Gibbs measures and Double Variational Principle for a potential depending on the first coordinate on XY model

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Background

- $\sigma : \mathbb{K}^{\mathbb{Z}} \to \mathbb{K}^{\mathbb{Z}} : \sigma((x_0, x_1, x_2, x_3, \ldots)) = (x_1, x_2, x_3, x_4, \ldots)$ $\mathbb{K}$: a set
- A shift with finite symbol : $(\mathcal{A}^{\mathbb{Z}}, \sigma)$

  $\mathcal{A}$: a finite set

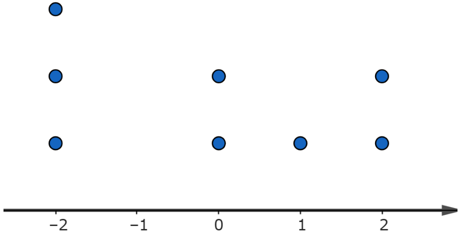

Figure: $\mathcal{A}^{\mathbb{Z}}$

- A shift with countable symbol : $(\mathbb{N}^{\mathbb{Z}}, \sigma)$

Background

- **XY model** : $(M^{\mathbb{Z}}, \sigma)$
  M : a connected and compact manifold

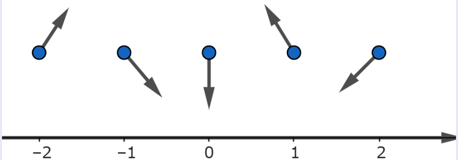

Figure: XY model where M is $\mathbb{S}^1$

- We want to understand the statistical property of XY model.
- See the invariant measure on XY model,
  in particular, "the Gibbs measure".
- Extend conclusions about a shift with finite symbol to XY model.

Background

Relation between Gibbs measures and Double Variational Principle

$\mathbb{K}$	$\mathcal{A}$: a finite set	M : a connected and compact manifold
The Ruelle operator $\mathcal{L}_{\varphi} f(x)$	$\sum_{\sigma y = x} e^{\varphi(y)} f(y)$	$\int e^{\varphi(ax)} f(ax)\ da$
Gibbs measures for a Hölder continuous function φ satisfy	Variational Principle	Double Variational Principle with potential?

- $\int e^{\varphi(ax)} f(ax)\ da$: [A.Baraviera, et al. 2011]
- Double Variational Principle with potential : [M.Tsukamoto, 2020]

Main result

- $\mathbb{K} = [0, 1]$
- $\sigma : [0,1]^{\mathbb{Z}} \to [0,1]^{\mathbb{Z}}$; $\sigma((x_m)_{m\in\mathbb{Z}}) = (x_{m+1})_{m\in\mathbb{Z}}$: the shift
- $\boldsymbol{d}$: the metric on $[0,1]^{\mathbb{Z}}$;

$$d(x,y) := \sum_{m\in\mathbb{Z}} 2^{-|m|}|x_m - y_m|, \quad (x = (x_m)_{m\in\mathbb{Z}},\ y = (y_m)_{m\in\mathbb{Z}})$$

- $\varphi : [0,1]^{\mathbb{Z}} \to [0,1]$; $\varphi((x_m)_{m\in\mathbb{Z}}) = x_0$: a potential
- $\forall N \in \mathbb{N}, \forall A_1, A_2, \ldots, A_N \in \mathscr{B}([0,1])$,

$$\boldsymbol{\mu}(A_1 \times A_2 \times \cdots \times A_N \times [0,1]^{\mathbb{N}})$$

$$:= \left(\frac{1}{\int_{[0,1]} e^{\varphi(x)}\ dx} \right)^N \int_{A_1} e^{\varphi(x_1)}\ dx_1 \int_{A_2} e^{\varphi(x_2)}\ dx_2 \cdots \int_{A_N} e^{\varphi(x_N)}\ dx_N.$$

Main result

Then, $\boldsymbol{d}$ and $\boldsymbol{\mu}$ satisfy Double Variational Principle with potential.

Background

Variational Principle

- $\mathscr{M}^T(\mathcal{X}) := \{\text{Invariant measures on } (\mathcal{X}, T)\}$ where
  $\mathcal{X}$ is a compact metric space and T is a continuous map on $\mathcal{X}$ to itself
- h_{top} : the topolpgical entropy
- h_μ : a measure-theoretic entropy
- $\varphi : \mathcal{X} \to \mathbb{R}$: the continuous potential
- $\mathscr{P}_{top}(\cdot)$: Topological pressure

1. $h_{top} = \sup_{\mu \in \mathscr{M}^T(\mathcal{X})} h_\mu$,
2. $\mathscr{P}_{top}(A) = \sup_{\mu \in \mathscr{M}^T(\mathcal{X})}(h_\mu + \int \varphi \, d\mu)$.

- Let $(\mathcal{X}, T) = (\mathcal{A}^{\mathbb{Z}}, \sigma)$.
  Then, if φ is Hölder, the Gibbs measure for φ is the unique measure which attains the supremum of ② [R.Bowen, 1975].

Background

It is known that

$$h_{top}([0,1]^{\mathbb{Z}}, \sigma) = \infty.$$

- It makes no sense to hold Variational Principle on XY model and Double Variational Principle is introduced to overcome this problem.
- Based on the conclusion about a shift with finite symbol, it is natural to suppose Gibbs measures on XY model relate to Double Variational Principle with potential.

Double Variational Principle with potential

Theorem 1 [M.Tsukamoto, 2020]

- $(\mathcal{X}, T)$: a dynamical system which has the marker property
- $\varphi : \mathcal{X} \to \mathbb{R}$: a continuous function
- $\mathscr{M}^T(\mathcal{X})$: the set of T-invariant Borel probability measures on $\mathcal{X}$
- $\mathscr{D}(\mathcal{X})$: the set of distance functions on $\mathcal{X}$

Then,

$$\begin{aligned} \mathrm{mdim}(\mathcal{X}, T, \varphi) &= \min_{d \in \mathscr{D}(\mathcal{X})} \sup_{\mu \in \mathscr{M}^T(\mathcal{X})} \left(\overline{\mathrm{rdim}}(\mathcal{X}, T, d, \mu) + \int_{\mathcal{X}} \varphi \, d\mu \right) \\ &= \min_{d \in \mathscr{D}(\mathcal{X})} \sup_{\mu \in \mathscr{M}^T(\mathcal{X})} \left(\underline{\mathrm{rdim}}(\mathcal{X}, T, d, \mu) + \int_{\mathcal{X}} \varphi \, d\mu \right). \end{aligned}$$

- We call this principle
  Double Variational Principle with potential.

$([0,1]^{\mathbb{Z}}, \sigma)$

From now, we consider the case

- **XY model** $([0,1]^{\mathbb{Z}}, \sigma)$: a dynamical system
  $\sigma : [0,1]^{\mathbb{Z}} \to [0,1]^{\mathbb{Z}}$; $\sigma((x_m)_{m\in\mathbb{Z}}) = (x_{m+1})_{m\in\mathbb{Z}}$: the shift
- $\boldsymbol{\varphi} : [0,1]^{\mathbb{Z}} \to [0,1]$; $\varphi((x_m)_{m\in\mathbb{Z}}) = x_0$: a potential

- $([0,1]^{\mathbb{Z}}, \sigma)$ has the marker property.

Calculation of $\mathrm{mdim}([0,1]^{\mathbb{Z}}, \sigma, \varphi)$ [M.Tsukamoto, 2020]

Denote the **mean dimension with potential** by $\mathrm{mdim}(\mathcal{X}, T, \varphi)$.

- $\mathrm{mdim}([0,1]^{\mathbb{Z}}, \sigma, \varphi) = 2.$

- $\forall N \in \mathbb{N}$, $\forall A_1, A_2, \dots, A_N \in \mathscr{B}([0,1])$,

$$\boldsymbol{\mu}(A_1 \times A_2 \times \cdots \times A_N \times [0,1]^{\mathbb{N}})$$

$$:= \left(\frac{1}{\int_{[0,1]} e^{\varphi(x)}\ dx} \right)^N \int_{A_1} e^{\varphi(x_1)}\ dx_1 \int_{A_2} e^{\varphi(x_2)}\ dx_2 \cdots \int_{A_N} e^{\varphi(x_N)}\ dx_N.$$

Mutual Information

- Fix a probability space $(\Omega, \mathbb{P})$ and
  assume that all random variables are defined on $(\Omega, \mathbb{P})$.
- X, Y : r. vs taking values in some measurable spaces $\mathcal{X}, \mathcal{Y}$
- $H(X) = -\sum_{x \in \mathcal{X}} \mathbb{P}(X = x) \log \mathbb{P}(X = x)$

Mutual information of finite sets

- $\mathcal{X}, \mathcal{Y}$: a finite set

Mutual information $I(X; Y)$;

$$I(X; Y) := H(X) - H(X|Y).$$

Mutual Information

1. Take finite measurable partitions of $\mathcal{X}, \mathcal{Y}$ (not necessarily finite sets) ;
   $\mathcal{P} = \{P_1, .., P_M\} : \bigcup_k P_k = \mathcal{X} \quad M < \infty \quad \mathcal{P}$ is a disjoint set,
   $\mathcal{Q} = \{Q_1, .., Q_N\} : \bigcup_k Q_k = \mathcal{Y} \quad M < \infty \quad \mathcal{Q}$ is a disjoint set.
2. For $x \in \mathcal{X}, y \in \mathcal{Y}$, $\tilde{\mathcal{P}}(x) := P_m, \ \tilde{\mathcal{Q}}(y) := Q_n \quad x \in P_m, \ y \in Q_n$
3. It is possible to consider mutual information $I(\mathcal{P} \circ X; \mathcal{Q} \circ Y)$ as the definition of Mutual information of finite sets.

Mutual information of any sets

- $\mathcal{X}, \mathcal{Y}$: a set

Mutual information $I(X;Y)$;

$$I(X;Y) := \sup_{\mathcal{P},\mathcal{Q}} I(\tilde{\mathcal{P}} \circ X; \tilde{\mathcal{Q}} \circ Y)$$

The rate distortion function

Definition of the rate distortion function

- $(\mathcal{X}, T)$: a dynamical system with a T-invariant measure μ
- $\varepsilon \in \mathbb{R}_{>0}$

Then, define the rate distortion function by

$$R(d, \mu, \varepsilon) = \inf_{N,X,Y} \frac{I(X;Y)}{N}, \quad \text{where}$$

- $N \in \mathbb{N}$
- $X, Y = (Y_0, ..., Y_{N-1})$: r. vs defined on a probability space $(\Omega, \mathbb{P})$, $\forall X, Y_n$, they take values in $\mathcal{X}$ and satisfy

$$X \text{ has the distribution } \mu,$$

$$\mathbb{E}\left(\frac{1}{N}\sum_{n=0}^{N-1} d(T^n X(\omega), Y_n(\omega))\right) < \varepsilon \quad (\omega \in \Omega), \tag{1}$$

and we call (1) the distortion condition.

The rate distortion dimension

Definition of the rate distortion dimension

- $(\mathcal{X}, T)$: a dynamical system with a T-invariant measure μ
- d : a metric on $\mathcal{X}$

Define the upper and the lower rate distortion dimensions by

$$\overline{\mathrm{rdim}}(\mathcal{X}, T, d, \mu) = \limsup_{\varepsilon \to 0} \frac{R(d, \mu, \varepsilon)}{\log(1/\varepsilon)},$$

$$\underline{\mathrm{rdim}}(\mathcal{X}, T, d, \mu) = \liminf_{\varepsilon \to 0} \frac{R(d, \mu, \varepsilon)}{\log(1/\varepsilon)}.$$

- If both of these limits coincide,
  we call the value the **rate distortion dimension** $\mathrm{rdim}(\mathcal{X}, T, d, \mu)$.

Calculation of the rate distortion dimension

Calcurate the rate distortion dimension to evaluate the right-hand side of Double Variational Principle for mean dimension with potential.

Setting

- **XY model** $([0,1]^{\mathbb{Z}}, \sigma)$: a dynamical system

  $\sigma : [0,1]^{\mathbb{Z}} \to [0,1]^{\mathbb{Z}}$; $\sigma((x_m)_{m\in\mathbb{Z}}) = (x_{m+1})_{m\in\mathbb{Z}}$: the shift

- $\boldsymbol{d}$: the metric on $[0,1]^{\mathbb{Z}}$;

$$d(x,y) := \sum_{m\in\mathbb{Z}} 2^{-|m|}|x_m - y_m|, \quad (x = (x_m)_{m\in\mathbb{Z}},\ y = (y_m)_{m\in\mathbb{Z}})$$

- $\boldsymbol{\varphi}$: $[0,1]^{\mathbb{Z}} \to [0,1]$; $\varphi((x_m)_{m\in\mathbb{Z}}) = x_0$: a potential

We want to calculate

$$\overline{\mathrm{rdim}}([0,1]^{\mathbb{Z}}, \sigma, d, \mu) + \int_{[0,1]^{\mathbb{Z}}} \varphi \, d\mu \text{ or } \underline{\mathrm{rdim}}([0,1]^{\mathbb{Z}}, \sigma, d, \mu) + \int_{[0,1]^{\mathbb{Z}}} \varphi \, d\mu$$

or both.

Calculation of the rate distortion dimension

Under this setting,
we can construct a Gibbs measure on XY model for a potential which depends on the first coordinate.

$\forall N \in \mathbb{N}$, $\forall A_1, A_2, \ldots, A_N \in \mathscr{B}([0,1])$,

$$\boldsymbol{\mu}(A_1 \times A_2 \times \cdots \times A_N \times [0,1]^{\mathbb{N}})$$
$$:= \left(\frac{1}{\int_{[0,1]} e^{\varphi(x)} \ dx} \right)^N \int_{A_1} e^{\varphi(x_1)} \ dx_1 \int_{A_2} e^{\varphi(x_2)} \ dx_2 \cdots \int_{A_N} e^{\varphi(x_N)} \ dx_N.$$

- $\varphi : [0,1]^{\mathbb{Z}} \to [0,1]$; $\varphi((x_m)_{m\in\mathbb{Z}}) = x_0$
- We call μ a **Gibbs measure** because it is constructed by the eigenvalue and the eigenfunction for the Ruelle operator $\mathcal{L}_\varphi$.

Calculation of the rate distortion dimension

First, calculate $\mathrm{rdim}(\mathcal{X}, \sigma, d, \mu)$.

- Take $\varepsilon \in \mathbb{R}_{>0}$.
- $X, Y = (Y_0, ..., Y_{N-1})$: r. vs defined on $(\Omega, \mathbb{P})$; $\forall \omega \in \Omega$, $X(\omega) := (X_m(\omega))_{m \in \mathbb{Z}} \in [0,1]^{\mathbb{Z}}$, $Y_k(\omega) := (Y_{k,m}(\omega))_{m \in \mathbb{Z}} \in [0,1]^{\mathbb{Z}}$
- X has the distribution μ.
- Y_k satisfies the distortion condition.

Then, by data processing inequality and independence of $X_0, \ldots X_{n-1}$,

$$I(X; Y) \geq \sum_{m=0}^{n-1} I(X_m; Y_{m,0}).$$

Moreover, because X and Y hold the distortion condition,

$$\frac{1}{n} \sum_{m=0}^{n-1} \mathbb{E}|X_m - Y_{m,0}| \leq \frac{1}{n} \mathbb{E} \left(\sum_{m=0}^{n-1} d(\sigma^m X, Y_m) \right) < \varepsilon.$$

Calculation of the rate distortion dimension

Define the value $r(\varepsilon)$ by $\boldsymbol{r(\varepsilon)} := \inf_{U,V} I(U;V)$ where

- U, V : the random variables which take values on $[0,1]$
- U has the distribution μ and V holds $\mathbb{E}|U-V| \le \varepsilon$.

Then, from

$$I(X;Y) \ge \sum_{m=0}^{n-1} I(X_m; Y_{m,0}), \quad \frac{1}{n}\sum_{m=0}^{n-1} \mathbb{E}|X_m - Y_{m,0}| < \varepsilon,$$

$$\boldsymbol{\frac{I(X;Y)}{n}} \ge \frac{1}{n}\sum_{m=0}^{n-1} r(\mathbb{E}|X_m - Y_{m,0}|) \ge r\left(\frac{1}{n}\sum_{m=0}^{n-1}\mathbb{E}|X_m - Y_{m,0}|\right) \ge \boldsymbol{r(\varepsilon)}.$$

Calculation of the rate distortion dimension

Consequently, because $R(d,\mu,\varepsilon) := \inf_{N,X,Y}(I(X;Y)/N)$,

$$R(d,\mu,\varepsilon) \geq r(\varepsilon).$$

On the other hand,

$$r(\varepsilon) \sim |\log \varepsilon| \quad (\varepsilon \to 0).$$

Therefore,

$$\liminf_{\varepsilon \to 0} \frac{R(d,\mu,\varepsilon)}{|\log \varepsilon|} \geq 1.$$

Hence, from the consequence of [Lindenstrauss, et.al 2018],

$$\limsup_{\varepsilon \to 0} \frac{R(d,\mu,\varepsilon)}{|\log \varepsilon|} \leq 1.$$

$$\therefore R(d,\mu,\varepsilon) \sim |\log \varepsilon| \quad (\varepsilon \to 0).$$

Calculation of the rate distortion dimension

Therefore,

$$\limsup_{\varepsilon\to 0} \frac{R(d,\mu,\varepsilon)}{\log(1/\varepsilon)} = \liminf_{\varepsilon\to 0} \frac{R(d,\mu,\varepsilon)}{\log(1/\varepsilon)} = 1.$$

Consequently,

$$\mathrm{rdim}([0,1]^{\mathbb{Z}}, \sigma, d, \mu) = 1.$$

Hence, because $\forall (x_m)_{m\in\mathbb{Z}} \in [0,1]^{\mathbb{Z}}$, $\varphi((x_m))_{m\in\mathbb{Z}} \leq 1$ and

the consequences of [M.Tsukamoto, 2020] and [Lindenstrauss, et.al 2018],

$$\mathrm{rdim}([0,1]^{\mathbb{Z}}, \sigma, d, \mu) + \int_{[0,1]^{\mathbb{Z}}} \varphi \; d\mu = 2.$$

The relation between the Gibbs measure and Double Variational Principle

$$\mathrm{mdim}([0,1]^{\mathbb{Z}},\sigma,\varphi)=\mathrm{rdim}([0,1]^{\mathbb{Z}},\sigma,d,\mu)+\int_{[0,1]^{\mathbb{Z}}}\varphi\ d\mu.$$

This indicates $\boldsymbol{\mu}$ satisfies
Double Variational Principle for mean dimension with potential.

A Gibbs measure on XY model μ

$\forall N\in\mathbb{N}$, $\forall A_1,A_2,\dots,A_N\in\mathscr{B}([0,1])$,

$$\boldsymbol{\mu}(A_1\times A_2\times\cdots\times A_N\times[0,1]^{\mathbb{N}})$$
$$:=\left(\frac{1}{\int_{[0,1]}e^{\varphi(x)}\ dx}\right)^N\int_{A_1}e^{\varphi(x_1)}\ dx_1\int_{A_2}e^{\varphi(x_2)}\ dx_2\cdots\int_{A_N}e^{\varphi(x_N)}\ dx_N.$$

- $\varphi:[0,1]^{\mathbb{Z}}\to[0,1]$; $\varphi((x_m)_{m\in\mathbb{Z}})=x_0$

Next goals

I try to

calculate both the mean dimension and the rate distortion dimension in the case of defining the measure with a potential which depends on $\boldsymbol{N} \in \mathbb{N}$ coordinates on the XY model.

AT Baraviera, LM Cioletti, Artur O Lopes, Joana Mohr, and Rafael Rigao Souza.
On the general one-dimensional xy model: positive and zero temperature, selection and non-selection.
Reviews in Mathematical Physics, Vol. 23, No. 10, pp. 1063–1113, 2011.

Rufus Bowen.
Equilibrium states and the ergodic theory of anosov diffeomorphisms.
Lecture notes in mathematics, Vol. 470, pp. 11–25, 1975.

Elon Lindenstrauss and Masaki Tsukamoto.
From rate distortion theory to metric mean dimension: variational principle.
IEEE Transactions on Information Theory, Vol. 64, No. 5, pp. 3590–3609, 2018.

References II

Masaki Tsukamoto.
Double variational principle for mean dimension with potential.
Advances in Mathematics, Vol. 361, p. 106935, 2020.

Peter Walters.
An introduction to ergodic theory, Vol. 79.
Springer Science & Business Media, 2000.