CIRM
Dynamique symbolique multidimensionnelle
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Homogeneity and stability of nonperiodic ground states

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Overview

Physical space: $\mathbf{Z}^d, d = 1, 2$

Configurations: $\Omega = \{1, ..., n\}^{\mathbf{Z}^d}$

Nonperiodic configurations in $\Omega \rightarrow$ ground-state configurations of some Hamiltonians

d=2, nonperiodic tilings (Robinson's tilings)
in general systems of finite type

d=1, substitution dynamical systems (Thue-Morse sequences),
Sturmian systems (Fibonacci sequences),
in general systems of minimal infinite type

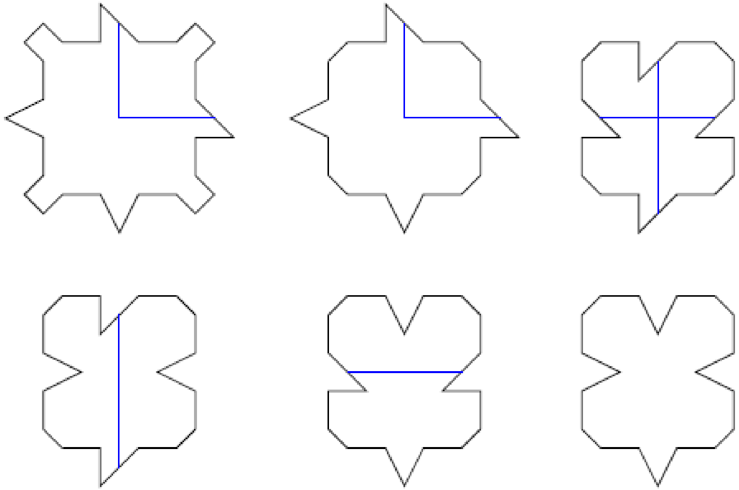
Main theme: stability of nonperiodic ground states

Main Open Problem

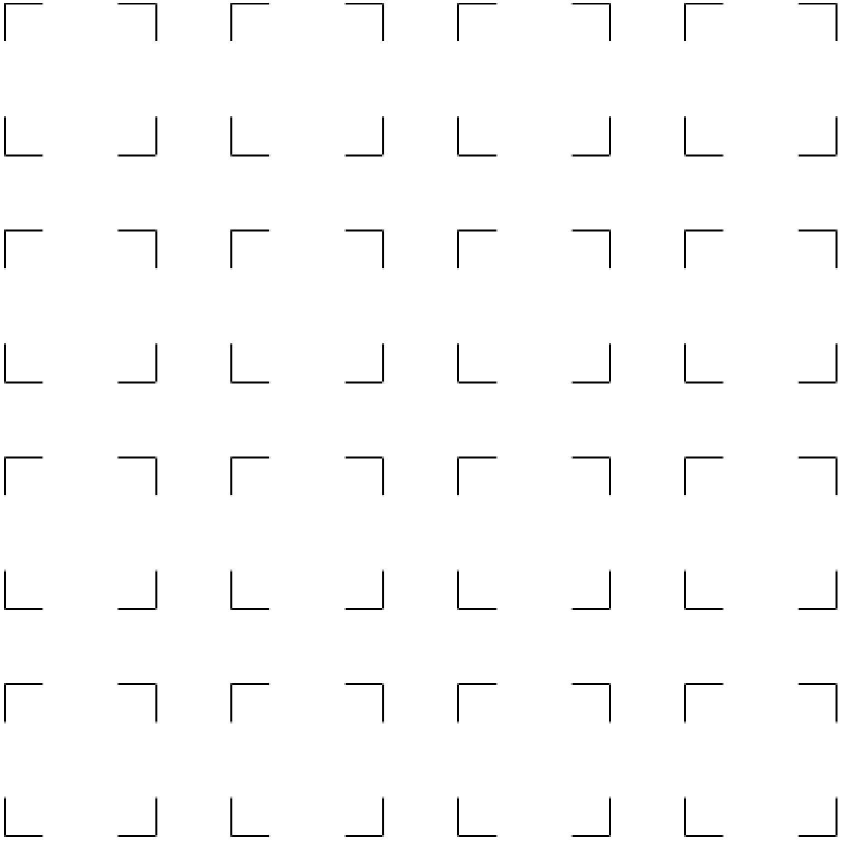
Does there exist a lattice-gas model with translation-invariant finite-range interactions without periodic ground-state configurations and non-periodic Gibbs state at low temperatures?

Raphael Robinson 1911 - 1995

6 (56) tiles which cover planes but only in a non-periodic way, 1971



Structure of an infinite tiling



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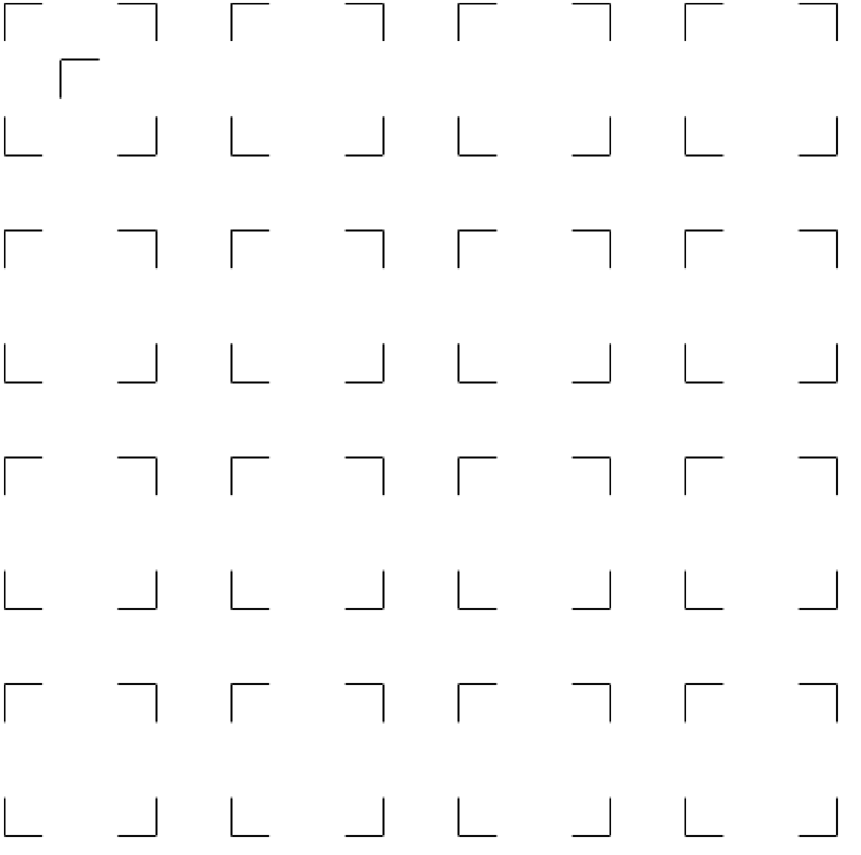

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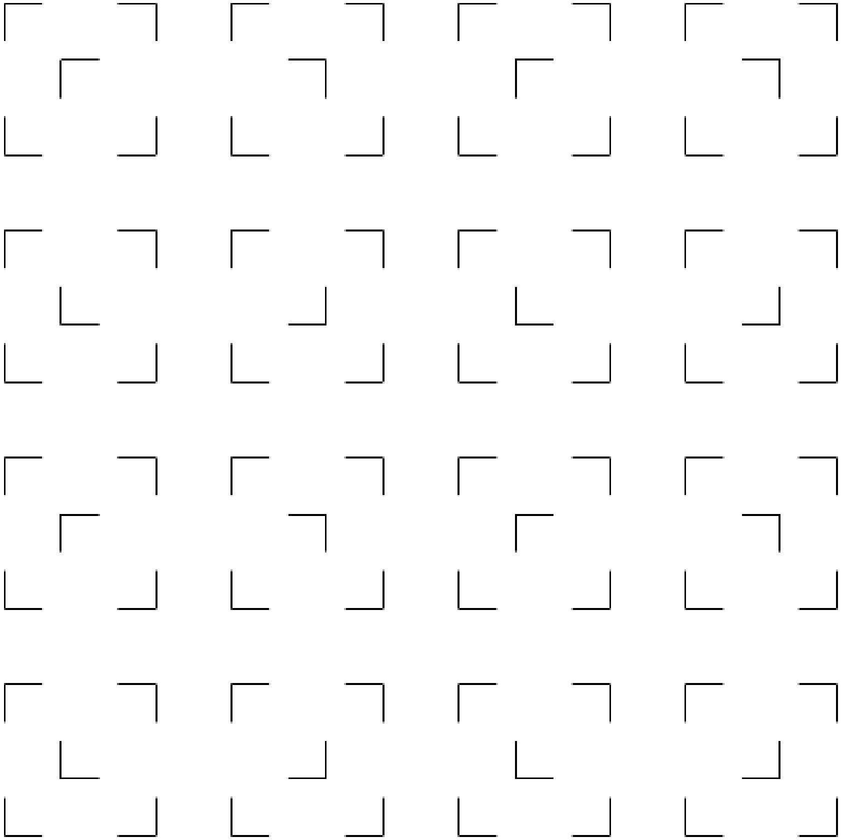

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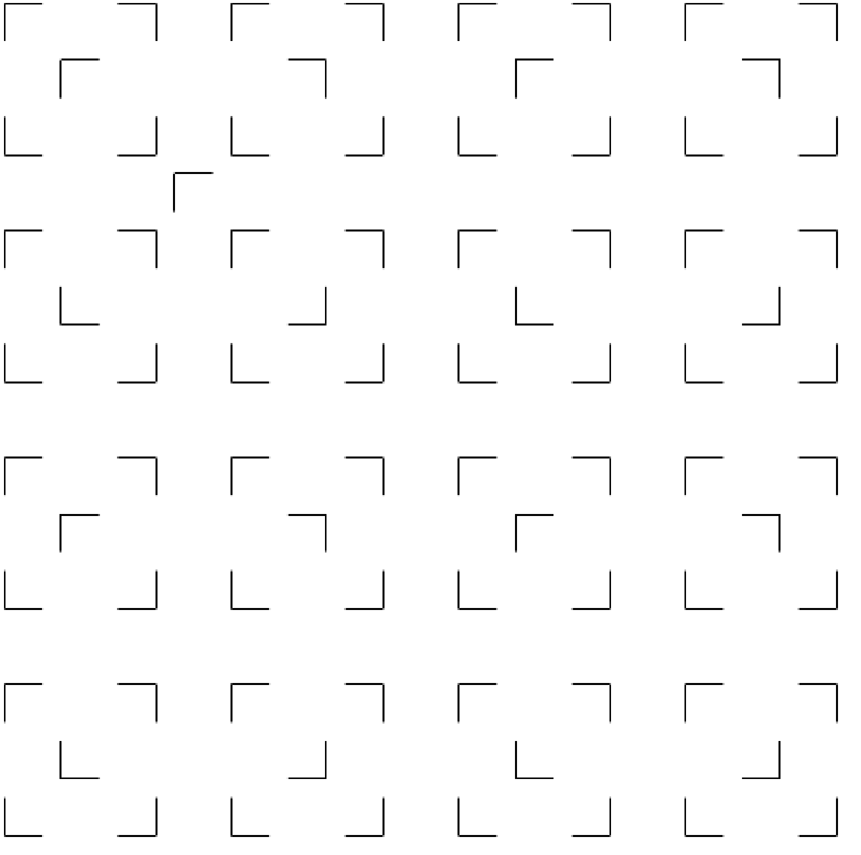

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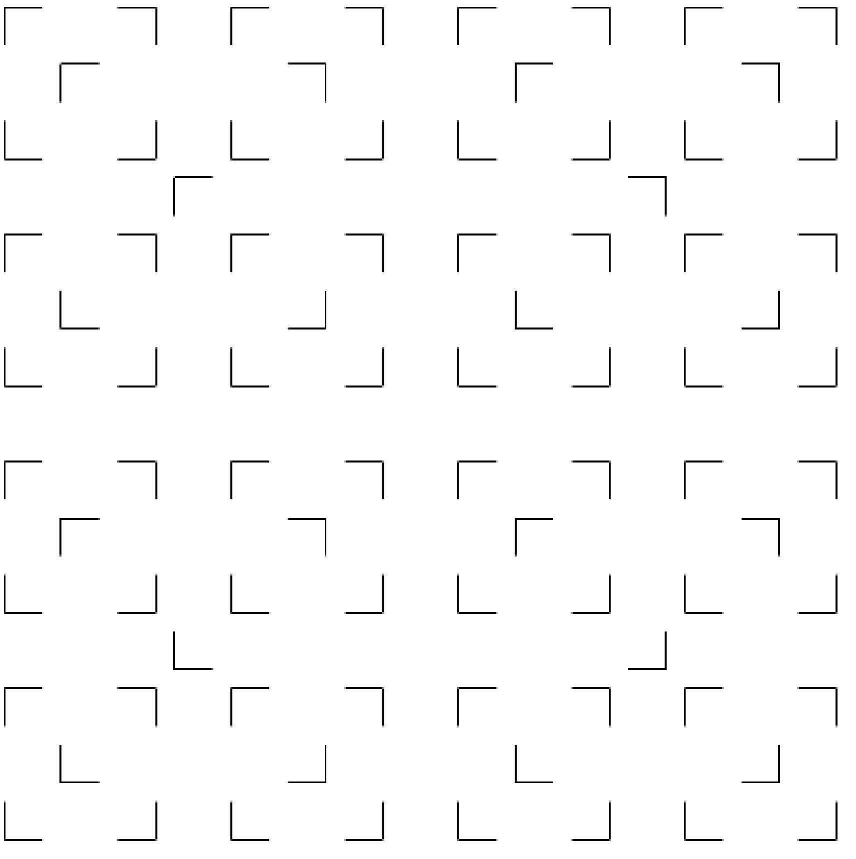

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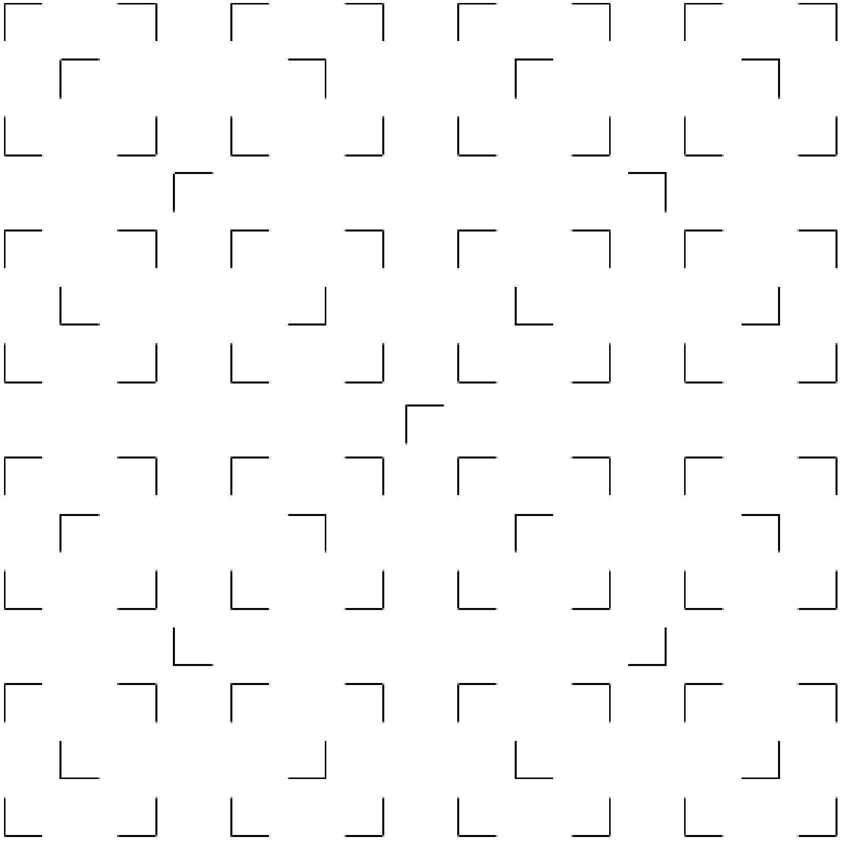

Configurations with period 2^{n+1} on sublattices $2^n Z^2$ $n \geq 1$

Global order from local rules

Robinson's tilings

All Robinson's tilings look the same. When you look at local subsets of the lattice you cannot say which Robinson's tiling it is.

Let us formalize this.

Let $X \in \Omega = \{1, ..., 56\}^{\mathbf{Z}^2}$ be a Robinson's tiling and T a translation on Ω,

$(T_a X)_i = X_{i-a}; a, i \in \mathbf{Z}^2$

$R = closure\{T_a X, a \in \mathbf{Z}^2\}$ in the product topology.

R is uniquely defined by a finite number of forbidden patterns – two neighboring tiles that do not match.

There exists the unique translation-invariant measure μ_R on R given by

$\mu_R = \lim_{L\to\infty} \frac{1}{L^2} \sum_{a \in \Lambda_L^0} \delta_{T_a X}$,

where $\delta_{T_a X}$ is a probability measure which assign 1 to $T_a X$.

(Ω, R, T, μ_R) is called a dynamical system of finite type.

Alternatively

we look at frequencies of finite patterns in any Robinson's tiling.

Let $X \in R$ and Λ_L^a be a square with the side lenght L and the center at $a \in \mathbf{Z}^2$, and $ar \in \{1, ..., \}^{\Lambda_{ar}}$ be a local pattern.

$$n_{ar}^L(X) = |\Lambda \subset \Lambda_L^a, \Lambda = T_b \Lambda_{ar}, b \in \mathbf{Z}^2; X(\Lambda) = ar(\Lambda_{ar})|$$

is the number of occurances of the pattern ar in Λ_L^a.

The following limit exists and is independent of X,

$$\omega_{ar} = \lim_{L \to \infty} \frac{n_{ar}^L}{L^2}.$$

This is the frequency of ar in any Robinson's tiling.

We obviously have

$$\mu_R(I_{ar}) = \omega_{ar}$$

for every $X \in R$, where $I_{ar}(X) = 1$ if $X(\Lambda_{ar}) = ar(\Lambda_{ar})$.

Strict Boundary Property of nonperiodic tilings (nonperiodic configurations in general) [1,2]

$X \in \Omega$ satisfies the **Strict Boundary Property** for a local pattern ar if there is a constant C_{ar} such that

$$|n_{ar}^{L}(X) - \omega_{ar} L^{2}| < C_{ar} L.$$

Such property is also called a rapid convergence to equilibrium of frequency of patterns [3,4].

In fact we need a following stronger version for local tilings.

Let $W \subset \Omega$ be a set of nonperiodic tilings for a given tiling set such that there is the unique translation-invariant probability measure supported by tilings.

W satisfies the **Strict Boundary Property** for a local pattern ar if for any local tiling Y of Λ_{L}^{0}, not necessarily extendable to the whole $\mathbf{Z}^{2}$

$$|n_{ar}^{L}(Y) - \omega_{ar} L^{2}| < C_{ar} L.$$

Open Problem

Are there any non-periodic tilings which satisfy the Strict Boundary Property for any local tiling?

Classical lattice-gas models based on tilings

tiles $\rightarrow$ particles

matching rules $\rightarrow$ interactions

If two tiles do not match, then the energy of interaction between corresponding particles is positive, say 1, otherwise the energy is zero

ground-state configurations – configurations which minimize the energy

forbidden patterns have positive energy

tilings $\rightarrow$ ground-state configurations

ground-state configurations have zero energy

Robinson's tilings → classical lattice-gas model
with finite-range,translational-invariant interactions
and without periodic ground-state configurations
with the unique translation-invariant
ground-state measure supported
by non-periodic ground-state configurations,

C. Radin, Phys. Lett. A,1986 [5]

Systems of finite type are defined by the absence of a finite number of finite patterns.
We construct hamiltonians by assigning a positive energy to forbidden patterns.

systems of finite type → lattice-gas models with finite-range interactions

ergodic measures → ground-state measures

However, there is a classical lattice-gas model with finite–range interactions which does not correspond to any system of finite type.

J. Miękisz, J. Stat. Phys. 1998 [6]

Strict Boundary Property for local excitations

$\Omega = \{1, ..., n\}^{\Lambda}$, Λ is a finite subset of $\mathbf{Z}^d$.

$\Phi_{\Lambda} : \Omega_{\Lambda} \to \mathbf{R}$ are interactions.

$H_{\Lambda} = \sum \Phi_{V \subset \Lambda}$ is Hamiltonian of our lattice-gas model.

Y is a **local excitation** of X, $X, Y \in \Omega$, $Y \sim X$
if $|i \in \mathbf{Z}^d, Y(i) \neq X(i)| < \infty$

For $Y \sim X$, $H(Y|X) = \sum_{\Lambda \subset \mathbf{Z}^d} (\Phi_{\Lambda}(Y) - \Phi_{\Lambda}(X))$
is a **relative Hamiltonian**.

$X \in \Omega$ is a **ground-state configuration**
if for any $Y \sim X$, $H(Y|X) \geq 0$

$H_{\Lambda} = \sum \Phi_{V \subset \Lambda}$ is **non-frustrated**
if there exists $X \in \Omega$ such that
for every Λ, $\Phi_{\Lambda}(X) = \min_{Y \in \Omega} \Phi_{\Lambda}(Y)$.

We may assume that the minimum is equal to 0
and all other interactions are equal to 1.

Strict Boundary Property for local excitations

X is a ground-state configuration, $Y \sim X$,
then $H(Y|X) = B(Y)$ is the number of broken bonds,
that is the number of Λ such that $\Phi_\Lambda(Y) = 1$.

$n_{ar}(Y|X)$ is the difference of the number of appearances of ar in Y and in X.

A classical lattice-gas model satisfies
Strict Boundary Property for local excitations
if for any local pattern ar, there exists C_{ar}
such that for every $Y \sim X$

$$|n_{ar}(Y|X)| < C_{ar} B(Y).$$

Theorem (J. Miękisz, J. Stat. Phys.1997 [1])

A unique ground-state measure of a finite-range Hamiltonian is stable against small perturbations of interactions of range smaller than r if and only if Strict Boundary Property is satisfied for patterns of diameter smaller than r

Theorem (J. Miękisz, C. Radin, Phys. Lett. 1986) [7])

Robinson's ground state is not stable against
an arbitrarily small chemical potential favoring
one type of particles corresponding to an arrow tile.

Low-temperature stability

Let H be a finite-range Hamiltonian with a unique non-periodic ground-state measure (based for non-periodic tilings for example)

Let $X \in \Omega$ be one of its ground-state configurations, Λ_L be a square of size L centered at the origin.

$$\Omega^X_{\Lambda_L} = \{Y \in \Omega, Y(\Lambda^c_L) = X(\Lambda^c_L)\}$$

$$\rho^X_{T,\Lambda_L} = \frac{e^{-\beta H(Y|X)}}{\sum_{Y \in \Omega^X_{\Lambda_L}} e^{-\beta H(Y|X)}}$$

is a finite-volume Gibbs measure, $\beta = 1/T$, T is the temperature of the system.

One can show that

$$\rho^X_{T,\Lambda_L} \to \rho^X_T$$

We would like to prove that

$$\rho^X_T(Y \in \Omega, Y(0) \neq X(0)) > 1 - \epsilon(\beta)$$

$$\epsilon(\beta) \to 0 \;\; as \;\; \beta \to \infty$$

ρ^X_T would be then a non-periodic Gibbs state, a perturbation of non=periodic ground-state measure.

Open Problem

Does there exist a lattice-gas model with translation-invariant finite-range interactions without periodic ground-state configurations and non-periodic Gibbs state at positive temperature?

so far

Theorem (JM, 1990)

There is a decreasing sequence of temperatures, T_n, such that if $T < T_n$, then there exists a Gibbs state with a period at least 2×6^n in both directions.

nonperiodic Gibbs states

van Enter, Miękisz, CMP 1990 [9]
nonperiodic Gibbs states for summable interactions in d=1

van Enter, Miękisz, Zahradnik, JSP 1998 [10]
nonperiodic Gibbs states for exponentially decaying interactions in d=3

One-dimensional lattice-gas models

It is known that one-dimensional lattice-gas models with finite-range interactions have at least one periodic ground-state configurations, see [11,12,13].

Therefore to force nonperiodicity we have to consider models with infinite-range potentials.

Thue-Morse sequences

substitutions

0 → 01
1 → 10

0

01

0110

01101001

0110100110010110

Let X be a Thue-Morse sequence, $X \in \{0,1\}^Z$

$$TM = closure(\{T_a X, a \in Z\})$$

$$\mu_{TM} = lim_{\Lambda \to Z} \frac{1}{|\Lambda|} \Sigma_{a \in \Lambda} \delta_{T_a} X$$

$$\lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} f(T^i(X)) = \int f d\mu_{TM}$$

(TM, T, μ_{TM}) is a uniquely ergodic dynamical system, Michael Keane, 1968

Characterizations of Thue-Morse sequences

Goal: looking for the minimal set of forbidden patterns

Gottschalk and Hedlung, 1964

TM is uniquely characterized by the absence of BBb,

where B is any word and b is its first character, 0 or 1

TM is uniquely characterized by the absence of infinitely many 4-point patterns
(Gardner, Radin, Miękisz, van Enter, J. Phys. A,1989) [14]

4-body Hamiltonian with Thue-Morse sequences as unique ground states

Theorem The following spin Hamiltonian has the Thue-Morse measure μ_{TM} as its unique ground state.

$$H = \sum_{j=-\infty}^{\infty} \sum_{p=0}^{\infty} \sum_{r=0}^{\infty} V(j,p,r)$$

$$= \sum_{j=-\infty}^{\infty} \sum_{p=0}^{\infty} \sum_{r=0}^{\infty} \exp[-(r+p)^2](\sigma_j + \sigma_{j+2^r})^2(\sigma_{j+(2p+1)2^r} + \sigma_{j+(2p+2)2^r})^2$$

Fibonacci sequences

substitutions

$0 \to 01$
$1 \to 0$

0	1
01	2
010	3
01001	5
01001010	8
0100101001001	13

$(F, T, \mu F)$ *is a uniquely ergodic system*

density of 0's = $\dfrac{2}{1+\sqrt{5}} = \gamma$

Another construction of Fibonacci sequences

Let $0 \leq \phi \leq 2\pi$ and let T_γ be a rotation by $2\pi\gamma$ on a unit circle.

If $T_\gamma^n(\phi) \in [0, 2\pi\gamma)$ mod 2π then let $a(n) = 0$, otherwise $a(n) = 1$.

Most homogeneous configurations

Definition 3. Let $X \in \{0,1\}^Z$ and $x_i \in Z$ be a position of a i-th 1 (a particle) in X. X is **most homogeneous** if there exists a sequence of natural numbers d_j such that $x_{i+j} - x_i \in \{d_j, d_j + 1\}$ for every $i \in Z$ and $j \in N$.

Fibonacci sequences are most homogeneous

$$d_j = [j(2+\gamma)], \ \ \gamma = \frac{2}{1+\sqrt{5}}, \ \ [y] \ \mathit{is\ an\ integer\ value\ of}\ y$$

Forbidden distances between 1's = 1,4,9,12,17,22,25,30,…

We also forbid 000

and construct non-frustrated Hamiltonian with Fibonacci sequences as only ground-state configurations

In an anologous way we construct two-body interactions for any Sturmian system.

van Enter, Koivusalo, Miękisz, J. Stat. Phys. 2021 [15]

Unfortunately they are unstable with respect to arbitrarily small chemical potentials which favor particles.

Głodkowski, Miękisz, preprint, 2024 [16]

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Mathematical models of quasicrystals

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