Directional Expansiveness for Tiling Dynamical Systems

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Expansiveness for $\mathbb{R}^d$-actions

Let (X, d) be a compact metric space, with metric d.

Definition 1

An $\mathbb{R}^d$-action T is called *weakly expansive* if there is an expansive constant $\delta > 0$ so that for any $x, y \in X$, if $d(T^{\boldsymbol{t}}x, T^{\boldsymbol{t}}y) < \delta$ for all $\boldsymbol{t} \in \mathbb{R}^d$, then $y = T^{\boldsymbol{t}_0}x$ for some $||\boldsymbol{t}_0|| < \delta$.

An $\mathbb{R}^d$-action T is called *strongly expansive* if there is an expansive constant $\delta > 0$ so that for any $x, y \in X$, there is a homeomorphism $\boldsymbol{h} = \boldsymbol{h}_{x,y} : \mathbb{R}^d \to \mathbb{R}^d$ with $\boldsymbol{h}(\mathbf{0}) = \mathbf{0}$ such that if $d(T^{\boldsymbol{t}}x, T^{\boldsymbol{h}(\boldsymbol{t})}y) < \delta$ for all $\boldsymbol{t} \in \mathbb{R}^d$, then $y = T^{\boldsymbol{t}_0}x$ for some $||\boldsymbol{t}_0|| < \delta$.

Directional expansiveness for $\mathbb{R}^d$-actions

For $1 \leq e < d$, an e-dimensional direction for T is a subspace $V \subseteq \mathbb{R}^d$. For $t > 0$ define $V^t := \bigcup_{\mathbf{r} \in V} B_t(\mathbf{r})$.

Definition 2

1. An $\mathbb{R}^d$-action T is called *weakly expansive* in the direction V if for all $\epsilon > 0$ there is an *expansive constant* $\delta > 0$ and expansive radius $t > 0$ such that if for some $x, x' \in X$, $d(T^{\boldsymbol{t}}x, T^{\boldsymbol{t}}x') < \delta$ for all $\boldsymbol{t} \in V^t$, then $x' = T^{\boldsymbol{t}_0}x$ for some $\boldsymbol{t}_0 \in \mathbb{R}^d$ with $||\boldsymbol{t}_0||_\infty < \epsilon$.

Definition 3

2. An $\mathbb{R}^d$-action T is called *strongly expansive* in the direction V if for all $\epsilon > 0$ there is an *expansive constant* $\delta > 0$ and expansive radius $t > 0$ so that for any $x, x' \in X$, there exists a continuous homeomorphism $\boldsymbol{h} = \boldsymbol{h}_{x,x'} : V \to V$ with $\boldsymbol{h}(\mathbf{0}) = \mathbf{0}$ such that if $d(T^{\boldsymbol{t}}x, T^{\boldsymbol{h}(\boldsymbol{t})}x') < \delta$ for all $\boldsymbol{t} \in V^t$, then $x' = T^{\boldsymbol{t}_0}x$ for some $\boldsymbol{t}_0 \in \mathbb{R}^d$ with $||\boldsymbol{t}_0||_\infty < \epsilon$.

Lemma 4

Strong expansiveness in a direction V implies weak expansiveness in a direction V.

Proposition 1

If an $\mathbb{R}^d$-action T is weakly (or strongly) expansive in a direction V, then it is weakly (or strongly) expansive in any direction W with $V \subseteq W$. If particular, T is weakly (or strongly) expansive as an $\mathbb{R}^d$-action.

Finite local complexity full tiling space

Fix $d = 2$.

Definition 5

Let p be a finite set of inequivalent tiles in $\mathbb{R}^2$. Assume there is given finite set p^2 of "*allowed*" 2-tile patches by tiles from p.

Let X_p denote the set of all tilings by tiles from p, and such that every 2-tile patch in $x \in X_p$ is in p^2.

Assuming $X_p \neq \emptyset$, we call X_p *finite local complexity* (abbreviated FLC) *full tiling space* on $\mathbb{R}^2$.

Example 6

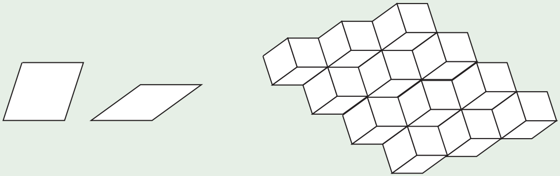


Figure: Unmarked rhombic tiles and a tiling.

Let p be the set of 20 tiles obtained by all rotations of these two tiles by multiples of $2\pi/10$. Let p^2 be the "edge-to-edge" condition.

The corresponding FLC full tiling space X_p is clearly non-empty.

$\mathbb{R}^2$-FLC tiling dynamical system

Definition 7

Let X_p be a FLC full tiling space on $\mathbb{R}^2$ with $\mathbb{R}^2$ translation action T. An FLC tiling space X is a closed T-invariant subset $X \subseteq X_p$. We call the translation action T on X a *FLC tiling dynamical system*.

Theorem 8 (Frank-Sadun, 2012)

For T a FLC tiling dynamical system, weak expansiveness implies strong expansiveness.

Theorem 9

Any $\mathbb{R}^2$-FLC tiling dynamical system T is strongly expansive.

Directional expansiveness for $\mathbb{R}^2$-FLC tiling dynamical systems

For $t > 0$ and $x \in X$ define an infinite strip $x[V^t]$ by

$$x[V^t] = \{\tau \in x : \tau \cap V^t \neq \emptyset\}.$$

Proposition 2

Let (X, T) be an FLC tiling dynamical system. Then V is a weakly expansive direction if and only if there is a $t > 0$ so that whenever $x, y \in X$ satisfy $x[V^t] = y[V^t]$, then $x = y$.

Proposition 3

If X is an FLC tiling space on $\mathbb{R}^2$, then a direction V is weakly expansive if and only if it is strongly expansive.

Penrose tiling

This version is based on a prototile set p called *arrowed rhomb Penrose tiles*.

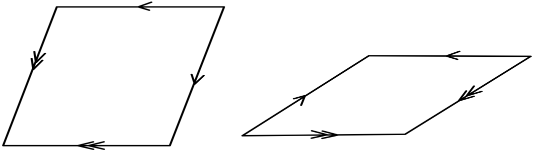


Figure: Penrose tiles ($72^\circ, 108^\circ$ and $36^\circ, 144^\circ$, respectively. We look at rotation of these)

X_p is the corresponding the FLC full tiling space. The tilings $x \in X_p$ are called the *arrowed rhomb Penrose tilings*.

Main result

Theorem 10

In the Penrose tiling dynamical systems (X_p, T), there are exactly 5 non expansive directions. They are the directions perpendicular to 5^{th} roots of unity.

Multigrid tiling

Multigrid

Let $\mathbf{v}_0, \mathbf{v}_1, \cdots, \mathbf{v}_4$ be the 5^{th} roots of unity, defined by

$$\mathbf{v}_j = [Re(e^{2\pi ij/5}), Im(e^{2\pi ij/5})], j = 0, 1, \cdots, 4 \tag{1}$$

For $0 \leq j < 5$, let

$$\ell_j(u) = \{\mathbf{t} \in \mathbb{R}^2 : \langle \mathbf{v}_j, \mathbf{t} \rangle = u\} = \{\mathbf{v}_j\}^{\perp} + u\mathbf{v}_j,$$

be a *line in a direction j* and let

$$L_j(u) = \bigcup_{k \in \mathbb{Z}} \ell_j(u + k).$$

Multigrid tiling

Multigrid (Continued)

Now, take $\mathbf{u} = (u_0, u_1, \cdots, u_4) \in \mathbb{R}^5$ and define the *multigrid* by

$$y(\mathbf{u}) = L_0(u_0) \cup L_1(u_1) \cup \cdots \cup L_4(u_4).$$

We call these 5-grid tilings. The tiles have at most 5 edges and parallel to some $\mathbf{v}_j^\perp$.
Let

$$Y = \{y(\mathbf{u}) : \mathbf{u} \in \mathbb{R}^5\}$$

be the set of all such grid tilings.

In this case, there are infinitely many prototiles.

5-grid tiling

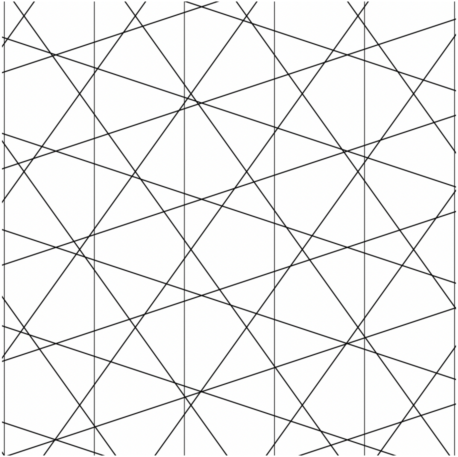


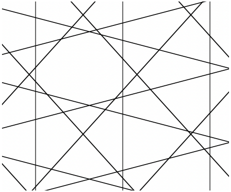
Figure: $y(\mathbf{u}) = L_0(u_0) \cup L_1(u_1) \cup \cdots \cup L_4(u_4)$

Nonsingular and singular tilings

Recall $y(\mathbf{u})$ is 5-grid tiling defined as before.

Definition 11

We call a 5-grid tiling $y(\mathbf{u}), \mathbf{u} \in \mathbb{T}^5$ *nonsingular* if no more than 2 grid lines cross at any point, and otherwise call it *singular*. We denote the nonsingular vectors by $\mathbb{T}^5_n$.



$\mathbb{R}^2$-action K on $\mathbb{T}^5$

Definition 12

Define a 5×2 matrix $V = \begin{bmatrix} \mathbf{v}_0 \\ \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_4 \end{bmatrix}$ with row vectors $\mathbf{v}_j$.

Define a $\mathbb{R}^2$-action $K = \{K^{\mathbf{t}}\}_{\mathbf{t} \in \mathbb{R}^2}$ on $\mathbb{T}^5$ by

$$K^{\mathbf{t}}\mathbf{u} = \mathbf{u} + V\mathbf{t} \mod 1. \tag{2}$$

Penrose condition

Penrose condition

We define the *Penrose condition* for $\mathbf{u} \in \mathbb{T}^5$ by

$$\mathbb{T}_0^5 = \{\mathbf{u} \in \mathbb{T}^5 : u_0 + \cdots + u_4 = 0 \mod 1\}, \tag{3}$$

with $\mathbb{T}_{0,n}^5 = \mathbb{T}_0^5 \cap \mathbb{T}_n^5$.

Later, we will see $\mathbf{u} \in \mathbb{T}_0^5$ correspond to Penrose tilings.

Let $L^{\mathbf{t}} = K^{\mathbf{t}}|_{\mathbb{T}_0^5}$.

Lemma 13 (Robinson (1996))

The $\mathbb{R}^2$-action $L^{\mathbf{t}}$ is minimal and uniquely ergodic.

$\mathbb{T}_0^5 \cong \mathbb{T}^4$ by making $u_3 = -(u_0 + u_1 + u_2 + u_4)$. $L^{\mathbf{t}}$ is a minimal rotation.

Duality

Definition 14

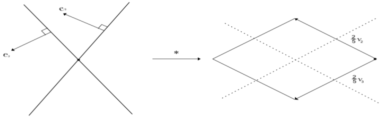
We say a tiling y^* is the *dual* of a grid tiling y if there are 1:1 correspondences,

$$\text{vertices of } y \leftrightarrow \text{faces of } y^*$$

$$\text{faces of } y \leftrightarrow \text{vertices of } y^*$$

$$\text{edges of } y \leftrightarrow \text{edges of } y^*$$

We call the duality geometric if the edges of y are perpendicular to the corresponding edges of y^*.



Dual to $y(\mathbf{u})$

Theorem 15 (de Bruijn, 1981)

If $\boldsymbol{u} \in \mathbb{T}^5_{0,n}$ then there is a unique geometric dual $y^*(\boldsymbol{u}) \in X_p$ with all edges length $\frac{2}{5}$, that is a Penrose tiling in X_p.

Dual to $y(\mathbf{u})$

Robinson (1996) showed that there is a unique translation $\boldsymbol{t}(\boldsymbol{u}) = \frac{2}{5}V^T\boldsymbol{u} \in \mathbb{R}^2$ so that $x(\boldsymbol{u}) = T^{\boldsymbol{t}(\boldsymbol{u})}y^*(\boldsymbol{u})$ satisfies $x(L^{\boldsymbol{t}}\boldsymbol{u}) = T^{\boldsymbol{t}}x(\boldsymbol{u})$.

Let $X_n = \{x(\boldsymbol{u}) : \boldsymbol{u} \in \mathbb{T}^5_{0,n}\}$. The map $\varphi : X_n \to \mathbb{T}^5_{0,n}$ satisfying $\varphi(x(\boldsymbol{u})) = \boldsymbol{u}$ is continuous (in the tiling to topology).

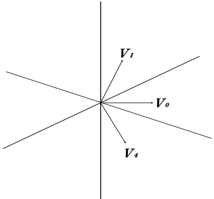
Type A

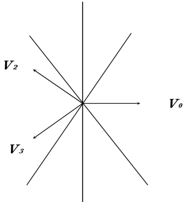
Let $\mathbf{u} \in \mathbb{T}_0^5 \backslash \mathbb{T}_{0,n}^5 := \mathbb{T}_{0,s}^5$ call $y(\mathbf{u})$ singular.

Definition 16

For $\mathbf{u} \in \mathbb{T}_{0,s}^5$ and assume all the multiple crossings in $y(\mathbf{u})$ are 3-fold. There are two *kinds*. In the *first kind*, two lines cross with angle $2\pi/5$, bisected by the third line. In the *second kind* lines cross with angle $4\pi/5$.

All 3-fold crossings occur along a single grid line, called a *spine*. We call $y(\mathbf{u})$ *Type A*.

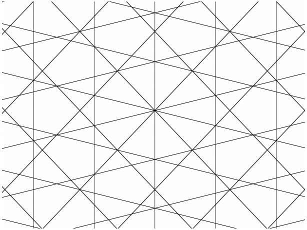




Type B

Definition 17

If there is a 5-fold crossing in $y(\boldsymbol{u})$ we call it *Type B*. Then there are infinite sequences of 3-fold crossings along each of the lines through the 5-fold crossing.



Penrose tiling and its dual

Let $X = \overline{X}_n$. The mapping φ extends to X.

Theorem 18 (Robinson)

The mapping $\varphi : X \to \mathbb{T}_0^5$ is an almost 1:1 factor mapping with

$$\mathrm{card}(\varphi^{-1}(\boldsymbol{u})) = \begin{cases} 1 & \text{if } \boldsymbol{u} \text{ is nonsingular,} \\ 2 & \text{if } \boldsymbol{u} \text{ is Type A,} \\ 10 & \text{if } \boldsymbol{u} \text{ is Type B.} \end{cases}$$

There are 2 ways to fill in a spine with Penrose tiles and 10 ways to fill in the dual of a grid tiling within a 5-fold crossing. These are obtained as the duals of $y(\boldsymbol{u})$ for small perturbations of $\boldsymbol{u} \in \mathbb{T}_{0,s}^5$.

Type A and its dual

The two kinds of crossings correspond to the vertices in dual up to its rotation.

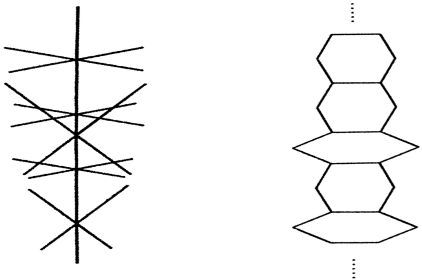


Figure: This is generally called a *worm*

3-fold crossings with small perturbations

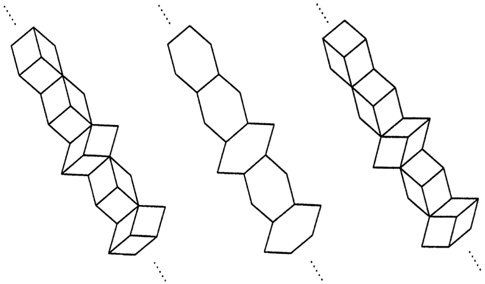
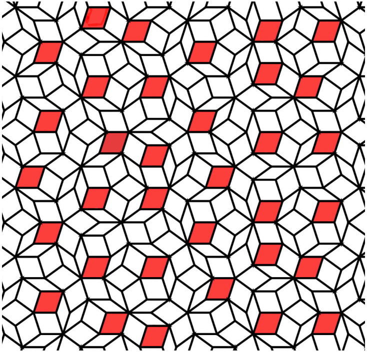


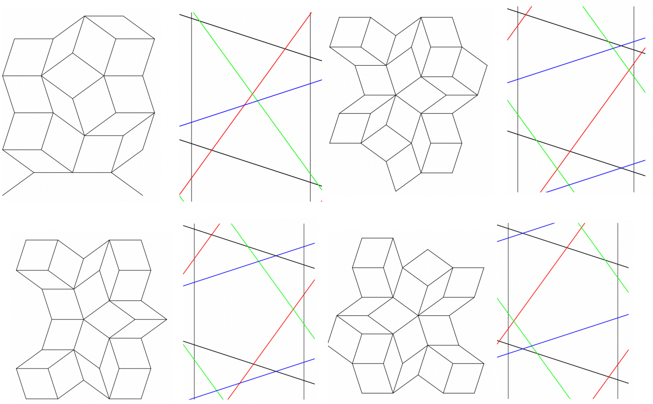
Figure: Any perturbation of $y(\mathbf{u})$ moves a spine above or below the origin.

A big patch of 24 Penrose Wang tiles

The red tiles divides a Penrose tiling into patches, called "Penrose Wang tile". There are 24 of them, and they cover any Penrose tilings. These are described in Grunbaum and Shephard [3].



Four examples of 24 Wang tiles and their duals



Tensor product of $\mathbb{Z}^2$

Let $\{0,1\} \otimes \{0,1\} := \{0 \otimes 0, 0 \otimes 1, 1 \otimes 0, 1 \otimes 1\}$. For $P \subseteq \{0,1\}^{\mathbb{Z}}$, $Q \subseteq \{0,1\}^{\mathbb{Z}}$ closed and S-invariant, we define

$$P \otimes Q = \{p \otimes q : p \in P, q \in Q\} \subseteq (\{0,1\} \otimes \{0,1\})^{\mathbb{Z}^2},$$

where

$$p \otimes q = \begin{bmatrix} & & \vdots & & \\ & & p_0 \otimes q_2 & & \\ \cdots & p_{-1} \otimes q_1 & p_0 \otimes q_1 & p_1 \otimes q_1 & \cdots \\ \cdots & p_{-1} \otimes q_0 & p_0 \otimes q_0 & p_1 \otimes q_0 & \cdots \\ & & p_0 \otimes q_{-1} & & \\ & & \vdots & & \end{bmatrix}$$

Call $P \otimes Q$ tensor product of P and Q.

Sturmian sequences

Definition 19

Fix $\alpha \in [0,1)/\mathbb{Q}$ and let $R_\alpha x = x + \alpha \mod 1$ be the irrational rotation on $\mathbb{T}$.
Define $\mathbf{w} = \mathbf{w}(x) \in \{0,1\}^{\mathbb{Z}}$ by

$$\mathbf{w}_n = \begin{cases} 0 & \text{if } R_\alpha^n x \in [0, 1-\alpha) \\ 1 & \text{if } R_\alpha^n x \in [1-\alpha, 1) \end{cases}$$

$W = \overline{\{\mathbf{w}(x) : x \in [0,1)\}}$ is a subshift $W \subseteq \{0,1\}^{\mathbb{Z}}$ called Sturmian subshift.

Choose $\alpha = \frac{\sqrt{5}-1}{2}$. We look at $\mathbf{w} \otimes \mathbf{w}$.

Sturmian sequences along the lines ℓ_2 and ℓ_4 through ℓ_0 and $_1$

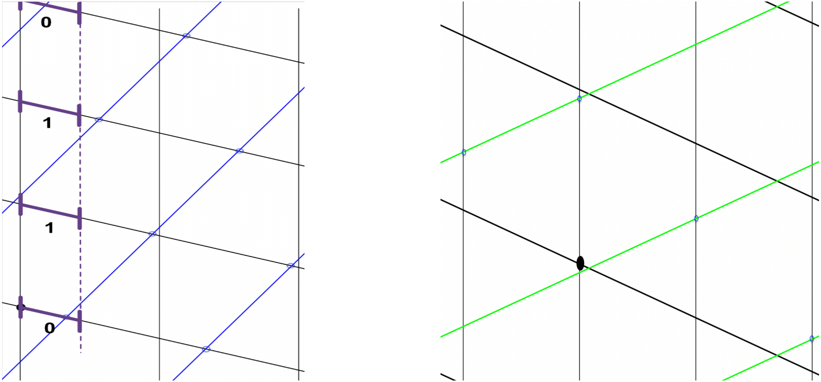


Figure: Sturmian sequences 0110 along the vertical row and Sturmian sequences 0110 along the horizontal row

Penrose 4-grid patches

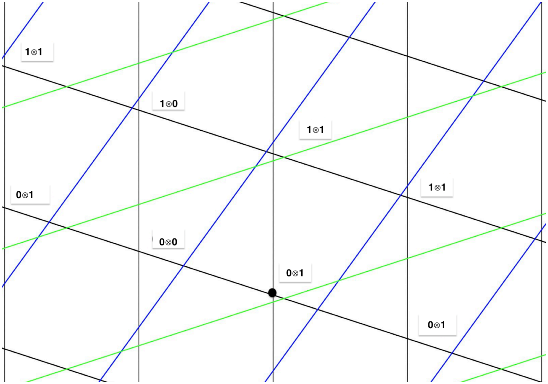


Figure: Tensor products in Penrose 4-grid patches

Dual of Penrose 5-grid patches

Lemma 20

Suppose $x \in X_n$. *Let* M *be a sufficiently wide enough strip not in the direction 0 or 1. Then* $x|_M$ *determines* x.

Lemma 21

If $x \in X_s$ *and* M *is a sufficiently wide strip not in the directions* $0, 1, 2, 3$ *or 4 then* $x|_M$ *determines* x.

Lemma 22

For any strip M *parallel to* ℓ_j, $j = 0, \cdots, 4$ *(no matter how wide) there exists* $x \in X_s$ *so that* $x|_M$ *does not determine* x.

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Thank you