Phase transitions on one-dimensional symbolic systems

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Phase Transitions

A phase transition is observed when one follows an evolution of a system depending on continuous external factors and a sharp change of the behavior of the system happens.

Fundamental goal in statistical physics: to understand the mechanism of this phenomenon.

One way to study phase transitions is by utilizing tools of thermodynamic formalism.

Thermodynamic Formalism

Let $\phi : X \to \mathbb{R}$ be a continuous potential associated with a symbolic dynamical system (X, σ) over a finite alphabet.

The topological pressure of ϕ is defined by

$$P_{\text{top}}(\phi) = \sup_{\mu \in \mathcal{M}} \{\mathsf{h}_\mu + \int \phi \, d\mu\},$$

where $\mathcal{M}$ is the set of all σ-invariant probability measures and h_μ is the measure-theoretic entropy of μ.

From the statistical physics point of view, $P_{\text{top}}(\phi)$ corresponds to the minimum of the free energy $E_\mu = -(\mathsf{h}_\mu + \int \phi \, d\mu)$.

A measure $\mu \in \mathcal{M}$ which minimizes the free energy (i.e. $P_{\text{top}}(\phi) = \mathsf{h}_\mu + \int \phi \, d\mu$) is called an equilibrium state of ϕ.

The Pressure Function

We introduce a parameter $t > 0$ (interpreted as the inverse temperature of the system) and study the equilibrium states of the potential $t\phi$.

When the temperature changes, the equilibrium of the system changes as well.
A phase transition refers to a qualitative change of the properties of a dynamical system as a result of the change in temperature.
Intuitively, this means co-existence of several equilibria at the same temperature.

We are interested in the values of t for which potential $t\phi$ has more than one equilibrium state.

Our main tool is the pressure function of ϕ, which is the map $p_\phi : t \mapsto P_{\text{top}}(t\phi)$.

Co-existence of several equilibria vs. regularity of the pressure

Walters (1992)

P_{top} is Gateaux differentiable at $\phi \iff \phi$ has a unique equilibrium state

- If the pressure function $p_\phi(t)$ is not differentiable at t_0 then $t_0\phi$ has at least two equilibrium states.
- Non-uniqueness of equilibrium states for $t_0\phi$ does not imply non-differentiability of $p_\phi(t)$ at t_0.
  Leplaideur (2015): there is a continuous ϕ on a mixing subshift of finite type such that $p_\phi(t)$ is analytic on some interval, but uniqueness of equilibrium states fails for two distinct values of t in that interval.
- $p_\phi(t)$ in not differentiable at $t_0 \iff t_0\phi$ has two equilibrium states with distinct entropies.

Types of Phase Transitions

- First-order phase transitions correspond to the points of non-differentiability of the pressure function. In this case we have co-existence of equilibrium states with different entropies.
- Higher-order phase transition correspond to the points where the pressure function is differentiable, but not analytic. Although non-uniqueness of equilibrium states may not appear at such points, they still indicate a sharp change in some property of the system.
- Another type of phase transitions is when the pressure function is differentiable (even analytic), but uniqueness of equilibrium states fails.
- A freezing phase transition occur when for some positive temperature value the systems reaches its ground state and then ceases to change.

Lack of Phase Transitions

Ruelle(1968)

If X is a mixing subshift of finite type then the pressure functional P_{top} acts real analytically on the space of Hölder potentials. Moreover, for each such potential there is only one equilibrium state.

In particular, when ϕ is Hölder:

- the pressure function $p_\phi(t)$ is analytic,
- $t\phi$ has a unique equilibrium state for any t,

and hence there are no phase transitions of any kind.

In order to allow the possibility of phase transitions one needs to consider potential functions that are merely continuous.

Questions of Interest

For continuous potentials on a one-dimensional shift space over a finite alphabet

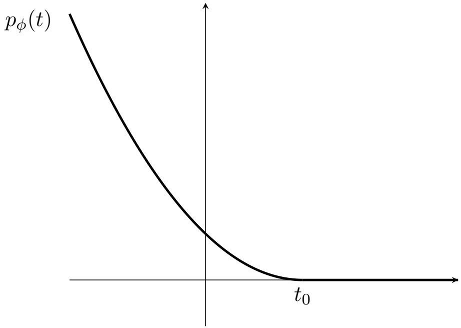
1. Are there any restrictions on a possible number and frequency of phase transitions?
2. Can any type of phase transitions occur?
3. What is the shape of the pressure function after the transitional value?
4. What is the behavior at zero temperature? (When do we have a freezing phase transition?)

Hofbauer Potentials

Lopes (1990s): a family of potentials on $\{0,1\}^{\mathbb{N}}$ exhibiting a first- or higher-order phase transition at some value t_0. (Built upon the previous work of Hofbauer)

The family of potentials is defined by $\phi_\gamma(x) = \gamma \log \frac{k}{k+1}$ if $d(x, \bar{0}) = 2^{-k}$.

The pressure function looks like this:



Methods:

Ruelle-Perron-Frobeneus operator;

direct evaluation of the pressure.

Higher-Order Phase Transitions

Leplaideur (2015): there are potentials on the transitive subshift of finite type given by the graph below which exhibit one of the following properties:

(a) there are two higher order phase transitions;
(b) the pressure is strictly convex after the transitional value;
(c) pressure function is analytic but two equilibrium states coexist.

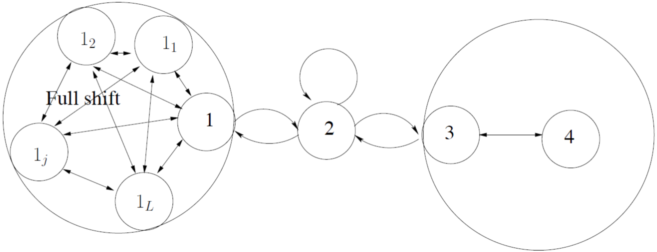


FIGURE: R.Leplaideur, *Chaos: butterflies also generate phase transitions*, J. Stat. Phys.(2015)

Multiple First-Order Phase Transitions

K., Quas, Wolf (2021)

For any given $\alpha > 0$ and any (finite or infinite) increasing sequence of positive real numbers $(t_n) \subset (\alpha, \infty)$ there is a continuous potential $\phi : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$ which has first-order phase transitions precisely at t_n.

Idea: build a potential ϕ so that the equilibrium states of $t\phi$ move among a sequence of disjoint subshifts when t changes.

- Consider the subshift X_n generated by $\{(\underbrace{00\ldots01}_{2^n}), (\underbrace{11\ldots10}_{2^n})\}$.
- For a suitable sequence of values (c_n) set ϕ to be constant c_n on each X_n
  We need: $P_{\text{top}}(t_n\phi|_{X_n}) = P_{\text{top}}(t_n\phi|_{X_{n+1}})$ and $P_{\text{top}}(t_n\phi|_{X_k}) < P_{\text{top}}(t_n\phi|_{X_n})$ whenever $k \notin \{n, n+1\}$.
- Make ϕ drop sharply outside $\bigcup X_n$ and force the equilibrium measures at all values of t to be supported on $\bigcup X_n$.

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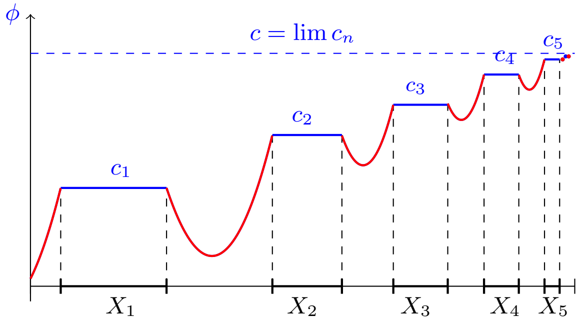

Define:
$\phi_n(x)$ in terms of $d(x, X_n)$ and $\phi(x) = \sup \phi_n(x)$.
Main Difficulty:
To ensure that the drop-off is sufficiently steep so that for any ergodic μ not supported on $\bigcup X_k$ we have $h_\mu + t_n \int \phi \, d\mu < P_{\text{top}}(t_n \phi|_{X_n})$.

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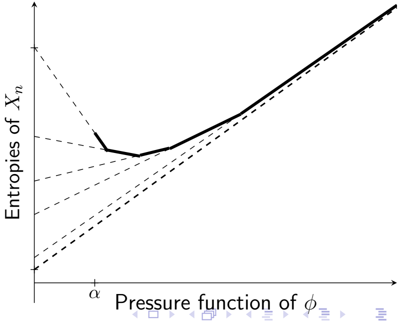
<u>Our Methods:</u>

Use the **pin-sequence space** $Z \subset \{0,1\}^{\mathbb{Z}}$:
for a pair (x, z) a 1 in z pins exactly the place in x where one block from X_ns ends and another one begins.

<u>We obtain:</u>

$p_\phi(t) = h(X_n) + c_n t$ for $t \in [t_{n-1}, t_n]$.



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Since $p_\phi(t)$ is convex Lipschitz, at most countably phase transitions are possible.
Taking $\{t_n\}$ to be infinite we get infinitely many phase transitions.
A convex function may have a countable dense set of points of non-differentiability.
Is it feasible for ϕ to have a dense set of phase transitions?

K. and Quas (2023)

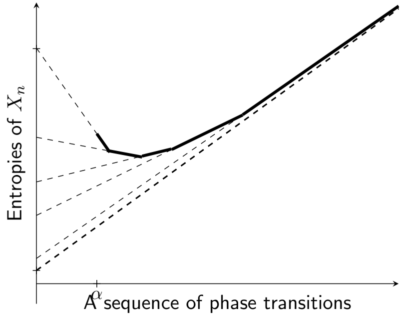
For any given countable set $S \subset (\alpha, \infty)$ there is a continuous potential ϕ whose phase transitions in (α, ∞) occur precisely at points in S.

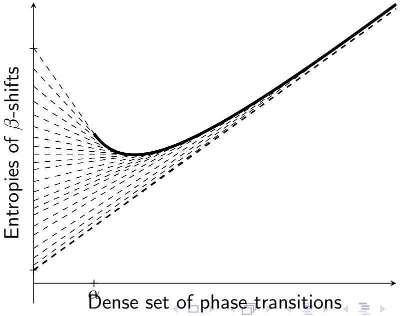
This is a corollary to our main result.

Multiple Phase Transitions

K. and Quas, (2022)

Let $\alpha > 0$ and let $f(t)$ be *any* convex Lipschitz function on (α, ∞) which has an asymptote $at + b$, $b \geq 0$. Then there exists a full shift on a finite alphabet and a continuous potential ϕ such that $p_\phi(t) = f(t)$ for all $t \in (\alpha, \infty)$.





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Consequences:

- There are no restrictions on a possible number and frequency of first order phase transitions
- One can also have any combination of higher order phase transitions
- Any shape of the pressure function after the transitional value is possible as long as it is convex.

What about an analytic pressure function with multiple equilibrium states?

Cardinality of Equilibrium States

The next theorem provides a flexible way of constructing systems of potentials with varying cardinalities of the equilibrium states.

K. and Quas, (2022)

Let $f(t)$ be a strictly convex differentiable function on (α, ∞) with a slant asymptote. Then for any $\ell \in \mathbb{N}$ and any upper semi-continuous function $N\colon (\alpha, \infty) \to \{1, \dots, \ell, \infty\}$, there exists a full shift on a finite alphabet and a continuous potential function ϕ such that

- $p_\phi(t) = f(t)$ for all $t \in (\alpha, \infty)$;
- the cardinality of the set of ergodic equilibrium states for $t\phi$ is exactly $N(t)$.

So far we were free to chose shifts, potentials and equilibrium states to obtain interesting phenomena.

Question: Can we specify the sets of equilibrium states at a phase transition?

Ground States

X is a shift space on a finite alphabet, $\phi : X \to \mathbb{R}$ is a continuous potential.

What is the behavior of $p_\phi(t)$ as $t \to \infty$ (the temperature is lowered to zero)?

The weak*-accumulation points of equilibrium states of $t\phi$ as $t \to \infty$ are called the ground states of ϕ.

Ground states are, in general, not unique even for Hölder potentials on a full shift [Chazottes, Hochman (2010)].

Although there might be multiple ground states, they all have the same free energy!

If μ_∞ is a ground state for ϕ then
$\int \phi\, d\mu_\infty = \max \left\{ \int \phi\, d\mu : \mu \in \mathcal{M} \right\}$ and $h_{\mu_\infty} = \max \left\{ h_\mu : \mu \text{ maximizes } \int \phi\, d\mu \right\}$

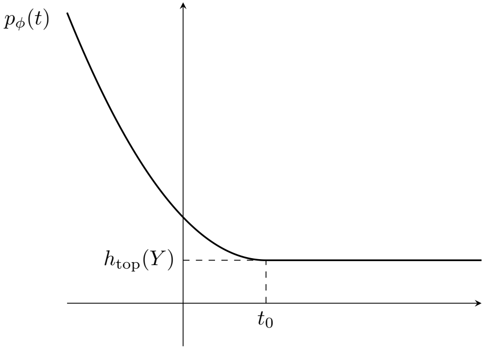
$p_\phi(t) = h_{\mu_{t\phi}} + t \int \phi\, d\mu_{t\phi} \underset{\text{large } t}{\approx} h_{\mu_\infty} + t \int \phi\, d\mu_\infty$, the asymptote is $h_{\mu_\infty} + t \int \phi\, d\mu_\infty$

Freezing Phase Transitions

We say that a potential ϕ has a freezing phase transition if its pressure function $p_\phi(t)$ reaches its asymptote at a finite value t_0.

Settings: X is a full (two- or one-sided) shift on a finite alphabet
$Y \subset X$ is a proper subshift

Question: Which continuous $\phi : X \to \mathbb{R}$ has a freezing phase transition at some temperature t_0 with ground states being the measures of maximal entropy of Y?



Potentials with Freezing Phase Transitions

This could be achieved using a continuous potential ϕ with the following properties:

1. $\phi(x) = 0$ for $x \in Y$ and $\phi(x) < 0$ for any $x \notin Y$;
2. $P_{\text{top}}(\phi) = h_{\text{top}}(Y)$;
3. The set of equilibrium states for ϕ is exactly the set of measures of maximal entropy for Y.

We now describe several different methods of obtaining such a potential.

General Existence

There is a very general result that guaranties the existence of ϕ.

K. and Thompson (2021)

Given a compact topological dynamical system (X, T) with positive entropy and upper semicontinuous entropy map, and any closed invariant subset $Y \subset X$ there exist a continuous potential $\phi : X \to \mathbb{R}$ satisfying (1), (2) and (3).

The proof is based on the theory of tangent functionals and relies on Israel's theorem.

It does not provide any information on the type of potentials which would generate a freezing phase transition.

Two-Sided Shifts (based on work with A. Quas)

Suppose X is a two-sided full shift on a finite alphabet and Y is any subshift of X.
Define

$$\phi(x) = \begin{cases} 0, & \text{if } x \in Y \\ a_j, & \text{if } \operatorname{dist}(x, Y) = 2^{-j} \end{cases},$$

where (a_j) is an increasing sequence of negative numbers which converges to zero.

Clearly ϕ is continuous, vanishes on Y, and negative off Y.

Using pinning sequence techniques we can choose (a_j) to satisfy properties (2)-(3).

$$a_j = h_{\text{top}}(Y) - \frac{\log n_j}{j} - \frac{2\log j}{j} - \frac{\log e}{j}, \text{ where } n_j = \#j\text{-words in } Y$$

This gives a range of potentials which generate a freezing phase transition on Y.

Our methods do not carry over to one-sided shifts.
We have different approach, however the sequence (a_j) is less optimal.

One-Sided Shifts (based on work with D. Thompson)

Suppose X is a one-sided full shift and Y is a *subshift of finite type*.

We have a construction of a potential ϕ satisfying properties (1)-(3).

The definition of ϕ is the same as before, with the sequence (a_j) given by

$$a_j = h_{\text{top}}(Y) - \frac{1}{k}\log n_k - \frac{c}{\sqrt{j}} \quad \text{for} \quad \frac{k(k-1)}{2} \le j < \frac{k(k+1)}{2},$$

where n_k is the number of words of length k in Y.

Our techniques are related to the ones by Hofbauer and Lopes.

Thank you!