Metallic Mean Wang tiles

talk at CIRM
April 1st, 2024
S. Labbé

I: self-similarity, aperiodicity, minimality arxiv:2312.03652

II: the dynamics of an aperiodic computer chip arxiv:2403.03197

① Definition (from Part II)

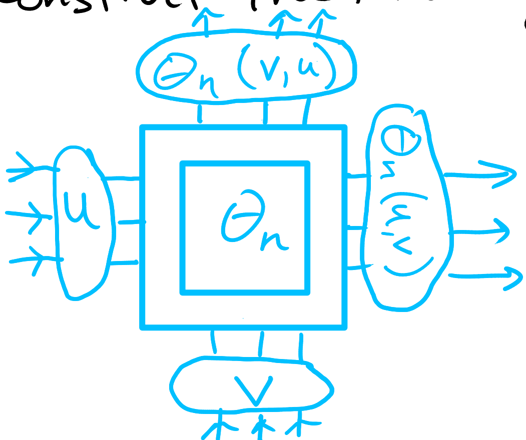
Let $n \geq 1$ integer. Let

$$V_n = \{(v_0, v_1, v_2) \in \mathbb{N}^3 \mid 0 \leq v_0 \leq v_1 \leq v_2 \leq n+1 \text{ and } v_1 \leq 1\}$$

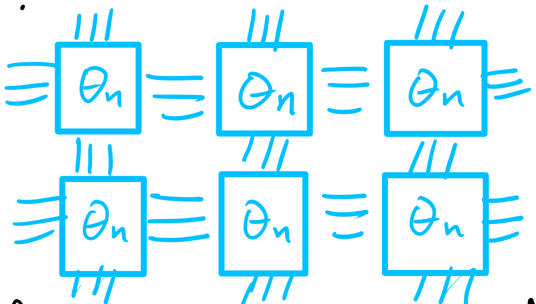
$$\begin{aligned} \Theta_n : \quad V_n \times V_n &\longrightarrow \mathbb{Z}^3 \\ (u_0,u_1,u_2),(v_0,v_1,v_2) &\longmapsto (r_0, r_1, r_2) \end{aligned}$$

where
$$\begin{cases} r_0 = u_0 \\ r_1 = \begin{cases} v_2 - n & \text{if } u_0 = 0, \\ 1 & \text{otherwise} \end{cases} \\ r_2 = \begin{cases} v_1 + u_0 & \text{if } v_0 = 0, \\ u_2 + 1 & \text{otherwise.} \end{cases} \end{cases}$$

and we construct the following computer chip



We connect many copies of this chip in large rectangles:



The different instances of the computer chip define a set of Wang tiles:

$$C_n = \left\{ \begin{array}{ccc} & \Theta_n(v,u) & \\ u & \boxed{} & \Theta_n(u,v) \\ & v & \end{array} \;\middle|\; u,v \in V_n \text{ and } \Theta_n(u,v), \Theta_n(v,u) \in V_n \right\}$$

Note:
$$\begin{aligned} \Theta_n(1u_1u_2, 1v_1v_2) &= 1\,1\,(u_2+1) =: 11\overline{u_2} \\ \Theta_n(0u_1u_2, 0v_1v_2) &= 0\,(v_2-n)\,v_1 \\ \Theta_n(0u_1u_2, 1v_1v_2) &= 0\,(v_2-n)(u_2+1) \\ \Theta_n(1u_1u_2, 0v_1v_2) &= 1\,1\,(v_1+1) \end{aligned}$$

There are

$W_n = \left\{ 11i \,\boxed{}^{11\bar{j}}_{11j}\, 11\bar{i} \;\middle|\; \begin{matrix} 1 \le i \le n \\ 1 \le j \le n \end{matrix} \right\} \subset C_n$ (white tiles),

$B_n = \left\{ 00i \,\boxed{}^{111}_{11n}\, 00\bar{i} \;\middle|\; 0 \le i \le n \right\} \subset C_n$ (blue tiles),

$Y_n = \left\{ 01i \,\boxed{}^{112}_{11\bar{n}}\, 01\bar{i} \;\middle|\; 1 \le i \le n \right\} \subset C_n$ (yellow tiles),

$G_n = \left\{ 00i \,\boxed{}^{111}_{11\bar{n}}\, 01\bar{i} \;\middle|\; 0 \le i \le n \right\} \subset C_n$ (green tiles),

$A_n = \left\{ 01i \,\boxed{}^{112}_{11n}\, 00\bar{i} \;\middle|\; 1 \le i \le n \right\} \subset C_n$ (antigreen tiles),

$J_n = \left\{ 00n \,\boxed{}^{000},\; 01n \,\boxed{}^{001},\; 01\bar{n} \,\boxed{}^{011} \right\}$
$\times \left\{ \boxed{}_{00n}\, 000,\; \boxed{}_{01n}\, 001,\; \boxed{}_{01\bar{n}}\, 011 \right\} \subset C_n$ (junction tiles).

Also $\widehat{B_n} \cup \widehat{Y_n} \cup \widehat{G_n} \cup \widehat{A_n} \subset C_n$ where $\widehat{c \,\boxed{}^{b}_{d}\, a} = d \,\boxed{}^{a}_{c}\, b$

<u>Proposition</u> C_n is the union of these subsets.

<u>Lemma</u> In a valid configuration $w: \mathbb{Z}^2 \to C_n$, the following tiles do not appear:

$A_n \cup \left\{ 00n \,\boxed{}^{111}_{11n}\, 00\bar{n},\; 00n \,\boxed{}^{000}_{01\bar{n}}\, 011 \right\}$ and their image under $\wedge$.

<u>Definition</u> Let $T_n \subset C_n$ be the remaining set of $n^2 + 2(n+n+n+1) + 7 = n^2 + 6n + 9 = (n+3)^2$ tiles.

<u>Theorem</u> When $n=1$, the set T_1 of 16 tiles is equivalent to the Ammann set of 16 Wang tiles.

<u>Proof</u> $1 \to 112$, $2 \to 111$, $3 \to 001$,
$4 \to 011$, $5 \to 012$, $6 \to 000$. $\square$

② Self-similarity (from Part I)

Theorem $\forall n \geq 1$, $\Omega_n = \{x : \mathbb{Z}^2 \to \mathcal{T}_n \text{ valid}\}$ is self-similar

i.e., $\Omega_n = \overline{\omega_n(\Omega_n)}^{\sigma}$ where ω_n is a recognizable 2-dimensional substitution.

Proof sketch

- Every configuration $x \in \Omega_n$ can be decomposed uniquely into "return blocks" with a unique junction tile at the bottom left of each block.
- These return blocks are instances of the θ_n chip! that is, $\exists$ map $\tau_n : V_n \to (V_n)^*$ s.t.

$$\{\text{return blocks}\} = \left\{ \begin{array}{ccc} & \tau_n(b) & \\ \tau_n(c) & \square & \tau_n(a) \\ & \tau_n(d) & \end{array} \;\middle|\; c \begin{array}{c} b \\ \square \\ d \end{array} a \in \mathcal{C}_n \right\}$$

Corollary $\forall n \geq 1$, Ω_n is aperiodic

Theorem $\forall n \geq 1$, the self-similarity is primitive

Theorem $\forall n \geq 1$, Ω_n is minimal (if $\phi \neq X \subset \Omega_n$ closed and shift-invariant then $X = \Omega_n$)

③ Explicit construction of valid configurations (from Part II)

Let $\beta > 0$ t.q. $\beta - \beta^{-1} = n \geq 1$ is an integer.

That is, β is a root of $x^2 - nx - 1$

$$\beta = \frac{n + \sqrt{n^2+4}}{2} = n + \cfrac{1}{n + \cfrac{1}{n + \dots}}$$

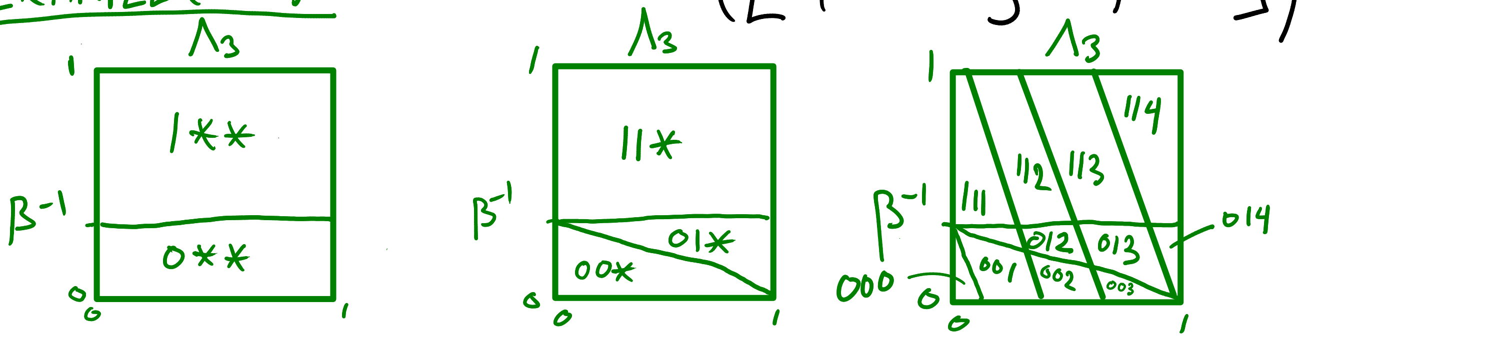
is the $\underline{n\text{-th metallic mean}}$ (Wikipedia, OEIS).

Alternate names:
- "silver means", in Schroeder 1991
- "metallic", de Spinadel 99
- "noble", BG 2013
- "Quadratic Pisot Unit"

Let $\Lambda_n : [0,1)^2 \to \mathbb{Z}^3$

$$(x,y) \mapsto \begin{pmatrix} \lfloor & y - \beta^{-1} + 1 \rfloor \\ \lfloor \beta^{-1}x & + y - \beta^{-1} + 1 \rfloor \\ \lfloor \beta x & + y - \beta^{-1} + 1 \rfloor \end{pmatrix}^t$$

EXAMPLE ($n=3$)



Lemma $\Lambda_n([0,1)^2) \subset V_n \setminus \{00\bar{n}\}$

Let $R_\beta : [0,1) \to [0,1)$, $x \mapsto \{x+\beta\}$ where $\{x\} = x - \lfloor x \rfloor$

Theorem $\forall n \geq 1$ $\forall (x,y) \in [0,1)^2$

$$\Lambda_n(R_\beta^j(y), R_\beta^i(x))$$

$$\Lambda_n(R_\beta^{i-1}(x), R_\beta^j(y)) \quad \boxed{t_{i,j,x,y}} \quad \Lambda_n(R_\beta^i(x), R_\beta^j(y)) \qquad \in T_n \subset C_n$$

$$\Lambda_n(R_\beta^{j-1}(y), R_\beta^i(x))$$

and the map
$$c_{(x,y)} : \mathbb{Z}^2 \to T_n, \quad (i,j) \mapsto t_{i,j,x,y} \qquad \in \Omega_n$$
is a valid configuration over the Wang tiles T_n.

4 A factor map (from Part II)

Theorem Let $d = (0,-1,1)$. The map

$$\Phi_n : \Omega_n \longrightarrow \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$$
$$w \longmapsto \lim_{k\to\infty} \frac{1}{2k+1} \sum_{i=-k}^{k} \begin{pmatrix} \langle \frac{1}{n} d, \mathrm{RIGHT}(w_{0,i}) \rangle \\ \langle \frac{1}{n} d, \mathrm{TOP}(w_{i,0}) \rangle \end{pmatrix}$$

is an almost 1-to-1 factor map (continuous and onto) satisfying $\Phi_n \circ \sigma^k = R_n^k \circ \Phi_n$ $\forall k \in \mathbb{Z}^2$ where

$$R_n : \mathbb{Z}^2 \times \mathbb{T}^2 \to \mathbb{T}^2, \quad (k, x) \mapsto x + \beta k.$$

Note Also, $\forall (x,y) \in [0,1)^2$, $\Phi_n(c_{(x,y)}) = (x,y)$.

5 Equations (from Part II)

There are equations satisfied by the tiles and the boundary of their rectangular tilings, but they are not sufficient to prove non-periodicity as it is nicely done for Kari/Culik tilings, unfortunately.

Metallic mean Wang tiles

Sébastien Labbé, CIRM, April 1st, 2024

(Some figures from [Lab23] and [Lab24])

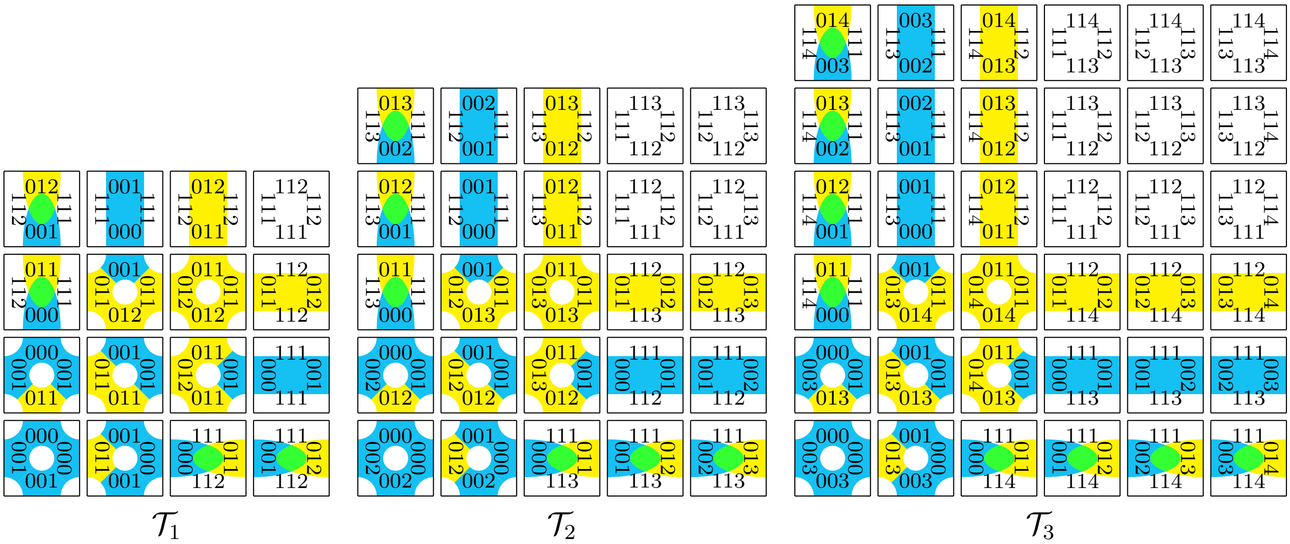


FIGURE 1. Metallic mean Wang tile sets $\mathcal{T}_n$ for $n = 1, 2, 3$.

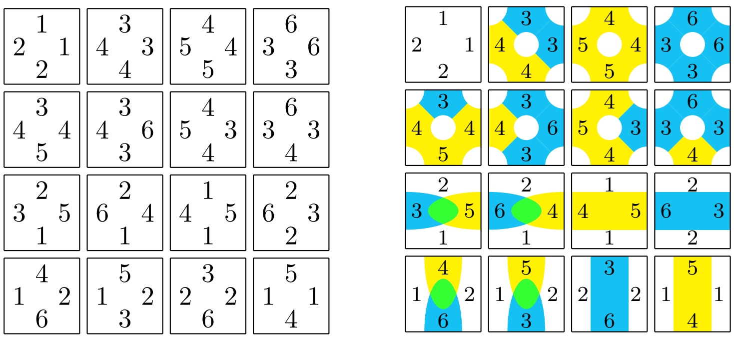


FIGURE 2. The Ammann set [GS87, p.595, Figure 11.1.13] is equivalent to $\mathcal{T}_1$.

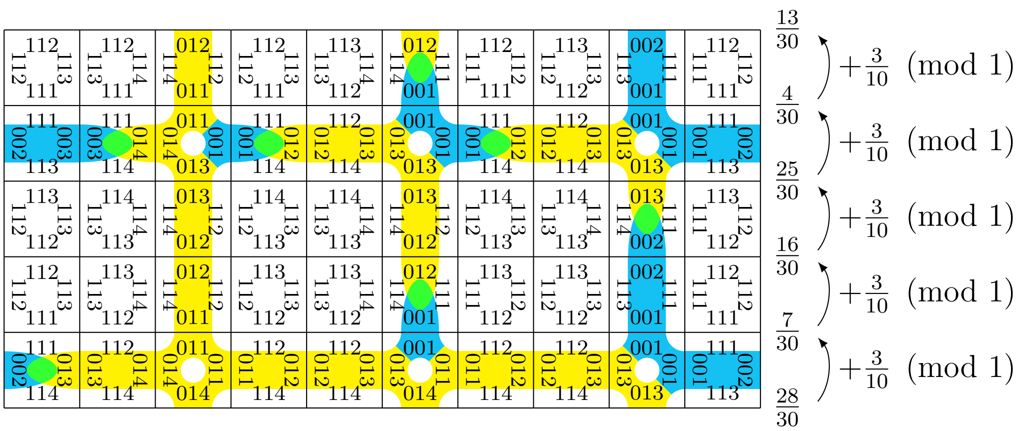


FIGURE 3. A valid rectangular tiling with the set $\mathcal{T}_n$ with $n = 3$. The numbers in the right margin are the average of the inner products $\langle \frac{1}{n} d, v\rangle$ over the vectors v appearing as top (or bottom) labels of a horizontal row of tiles and where $d = (0, -1, 1)$.

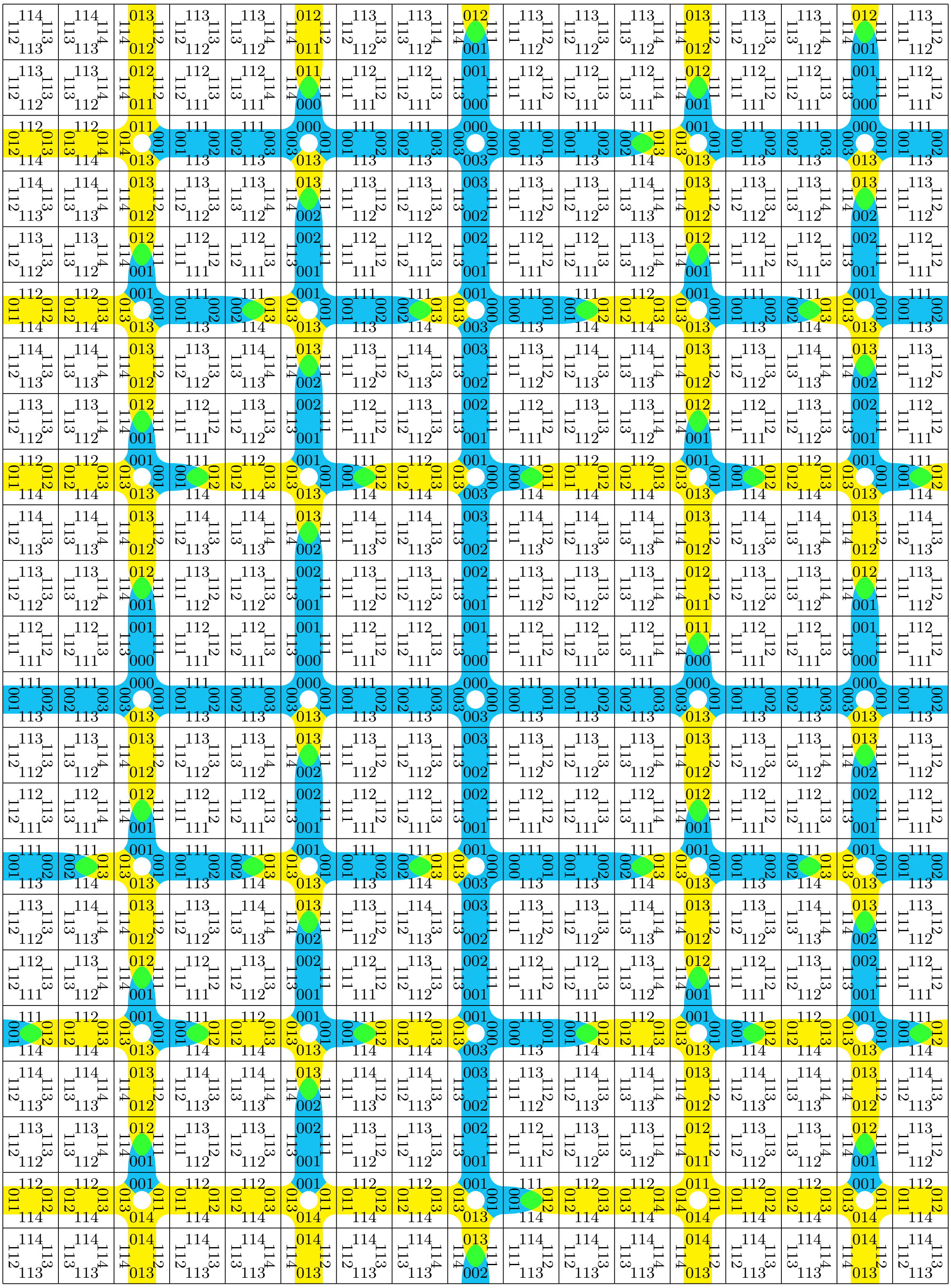


FIGURE 4. A valid 17×23 pattern with Wang tile set $\mathcal{T}_3$.

REFERENCES

[GS87] Branko Grünbaum and G. C. Shephard. *Tilings and patterns*. W. H. Freeman and Company, New York, 1987.

[Lab23] Sébastien Labbé. Metallic mean Wang tiles I: self-similarity, aperiodicity and minimality. December 2023. `arxiv:2312.03652` .

[Lab24] Sébastien Labbé. Metallic mean Wang tiles II: the dynamics of an aperiodic computer chip. March 2024. `arxiv:2403.03197` .