Support stability of measures for subshift of finite type

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Summary

- Motivation
- One dimensional case
- Two dimensional case
- Conclusions

Motivation

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Motivation

- Ergodic Optimization

  $P_f(\mu) = \int f d\mu$

  Jenkinson, Contreras, Lopes, etc

One Dimension

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Theorem 1

Given an aperiodic irreducible subshift of finite type, $X = SFT(L) \subset \Sigma^{\mathbb{Z}}$ where L is a set of forbidden words of length 2 and the Lipschitz function

$$f(x) = \begin{cases} -1 & \text{if } x_0x_1 \in L \\ 0 & \text{otherwise} \end{cases},$$

there exists an $\varepsilon > 0$ such that for all g with $\|f - g\|_{Lip} < \varepsilon$, every g-maximizing measure μ on $\Sigma^{\mathbb{Z}}$ is such that $\mu(X) = 1$.

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- Let μ_g be an ergodic g-maximizing measure on $\Sigma^{\mathbb{Z}}$, suppose (for a contradiction) that its support is contained in X^c. If $\mu_g(I^c) \geq \frac{1}{2N}$, using the fact that $|f(x) - g(x)| \leq \varepsilon$ for all x and $\int f d\mu_g = -\mu_g(I^c) \leq -\frac{1}{2N}$ we get

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- $$\int g d\mu_g = \int f d\mu_g + \int (g - f) d\mu_g \leq -\frac{1}{2N} + \varepsilon.$$

  If ν is any invariant probability measure supported on X, we have:

  $$\int g d\mu_g < -\frac{1}{3N} < -\varepsilon \leq \int g d\nu$$

  contradicting the assumption that μ_g is a g-maximizing measure.

- Now if $\mu_g(I^c) < 1/(2N)$, using the Birkhoff ergodic theorem, we have for μ_g-almost every x,

$$\lim_{n\to\infty} \frac{1}{n} S_n 1_{I^c}(x) = \int 1_{I^c}\, d\mu_g = \mu_g(I^c).$$

Hence for all sufficiently large n, the number of j in the range 0 to $n-1$ such that $x_j x_{j+1} \in L$ is fewer than $n/(2N)$. Hence x must almost surely contain infinitely many subwords of length $N+1$ with no forbidden blocks.

- Using the technique called coupling and splicing we are able find a empirical invariant probability measure ν with support in X such that

$$\int g\, d\mu_g < \int g\, d\nu,$$

establishing that μ_g is not g-maximizing.

Two Dimension

Two Dimension

Theorem 2

There exists a shift of finite type $X = \mathrm{SFT}(L) \subset \Sigma^{\mathbb{Z}^2}$ where L is a set of 2×2 forbidden words with the following property. The Lipschitz function

$$f(x) = \begin{cases} -1 & \text{if} \quad \begin{smallmatrix} x_{01} & x_{11} \\ x_{00} & x_{10} \end{smallmatrix} \in L \\ 0 & \text{otherwise} \end{cases}$$

is uniquely, but unstably, optimized by a measure supported on X. That is, for all $\varepsilon > 0$, there exists $g \in \mathrm{Lip}(X)$, such that $\|f - g\|_{Lip} < \varepsilon$, and μ_g, the g-maximizing measure, is supported on X^c.

Two Dimension

Two Dimension

- Using the Robinson Tiling

Two Dimension

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- Generate a perturbation of the penalty function whose maximizing T-invariant measure is supported on a periodic set.

Robinson Tiling

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1. at each intersection of two tiles, all arrow heads must meet arrow tails and vice versa; and
2. there is a translate of the sub-lattice $2\mathbb{Z} \times 2\mathbb{Z}$ at which every tile is a cross (one of tiles 1–4) (in addition to other crosses in the tiling)

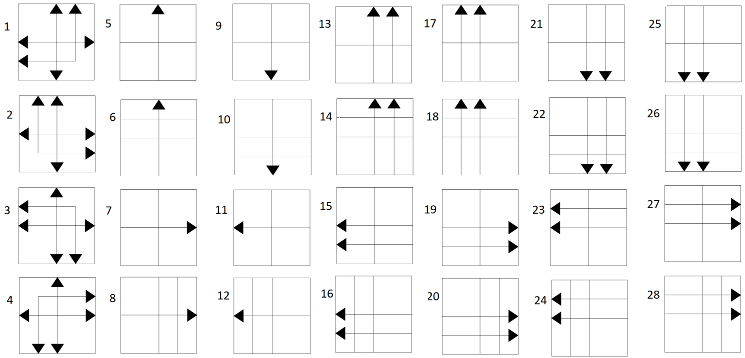


Figure: Robinson's tiles.

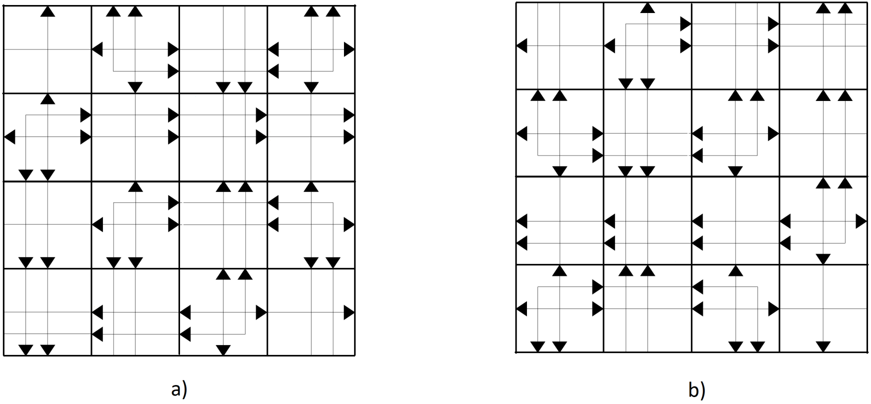


Figure: Configurations in X

Proposition

(T, X) is uniquely ergodic.

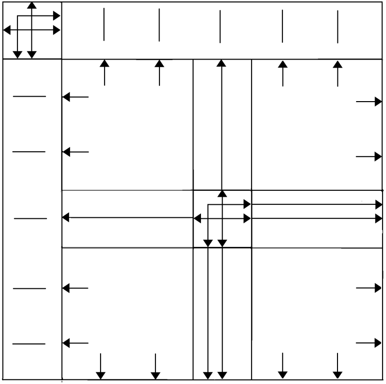
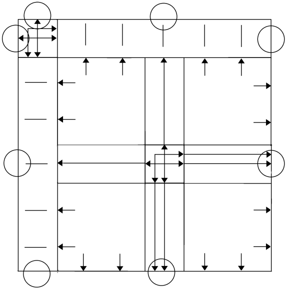


Figure: Square $x^{(N)}$

Proposition

The periodic extension of $x^{(N)}$, has 8 errors per period for every $N \in \mathbb{N}$, and the function f, defined in theorem 2, detects only 7 errors.



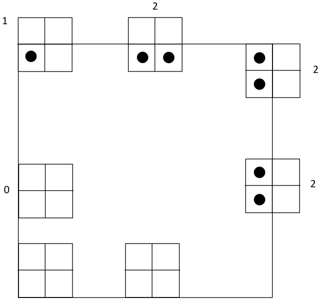


Figure: a) Errors in the periodic configuration and b) The dots indicate those translations of the configuration such that the function f sees an error that is such that f takes the value -1.

- $g = f + \varepsilon h$, where h is the following function:

$$h = 1_{\tau_{25} \cup \tau_{27}} - 1_{\tau_{13} \cup \tau_{15}}.$$

  Here, τ_{**} means Tile $**$ in the set Σ (Figure 1).

- $$\int f d\mu_{per} = \frac{-7}{(2^N)^2} < 0 \text{ and } \int f d\mu = 0,$$

- $$\int g d\mu = 0 \text{ and } \int g \, d\mu_{per} > \frac{-7}{(2^N)^2} + \varepsilon \frac{1}{2^N} > 0$$

- Conclusion

- References

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A. Johnson and K. Madden. Putting the pieces together: Understanding Robinson's nonperiodic tilings. Coll. Math. J., 28:172 - 181, 1997.