Some advances in the selection problem in ergodic optimization

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April 2, 2024

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Thermodynamic formalism for dynamical systems

Orbits

Goal of dynamics : describe orbits for (X, T), X compact metric space, $T : X \to X$ continuous.
Usually too difficult. Then describe only almost all trajectories. For that purpose, need to select some special invariant measure = aim for thermodynamic formalism.

Definition

(X, T)= Dynamical System, $A : X \to \mathbb{R}$ continuous called *potential*.

1. Pressure = $P(A) := \sup_{\mu\ T-inv} \left\{h_\mu(T) + \int A d\mu\right\}$,
2. Any measure realizing maximum = equilibrium state for A.

$h_\mu(T)$= Kolmogorov entropy.

Theorem (Ruelle)

If (X, T) uniformly hyperbolic and A Hölder continuous, then there exists a unique equilibrium state for A. Furthermore, $\mathcal{P}(\beta) = P(\beta.A)$, $\beta \in \mathbb{R}$ is analytic.

my settings : subshift of finite type + A Lipschitz. Easy to check $\mathcal{P}(\beta)$ is convex with asymptote at $+\infty$:

Set μ_β for unique equilibrium state for $\beta.A$.

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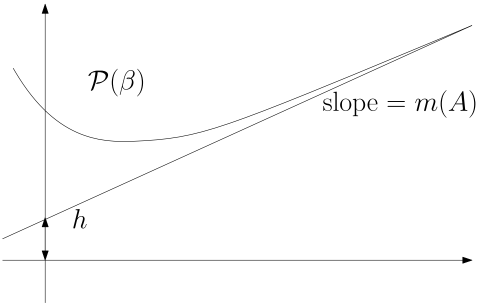

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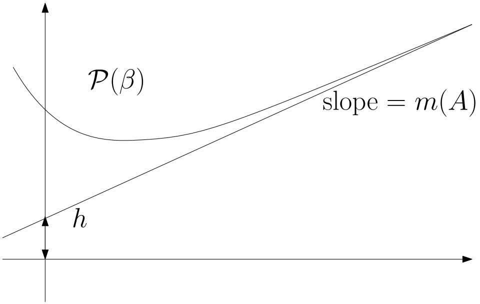

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On the road to ergodic optimization

Here $m(A) = \max_{\mu \; T-inv} \left\{ \int A d\mu \right\}$.

Definition

An invariant measure is said to be A-maximizing if it realizes maximum.

h= maximal entropy among A-maximizing measures.

Theorem (Folklore)

Any accumulation for μ_β is A-maximizing with entropy h.

Question

Is there convergence for μ_β ? If yes why ? If not why ? How selects different accumulation points.

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One example (Baraviera-L-Lopes)

Considering in $\Sigma := \{0, 1, 2\}^{\mathbb{N}}$ and

$$A(x) = \begin{cases} -d(x, 0^\infty) & \text{if } x \in [0] \\ -3d(x, 1^\infty) & \text{if } x \in [1] \\ -\alpha < 0 & \text{if } x \in [2]. \end{cases} .$$

Theorem

Let ρ be the golden mean $\rho := \dfrac{1+\sqrt{5}}{2}$. Then

1. *for $\alpha > 1$, μ_β converges to $\frac{1}{2}(\delta_{0^\infty} + \delta_{1^\infty})$ as β goes to $+\infty$,*
2. *for $\alpha = 1$, μ_β converges to $\frac{1}{1+\rho^2}(\rho^2\delta_{0^\infty} + \delta_{1^\infty})$ as β goes to $+\infty$,*
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A good way to see ergodic optimization

The Aubry set

On one island = can walk without being wet. To go from 1 island to another, need to go to the water = to pay.

Definition

Seeing A as a cost, Aubry set = some islands in X, invariant for T where we can walk without paying.

Good tool for that = calibrated subaction.
Remind transfert operator gives all informations in thermodynamic formalism:

$$L(g)(x) = \sum_{y,\ T(y)=x} e^{\beta A(y)} g(y).$$

$e^{\mathcal{P}(\beta)}$= unique single dominating eigenvalue, H_β= eigenfunction, ν_β= eigenmeasure (for L^*), $\mu_\beta = H_\beta \otimes \nu_\beta$.

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Calibrated Subaction

$$e^{\mathcal{P}(\beta)} H_\beta(x) = \sum_y e^{\beta.A(y)} H_\beta(y)$$

$$e^{\beta\left(\frac{1}{\beta}\mathcal{P}(\beta)+\frac{1}{\beta}\log H_\beta(x)\right)} = \sum_y e^{\beta(A(y)+\frac{1}{\beta}\log H_\beta(y))}$$

$$\frac{1}{\beta}\log + \beta \to +\infty$$

$$m(A) + V(x) = \max_y \{V(y) + A(y)\}$$

Rewritten as $\boxed{A(y) + V(y) - V \circ \underbrace{T(y)}_{=x} - m(A) \leq 0}$.

Up to coboundary + constant =no change for equilibrium states, A always non positive. Aubry set ≈ bigger invariant set where $A \equiv 0$.

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First Result

Theorem (L-Mengue, in progress)

If the Aubry set is a subshift of finite type, then the pressure function converges exponentially fast to the asymptote:

$$\lim_{\beta\to+\infty} \frac{1}{\beta} \log(\mathcal{P}(\beta) - h) = \gamma < 0.$$

Furthermore, γ is identified as the unique eigenvalue for the matrix of cost between the irreducible components for the Aubry set, within the Max-Plus formalism

Max-plus : $\oplus = \max$, $\otimes = +$.
$\lim \frac{1}{\beta} \log(e^{a\beta} + e^{c\beta}) = \max\{a, c\}$. $+ \to \max$.
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Another good way to see ergodic optimization

Pertubations

Aubry set has zero measure for every μ_β.

Question

Can we however see emerge the properties of the convergence/accumulation point from the transfer operator at $\beta < +\infty$, as island emerging when tide is going out ?

Corollary

If we know that for some A μ_β converge, can we get the same result if we change a little bit A ?

$$L(g)(x) = \sum_{y,\ T(y)=x} e^{\beta A(y)} g(y).$$

Need control on Lipschitz norm for the potential: Blows up as $\beta \to +\infty$.

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Result 2

Only for $X = \{0,1\}^{\mathbb{N}}$= full 2 shift & A= special Walters potential. First Theorem holds.

Theorem (L-Mengue, in progress)

If B_β is a family of Lipschitz potentials going exponentially to 0 faster than $e^{\gamma.\beta}$ (= speed for pressure), then,

$$\lim_{\beta\to+\infty} \mu_{\beta.A+B_\beta} = lim_{\beta\to+\infty}\mu_{\beta.A},$$

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More precisely:

Set Ω=Aubry set = subshift of finite type.

Set $S(x, y)$= Mañé potential

$$S(x, y) = \lim_{\varepsilon \to 0} \sup\{S_n(A)(z), \sigma^n(z) = y, \ d(x, y) \leq \varepsilon\}.$$

Then,

1. Irreducible components for Ω as subshift of finite type coincide with equivalence classes for
$$S(x, y) + S(y, x) = 0.$$
2. Set $\Omega = \underbrace{\Omega_1 \sqcup \ldots \sqcup \Omega_k}_{\text{entropy } h} \sqcup \underbrace{\Omega_{k+1} \sqcup \ldots \sqcup \Omega_l}_{\text{entropy} < h}$.
3. Set $a_{ij} := \sup_{x \in \Omega_j} \sup_{y \notin \Omega_i, \ \sigma(y) \in \Omega_i} S(x, y)$.

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Then

If $V =$ accumulation point for $\frac{1}{\beta}\log(H_\beta)$ = one selected subaction.

$$\gamma \otimes \begin{bmatrix} V(\Omega_1) \\ \vdots \\ V(\Omega_k) \end{bmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ \vdots & \vdots & \vdots & \vdots \\ a_{k1} & a_{k2} & \dots & a_{kk} \end{pmatrix} \otimes \begin{bmatrix} V(\Omega_1) \\ \vdots \\ V(\Omega_k) \end{bmatrix}.$$

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And next ?

Questions

1. Can we extend this when Ω not of finite type ?
2. Can we get convergence for $\frac{1}{\beta}\log H_\beta$ to some special eigenvector ?
3. What is the role of components with smaller entropy ?
4. Can we detect/decide convergence in function of the phase diagram that gives γ ?

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Thank You