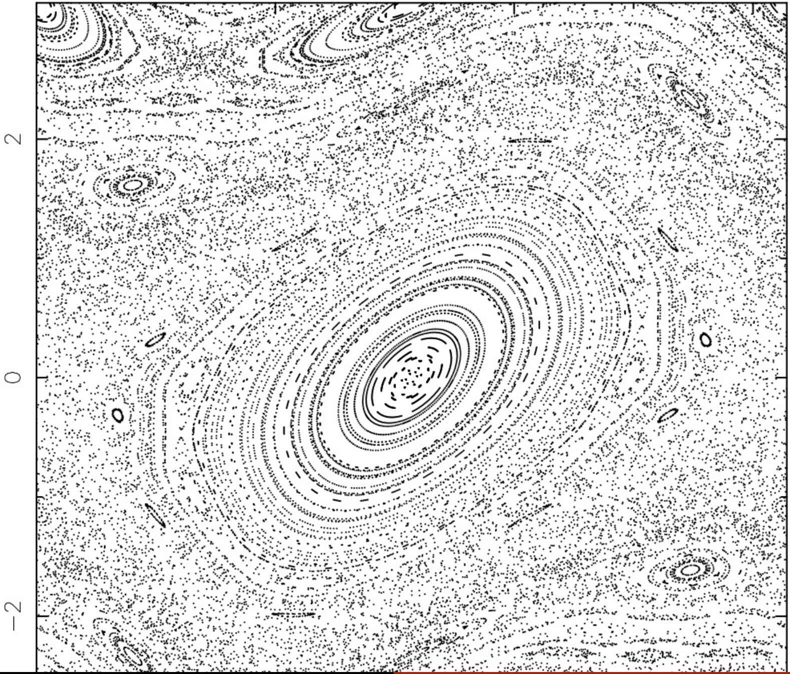
$$\frac{\partial E_\omega}{\partial y}(\theta_{k-1}, \theta_k) + \frac{\partial E_\omega}{\partial x}(\theta_k, \theta_{k+1}) = 0.$$

$$\frac{\partial E_\omega}{\partial x}(\theta_k, \cdot)^{-1}(-\frac{\partial E_\omega}{\partial y}(\theta_{k-1}, \theta_k)) = \theta_{k+1}.$$

Euler-Lagrange map $\begin{pmatrix} \theta_k \\ \theta_{k-1} \end{pmatrix} \mapsto \begin{pmatrix} \theta_{k+1} \\ \theta_k \end{pmatrix}$

extends to a homeo. $\varphi\colon \Omega \times \mathbb{R} \to \Omega \times \mathbb{R}$.

Periodic model $\varphi =$ standard map $\begin{pmatrix} \theta \\ p \end{pmatrix} \mapsto \begin{pmatrix} \theta + p \mod \mathbb{Z} \\ p + K 2\pi \sin(\theta 2\pi) \end{pmatrix}$



2
0
−2

Theorem (Aubry-Le Daeron; Mather, 1983)

For the energy $E(x, y) = W(x - y) + V(x)$ *with* V *1-periodic:* $V(\cdot + 1) = V(\cdot)$

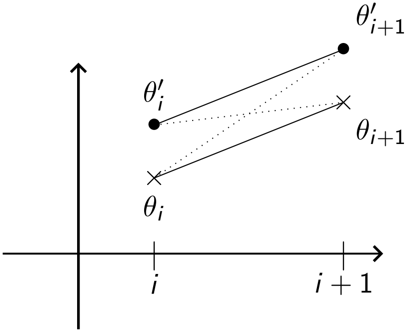
i) *Any minimizing configuration* $(\theta_n)_n$ *has a rotation number,*

$$\lim_{n \to \pm\infty} \frac{\theta_n - \theta_0}{n} = \rho.$$

ii) *Any real* $\rho \geq 0$ *is the rotation number of some minimizing configuration.*

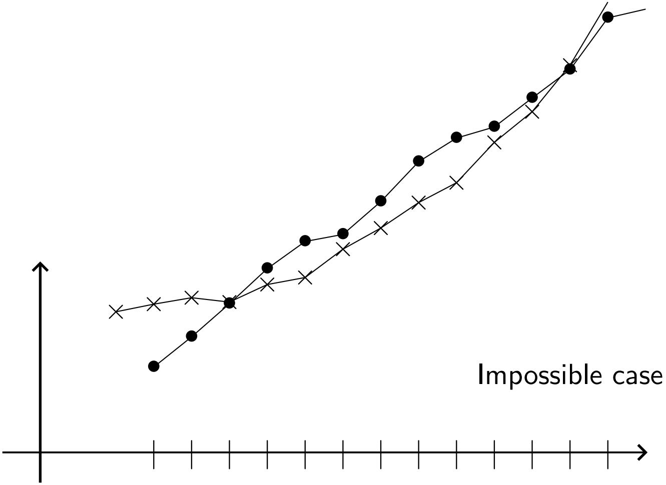
By the **twist condition**: For any $\theta_i < \theta'_i < \theta_{i+1} < \theta'_{i+1}$:

$$E(\theta_i, \theta_{i+1}) + E(\theta'_i, \theta'_{i+1}) < E(\theta_i, \theta'_{i+1}) + E(\theta'_i, \theta_{i+1}).$$



Lemma (Aubry-crossing lemma)

Two minimizing configurations cross each other at most once.



Theorem (Gambaudo-Guiraud-P.,06)

For an quasicrystalline interaction $E(x, y) = W(x - y) + V(x)$*, V strongly pattern equivariant.*

i) Any minimizing configuration $(\theta_n)_n$ *has a rotation number,*

$$\lim_{n\to\pm\infty} \frac{\theta_n - \theta_0}{n} = \rho.$$

ii) Any real $\rho \geq 0$ *is the rotation number of some minimizing configuration.*

Theorem (Aubry; Mather, 1983)

For the periodic model, the Euler-Lagrange map $\varphi\colon \mathbb{R}/\mathbb{Z}\times\mathbb{R}\to\mathbb{R}/\mathbb{Z}\times\mathbb{R}$, *there is closed set* $\Xi\subset\mathbb{R}/\mathbb{Z}\times\mathbb{R}$ φ*-invariant* ($\varphi(\Xi)=\Xi$).

- Ξ *is the graph of some Lipschitz map* $f:\Xi_0\subset\mathbb{R}/\mathbb{Z}\to\mathbb{R}$

$$\Xi=\{(x,f(x)):x\in\Xi_0\}.$$

- *An element in* Ξ *has a rotation number* $\rho\geq 0$.
  If $\rho\in\mathbb{Q}$, Ξ *is finite.*
  If $\rho\in\mathbb{R}\setminus\mathbb{Q}$, Ξ *is a Cantor set or* $\Xi_0=\mathbb{R}/\mathbb{Z}$.

Ξ is called a Aubry-Mather set.

The ground energy is

$$\overline{E_\omega} := \lim_{n\to+\infty} \inf_{x_0,\dots,x_n\in\mathbb{R}} \frac{1}{n}\sum_{i=0}^{n-1} E_\omega(x_i, x_{i+1}).$$

Existence by subadditivity,
$\overline{E_\omega} = \bar{E}$ independent of ω by strict ergodicity.

A configuration $(\theta_n)_n$ is said calibrated (at the level $\bar{E}$) if for any $m \le n$

$$\sum_{i=m}^{n-1} \left[E_\omega(\theta_i, \theta_{i+1}) - \bar{E}\right] \le \inf_{\ell\ge 1} \inf_{y_0=\theta_m,\dots,y_\ell=\theta_n} \sum_{i=0}^{\ell-1} \left[E_\omega(y_i, y_{i+1}) - \bar{E}\right].$$

A calibrated configuration for $E_\omega(x,y) - \lambda(x-y)$ is a minimizing configuration for E_ω.

Aubry-Mather theory

Theorem (Aubry, Mather, 83, 89, ...)

For the energy $E_\lambda(x, y) = W(x - y) - \lambda(x - y) + V(x)$ with V 1-periodic.

- $\lambda \in \mathbb{R} \mapsto \overline{E_\lambda}$ *is a* C^1 *convex function.*
- $\forall \lambda \in \mathbb{R}$, *there exists a calibrated configuration for* $E_\lambda(x, y)$, *with rotation number*

$$\rho = \lim_{n \to \pm\infty} \frac{\theta_n - \theta_0}{n} = -\frac{d\overline{E_\lambda}}{d\lambda}.$$

- *If* $\rho \in \mathbb{Q}$, *then* $\Lambda(\rho) := \{\lambda : \rho = -\frac{d\bar{E}_\lambda}{d\lambda}\}$ *has non empty interior. If* $\rho \notin \mathbb{Q}$, $Int(\Lambda(\rho)) = \emptyset$.

Any $\rho \geq 0$ is the rotation number of a calibrated configuration.

Aubry-Mather theory

Theorem (Garibaldi-P.-Thieullen, 2017)

For an quasicrystalline interaction $E_\omega(x, y) = W(x - y) + V_\omega(x)$, *with* V_ω *strongly pattern equivariant.*
There exist configurations $(\theta_n)_{n\in\mathbb{Z}}$ *calibrated for* $\overline{E_\omega}$ *with bounded jumps:*

$$\sup_{n\in\mathbb{Z}} |\theta_{n+1} - \theta_n| < +\infty.$$

Weak KAM solution:[Chou-Griffiths 1986, Fathi 16] A C^0 function $u\colon \mathbb{R} \to \mathbb{R}$ such that

$$u(y) + \bar{E} \leq u(x) + E_\omega(x, y) \quad \forall y, x \in \mathbb{R}$$

and the inequality is sharp:

$$u(y) + \bar{E} = \min_x [u(x) + E(x, y)], \forall y \ \text{(backward)}$$

u plays the role of a discrete viscosity solution for the Hamilton-Jacobi equation (sometimes also called calibrated solution or corrector).

Analogy in the $(\min, +)$-algebra:

$$\text{“}a.b\text{''} = a + b$$

$$\text{“}a + b\text{''} = \min(a, b)$$

$$\text{“}\bar{E}.u(\cdot)\text{''} = \text{“}\sum_x u(x).E(x, \cdot)\text{''}$$

u is the eigenfunction of a linear operator.

weak KAM theory: some ideas

A configuration $(\theta_n)_n$ such that for some w-KAM solution

$$u(\theta_{n+1}) + \bar{E} = u(\theta_n) + E(\theta_n, \theta_{n+1}), \quad \forall n \in \mathbb{Z}$$

is calibrated

Indeed $\forall \theta_n = y_0 < y_1 < \ldots < y_\ell = \theta_m$

$$\begin{aligned}
\sum_{i=0}^{\ell-1} E_\omega(y_i, y_{i+1}) - \bar{E} \geq & \sum_{i=0}^{\ell-1} u(y_{i+1}) - u(y_i) \\
= & u(y_\ell) - u(y_0) = u(\theta_m) - u(\theta_n) \\
= & \sum_{i=n}^{m-1} u(\theta_{i+1}) - u(\theta_i) = \sum_{i=n}^{m-1} E_\omega(\theta_i, \theta_{i+1}) - \bar{E}.
\end{aligned}$$

Interpretation: u creates a force on the left hand side of any finite box $[a, b]$ so that, if the particles $(\theta_n)_n$ at the left hand side are erased and replaced by the effective force, the configuration stay unchanged. **Analogy**: $u(y) - u(x) \leq S(x, y)$, for the Mañé potential

Periodic model. There is a cont. periodic (hence *bounded*) solution u, defining $u\colon \Omega \to \mathbb{R}$.

Almost-periodic and quasicrystalline models

Existence of a **bounded** solution is still unclear.

Theorem (GPT 23)

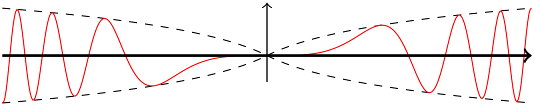
If $\inf_x E_\omega(x,x) > \bar{E}$*, then there exists a Lipschitz w-KAM solution* $u_\omega\colon \mathbb{R} \to \mathbb{R}$*.*

Theorem (GPT 23, Quasicrystalline model)

When the tiling is linearly repetitive, $\exists \epsilon \in \{\pm 1\}$, $\gamma > 0$ *s.t.* $\forall$ *w-KAM solutions* $u_\omega\colon \mathbb{R} \to \mathbb{R}$ *belongs to one of the following case*

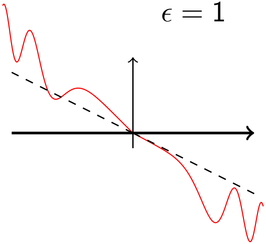
1. *sublinear growth* $\lim_{x\to\pm\infty} u_\omega(x)/x = 0$
2. $\limsup_{x\to+\infty} u_\omega(\epsilon x)/|x| \le -\gamma$, $\liminf_{x\to-\infty} u_\omega(\epsilon x)/|x| \ge \gamma$.
3. $\limsup_{x\to+\infty} u_\omega(\epsilon x)/|x| \le -\gamma$, $\lim_{x\to-\infty} u_\omega(\epsilon x)/|x| = 0$.

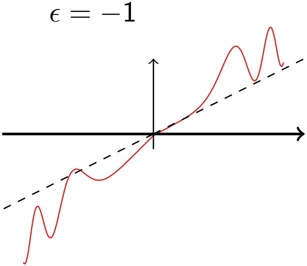
Type I



Sublinear growth at $\pm\infty$.

Type II



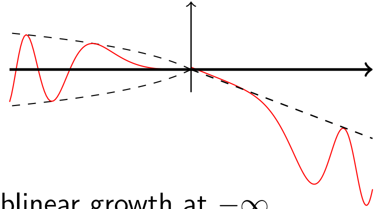


At least (negative) linear growth at $-\infty$, at most (negative) linear growth at $+\infty$.

At most (positive) linear growth at $-\infty$, at least (positive) linear growth at $+\infty$.

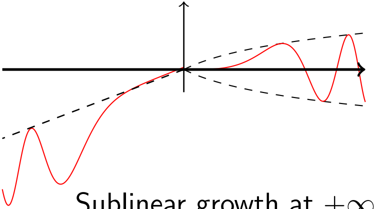
Type III

$\epsilon = 1$



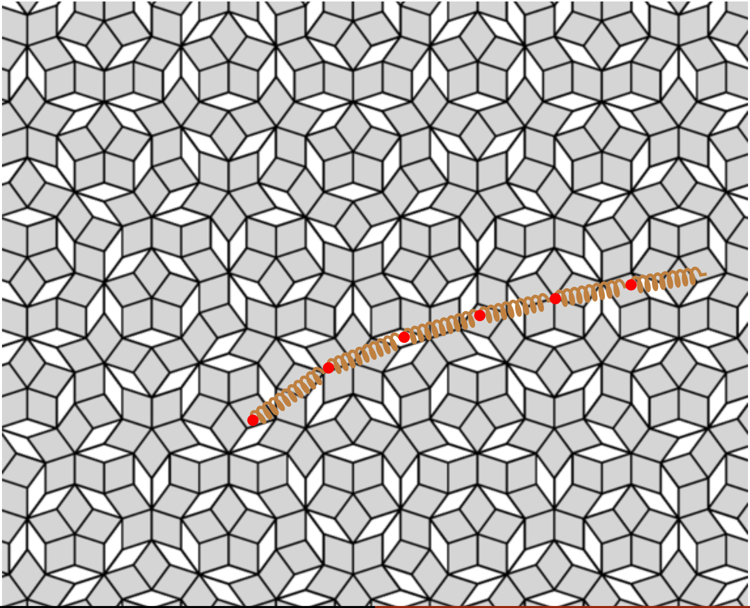
Sublinear growth at $-\infty$, at most (negative) linear growth at $+\infty$.

$\epsilon = -1$

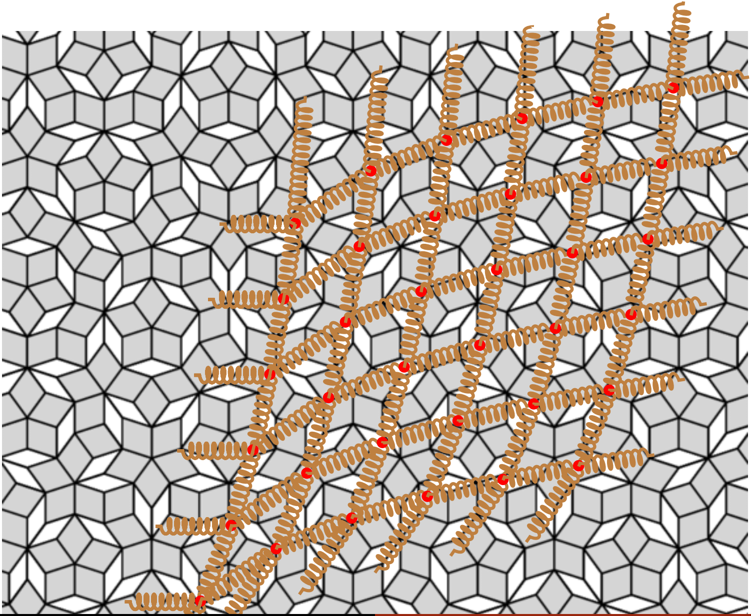


Sublinear growth at $+\infty$, at most (positive) linear growth at $-\infty$.

FK in higher dimension $d \geq 1$



FK in higher dimension $d \geq 1$



FK in higher dimension $d \geq 1$

$\theta \colon \mathbb{Z}^r \to \mathbb{R}^d \qquad (\theta_z)_{z\in\mathbb{Z}^r} \in (\mathbb{R}^d)^{\mathbb{Z}^r}$

Total (formal) energy:

$$\mathcal{E}((\theta_z)_z) = \sum_{i,j\in\mathbb{Z}^r} a(i-j)\|\theta_i - \theta_j - \gamma(i-j)\|^{2|i-j|} + \sum_{i\in\mathbb{Z}^r} V(\theta_i),$$

where

$\gamma \colon \mathbb{Z}^r \to \mathbb{R}^d$;

$a \colon \mathbb{Z}^r \to \mathbb{R}_+$ with finite support;

$V \colon \mathbb{R}^d \to \mathbb{R}$ smooth.

FK in higher dimension $d \geq 1$

A configuration $(\theta_z)_{z\in\mathbb{Z}^r} \in (\mathbb{R}^d)^{\mathbb{Z}^r}$ is minimizing, if

$$\mathcal{E}((\theta_z)_z) \leq \mathcal{E}((\eta_z)_z)$$

for any configuration $(\eta_z)_z$ such that $\eta_z = \theta_z$ except finitely many $z \in \mathbb{Z}^r$.

A configuration $(\theta_z)_{z\in\mathbb{Z}^r} \in (\mathbb{R}^d)^{\mathbb{Z}^d}$ is equilibrium (or a critical point), if $\forall z \in \mathbb{Z}^r$,

$$\begin{aligned}\nabla_{\theta_z}\mathcal{E}((\theta_z)_z) =& \nabla_{\theta_z}\left(\sum_{i,j\in\mathbb{Z}^d} a(i-j)|\theta_i - \theta_j - \gamma(i-j)|^{2|i-j|}\right) + \nabla_{\theta_z}V(u_z)\\ =& 0\end{aligned}$$

FK in higher dimension

Theorem (de la Llave-Valdinoci, 07, $d = 1$, $r \geq 1$)

When $V : \mathbb{R} \to \mathbb{R}$ is $\mathbb{Z}$-periodic.
For any vector $\rho \in \mathbb{R}^r$, there exists a minimizing configuration $(\theta_z)_z \in (\mathbb{R})^{\mathbb{Z}^r}$, such that

$$\forall z \in \mathbb{Z}^r, \quad |\theta_z - \langle \rho, z \rangle| \leq 1.$$

Moreover, if $\langle \rho, j \rangle \geq n$ for some $j \in \mathbb{Z}^r$, $n \in \mathbb{Z}$, then

$$\forall z \in \mathbb{Z}^r, \quad \theta_{z+j} \geq \theta_z + n.$$

The minimizing configurations and their translations are organized in a lamination Extension by changing $\mathbb{Z}^r$ with G finitely generated and residually finite group.

FK in higher dimension

Theorem (Garibaldi-P-Thieullen 17 $r = 1, d \geq 1$)

For

$$\mathcal{E}_{\omega^*}((\theta_z)_{z\in\mathbb{Z}}) = \sum_{i\in\mathbb{Z}} a\|\theta_i - \theta_{i+1} - \gamma\|^2 + V_{\omega^*}(\theta_i),$$

where $V_{\omega^*}\mathbb{R}^d \to \mathbb{R}^d$ is almost-periodic with respect a strictly ergodic flow $\mathbb{R}^d \overset{\tau}{\curvearrowright} \Omega(\omega_*)$,
then $\exists$ $\omega \in \Omega(\omega^*)$, and a calibrated configuration $(\theta_z)_z \in (\mathbb{R}^d)^{\mathbb{Z}}$ for $\mathcal{E}_\omega$.

Existence of minimizing configurations for each $\omega \in \Omega(\omega^*)$ is unclear.

FK in higher dimension $d \geq 1$

Theorem (Treviño, 19, $d \geq 1$, $r \geq 1$)

When V is strongly pattern equivariant with respect to $\mathbb{R}^d$ tiling of finite translation type.
For any $\rho \in \mathrm{Hom}(\mathbb{Z}^r, \mathbb{R}^d)$, for any large enough $\lambda \gg 1$ there exists an equilibrium configuration $(\theta_z)_z \in (\mathbb{R}^d)^{\mathbb{Z}^r}$, for the energy

$$\mathcal{E}((\theta_z)_z) = \sum_{i \in \mathbb{Z}^r} \sum_{\|j-i\|=1} \|\theta_i - \theta_j - \gamma\|^2 + \lambda V_{\omega^*}(\theta_i),$$

such that

$$\sup_{z \in \mathbb{Z}^r} \|\theta_z - \rho(z)\| < +\infty.$$

J. Du-X. Su, 20: These configurations are not minimizing for $\lambda \gg 1$.