Computer-Powered Chaos in Lattice Models

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Thermodynamic Formalism 101

Gibbs Measures on Finite Spaces

- Ω a *finite* set of *states*.
- $E : \Omega \to \mathbb{R}^+$ an *energy* function.
- β the inverse temperature.

Theorem (Variational Principle)

The distribution $\mu_\beta(\omega) \propto \exp(-\beta E(\omega))$ *is the only maximiser of* $\mu \mapsto H(\mu) - \beta\mu(E)$, *with* $H(\mu) := \sum -\log_2(\mu(\omega))\mu(\omega)$ *the entropy.*

We call μ_β *a Gibbs measure.*

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We call μ_β a Gibbs measure.

- At high temperatures, as $\beta \to 0$, we converge to the uniform distribution $\mathcal{U}(\Omega)$, that maximises H.
- At low temperatures, as $\beta \to \infty$, we converge to the uniform distribution $\mathcal{U}(\Omega^*)$, that maximises H among measures of minimal energy, supported by $\Omega^* := \arg\min(E)$.

Invariant Gibbs Measures on Lattice Models

- $\Omega_{\mathcal{A}} := \mathcal{A}^{\mathbb{Z}^d}$ the phase space, with $\mathcal{A}$ a finite alphabet.
- $\mathbb{Z}^d \overset{\sigma}{\curvearrowright} \Omega_{\mathcal{A}}$ the shift action, so that $\sigma^x(\omega)_y = \omega_{y-x}$ for any $x, y \in \mathbb{Z}^d$ and $\omega \in \Omega_{\mathcal{A}}$.
- $\mathcal{M}_\sigma(\Omega_{\mathcal{A}})$ the set of invariant measures on $\Omega_{\mathcal{A}}$, such that $\mu \circ \sigma^x = \mu$ for any $x \in \mathbb{Z}^d$.
- $\varphi : \Omega_{\mathcal{A}} \to \mathbb{R}^+$ a continuous potential, the contribution of $0 \in \mathbb{Z}^d$ to the energy.

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Definition (Pressure Function)

Define the pressure $p_\mu(\beta) := h(\mu) - \beta\mu(\varphi)$,
with $h(\mu) := \lim \frac{1}{n^d} H\left(\mu_{[\![0,n-1]\!]^d}\right)$ the entropy per site.

Let $\mathcal{G}_\sigma(\beta) := \arg\max_{\mu \in \mathcal{M}_\sigma} p_\mu(\beta)$ the set of Gibbs measures.

- φ has finite range if it is *locally constant*, if $\varphi(\omega)$ only depends on $\omega_{[\![-r,r]\!]^d}$.

Limit Behaviour for Ground States

- We call $(\mu_\beta \in \mathcal{G}_\sigma(\beta))_{\beta>0}$ a *cooling trajectory* of the model.
- Denote $\mathcal{G}_\sigma(\infty) := \mathsf{Acc}_{\beta\to\infty}\,\mathcal{G}_\sigma(\beta)$ the set of *ground states*, of accumulation points of all the cooling trajectories.
- $\mathcal{G}_\sigma(\infty)$ is a connected compact set (for the weak-$*$ topology).

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Lemma

Assume that $X := \left\{\omega \in \Omega_\mathcal{A}, \forall x \in \mathbb{Z}^d, \varphi \circ \sigma^x(\omega) = 0\right\} \neq \emptyset$.
Then $\mathcal{G}_\sigma(\infty) \subset \mathcal{M}_\sigma(X)$*, and the ground states have maximal entropy* h *in* $\mathcal{M}_\sigma(X)$.

- Measures that maximise h in $\mathcal{M}_\sigma(X)$ are not necessarily in $\mathcal{G}_\sigma(\infty)$.

Tilings With Local Rules

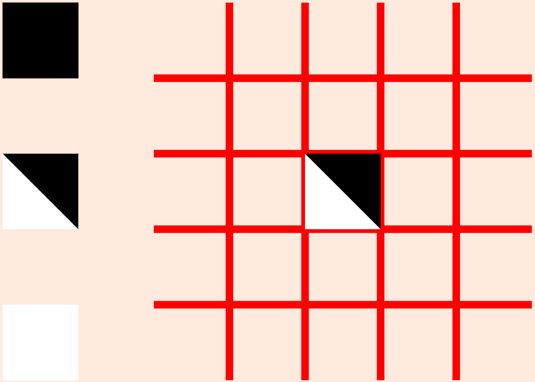
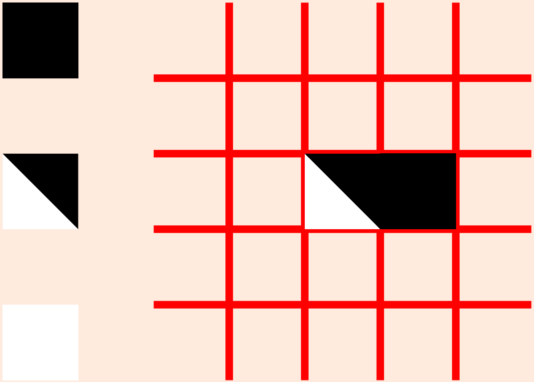


Figure 1: In this example, there is a unique way to globally extend the tiling.

Formally, the set $\mathcal{F}$ of *forbidden patterns* induces a set of admissible tilings $X_{\mathcal{F}} \subset \Omega_{\mathcal{A}}$.

Tilings With Local Rules



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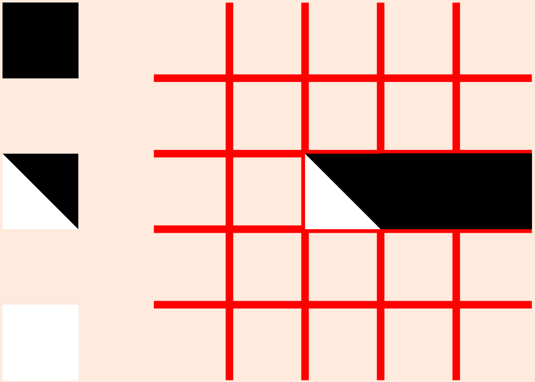

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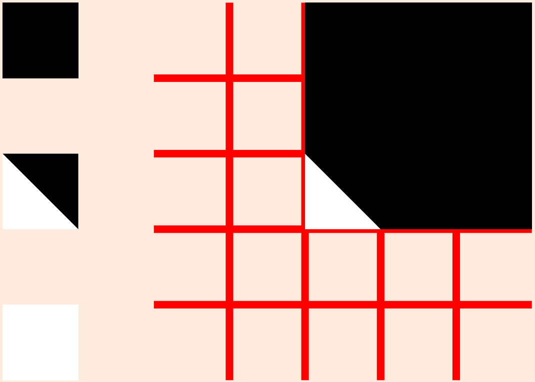

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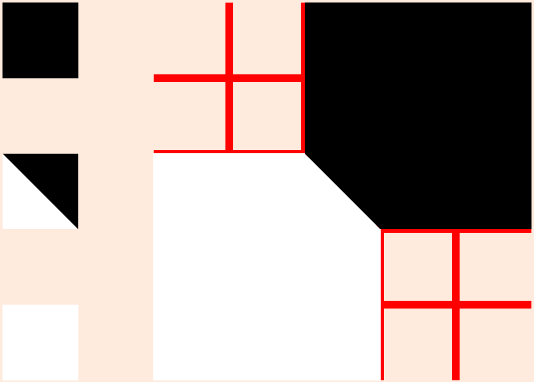

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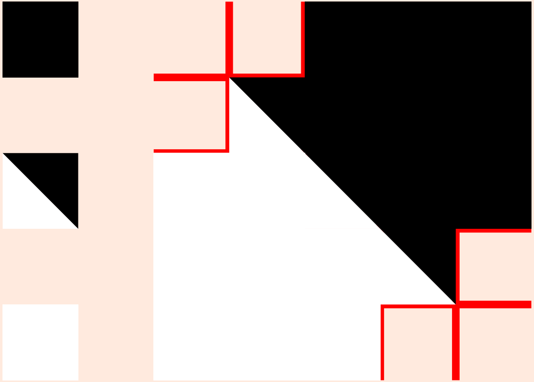

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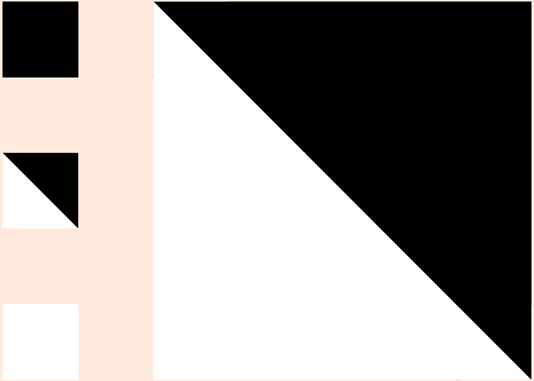

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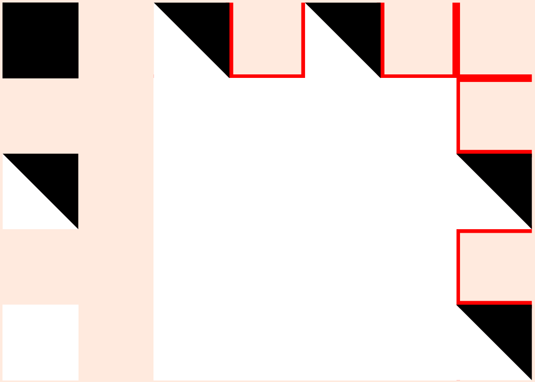

Figure 2: This example is locally but not globally admissible.

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Joining Thermodynamics and Combinatorics

Lemma

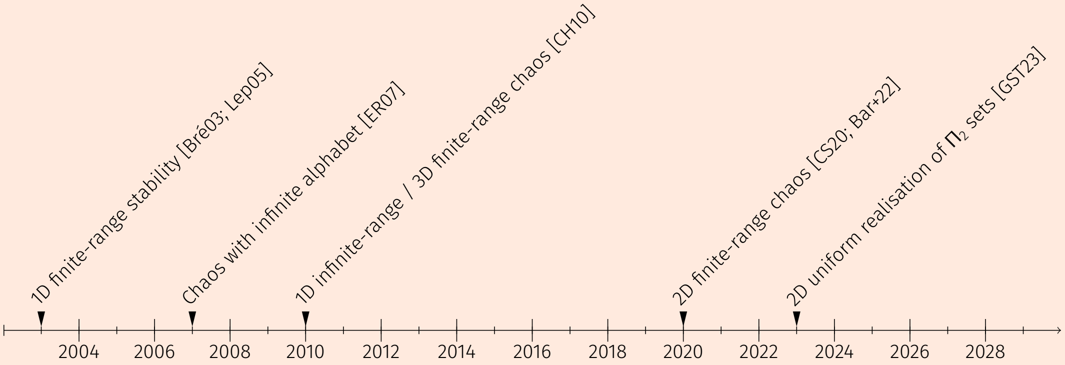
Assume that $X_{\mathcal{F}} \neq \emptyset$, and let $\varphi := \mathbb{1}_{\mathcal{F}\text{ covers }0}$ the induced finite-range potential. Then $\mathcal{G}_\sigma(\infty) \subset \mathcal{M}_\sigma(X_{\mathcal{F}})$, and the ground states have maximal entropy h in $\mathcal{M}_\sigma(X_{\mathcal{F}})$.

What can we ask about $\mathcal{G}_\sigma(\infty)$?

Overview of Chaoticity Results

Timeline

Are there models with chaotic temperature dependence? [NS03]



Stability and Chaos

Definition (Stability)

A model is stable if all the cooling trajectories converge to the same limit.

Definition (Chaoticity)

A model is chaotic if there is no converging cooling trajectory.

Definition (Uniformity)

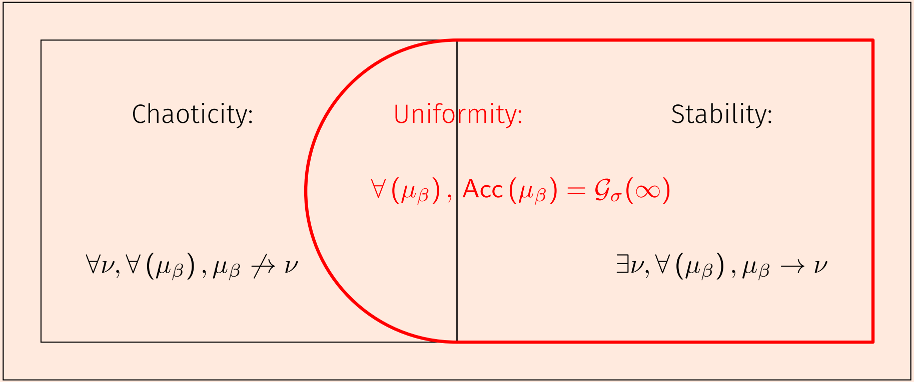
A model is uniform if all the cooling trajectories have the same accumulation set.

Recap of Behaviours

Chaoticity:	Stability:
$\forall \nu, \forall (\mu_\beta), \mu_\beta \not\to \nu$	$\exists \nu, \forall (\mu_\beta), \mu_\beta \to \nu$

Figure 3: Inventory and comparison of model behaviours.

Recap of Behaviours



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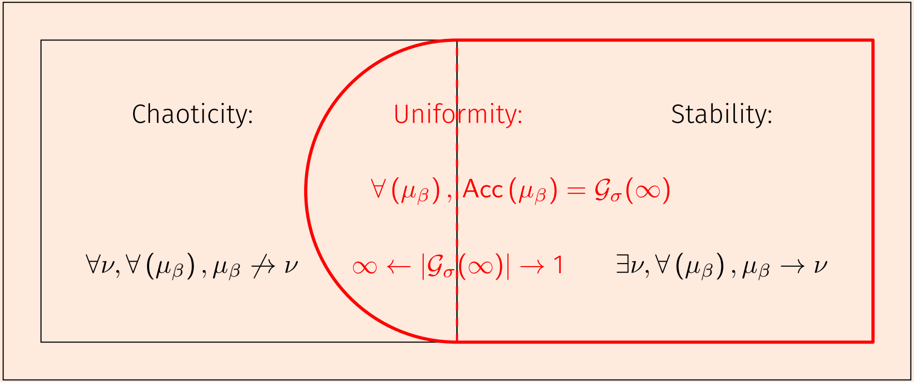

Figure 3: Inventory and comparison of model behaviours.

The Infinite-Alphabet Case [ER07]

- Continuous spin alphabet $\mathcal{A} = \mathbb{R}/2\pi\mathbb{Z}$,
- Potential made of infinitely nested (anti)ferromagnetic wells:

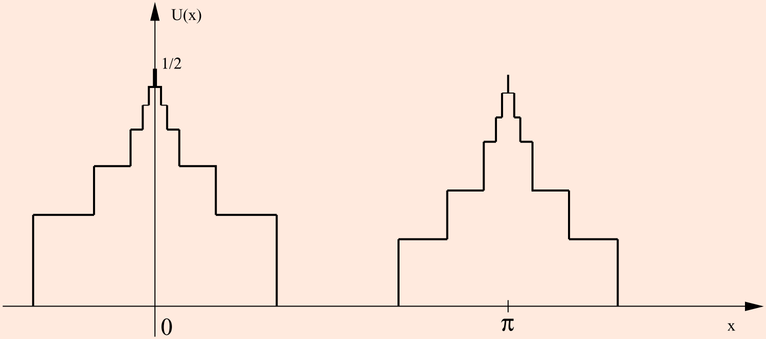

Figure 4: Interaction $U(\theta - \theta')$ between neighbouring spins on the grid.

General Idea for Chaoticity

We have two measures $\mu \neq \mu'$ s.t. $d(\mu, \mu') \geq r$ and:

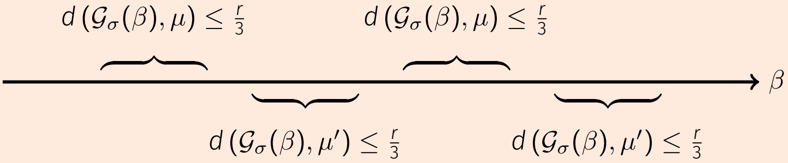

Figure 5: Alternating between mutually exclusive adherence values on non-overlapping intervals.

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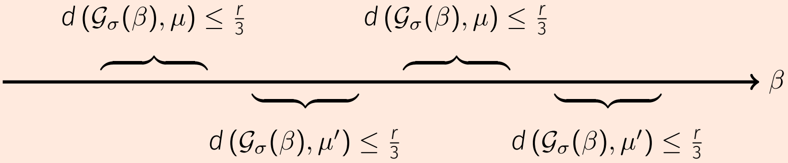

Figure 5: Alternating between mutually exclusive adherence values on non-overlapping intervals.

Thus $\mathrm{Acc}(\mu_\beta)$ intersects the disjoint neighbourhoods of both μ and μ'.

Locally Admissible Typical Behaviours

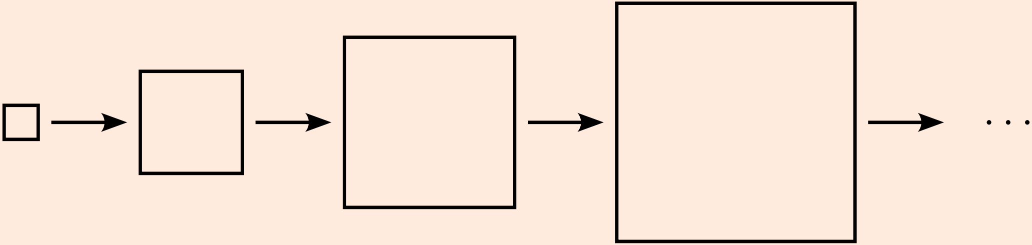

Figure 6: Each temperature range will correspond to a scale of locally admissible tilings.

General Idea for Uniformity

We want (μ_n) and $\varepsilon_n \to 0$ s.t.:

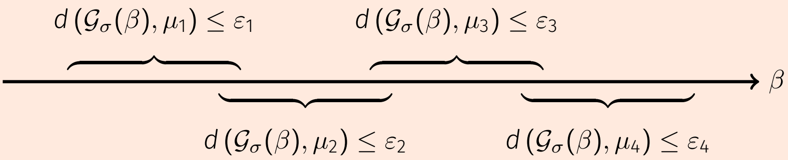

Figure 7: Contracting tube of measures with overlapping intervals.

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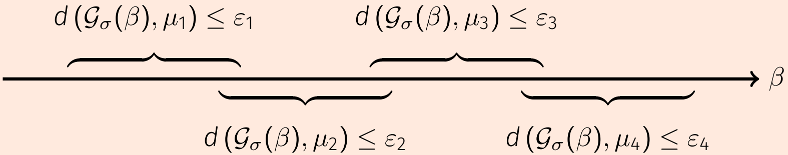


Figure 7: Contracting tube of measures with overlapping intervals.

Thus $\mathrm{Acc}(\mu_\beta) = \mathcal{G}_\sigma(\infty) = \mathrm{Acc}(\mu_n)$.

Realisation Result on the Limit Set [GST23]

Proposition (Obstruction)

In every uniform model with computable interactions, the set of ground states $\mathcal{G}_\sigma(\infty)$ is compact, connected and Π_2-computable.

In the general non-uniform case, the computability bound becomes Π_3.

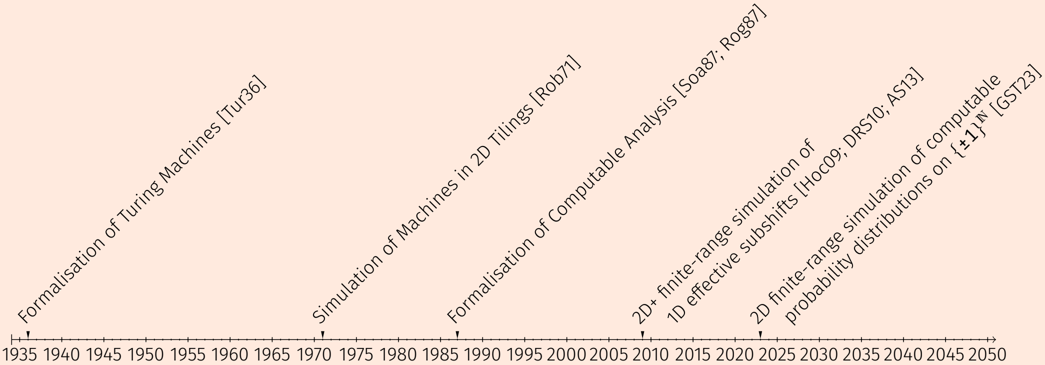
Theorem (Realisation)

Given a connected Π_2-computable compact set K of probability measures on $\{\pm 1\}^{\mathbb{N}}$, there exists a 2D uniform model with zero-one finite-range interactions, for which $\mathcal{G}_\sigma(\infty)$ is computably and affinely homeomorphic to K.

In particular, for any non-singleton set K, the model is (uniformly) chaotic.

Computability is Everywhere

Timeline


Formalisation of Turing Machines [Tur36]
Simulation of Machines in 2D Tilings [Rob71]
Formalisation of Computable Analysis [Soa87; Rog87]
2D+ finite-range simulation of 1D effective subshifts [Hoc09; DRS10; AS13]
2D finite-range simulation of computable probability distributions on $\{\pm 1\}^{\mathbb{N}}$ [GST23]
1935 1940 1945 1950 1955 1960 1965 1970 1975 1980 1985 1990 1995 2000 2005 2010 2015 2020 2025 2030 2035 2040 2045 2050

Computability is Everywhere

Turing Machines

Turing Machines

Turing machines are a model equivalent to real-life computers and algorithms.

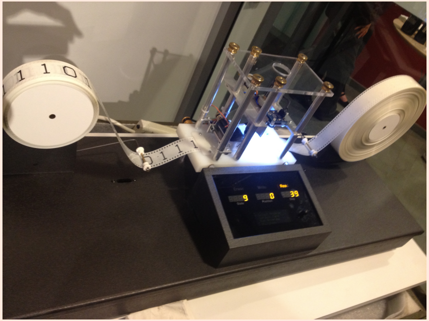


Figure 8: Real-life Turing machine (Source: wikimedia.org)

Formally, M is made of:

- internal states Q,
- an initial state $q_0 \in Q$,
- accepting states $Q_A \subset Q$,
- rejecting states $Q_R \subset Q$,
- an input alphabet $\mathcal{A}$,
- a tape alphabet $\Gamma \supset \mathcal{A} \sqcup \{\#\}$,
- a transition function $\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R\}$.

Tileset of Space-Time Diagrams

A Turing machine $M = (Q, q_0, Q_A, Q_R, \mathcal{A}, \Gamma, \delta)$ can be simulated by a Wang tileset:

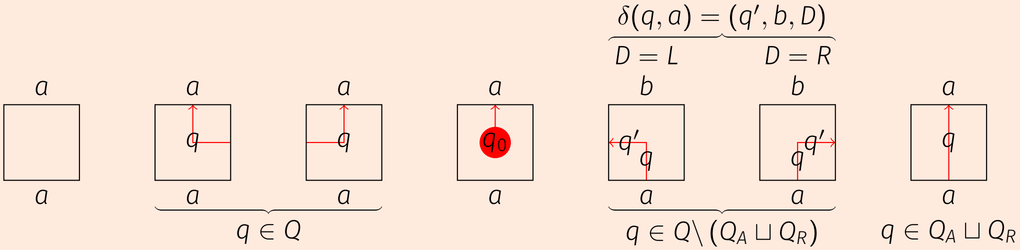


Figure 9: Turing space-time diagram Wang tiles for each letter $a \in \Gamma$.

The Halting Problem

Can we algorithmically decide if the machine M halts on the input u?

Lemma (Diagonal Argument)

The halting problem is not decidable.

- *Assume it is with a machine H, and use it to define D so that:*
  - *if M halts on its own code $u = \langle M \rangle$ as the input, then D loops forever on $\langle M \rangle$,*
  - *else, D stops once it has determined the other computation doesn't end.*
- *We feed the code of the machine $\langle D \rangle$ to itself.*
- *If D halts on $\langle D \rangle$, then by construction it means that H says D doesn't halt on $\langle D \rangle$, and conversely [...], hence a paradox.*

Computability is Everywhere

Simulacra and Simulation

Canonical Robinson Tiling

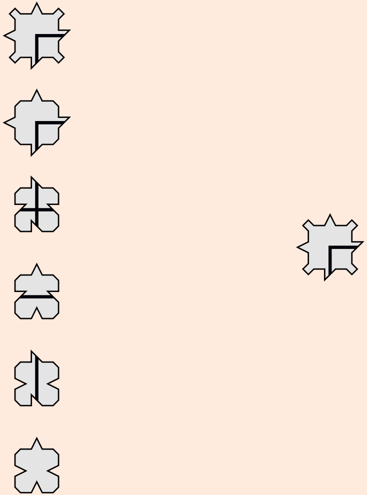
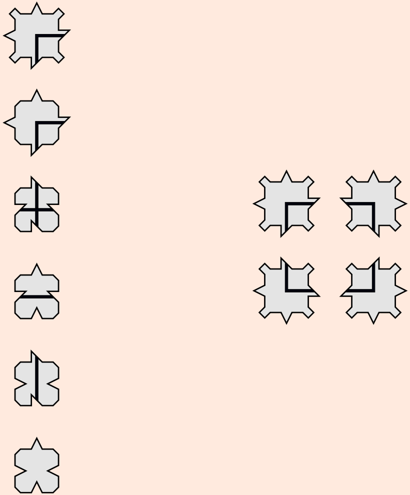


Figure 10: Hierarchical structure of the Robinson tiling.

Canonical Robinson Tiling



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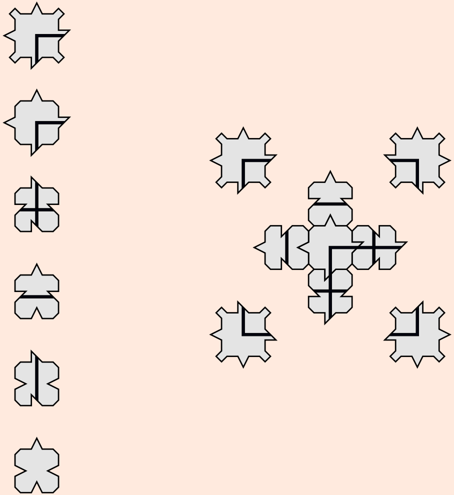

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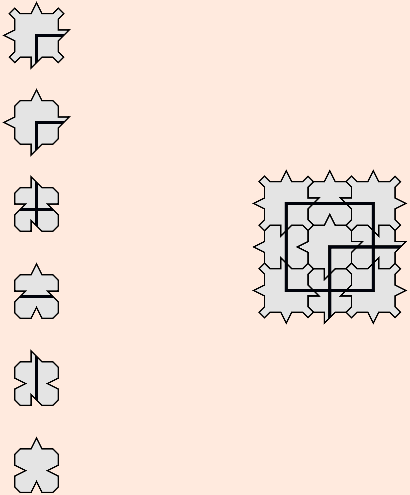

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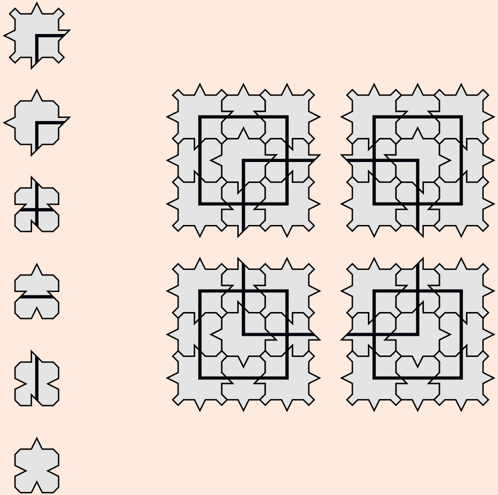

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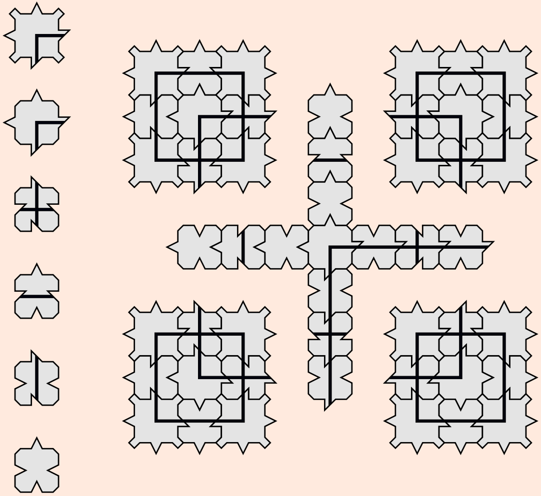

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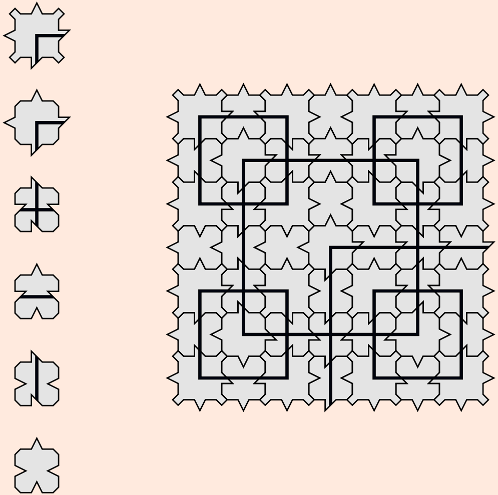

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Simulating Tilesets

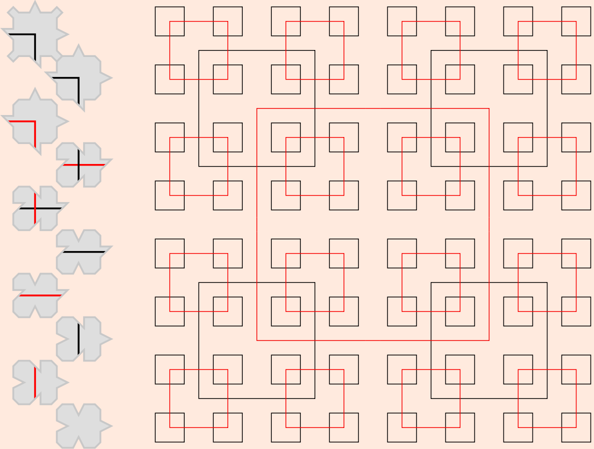
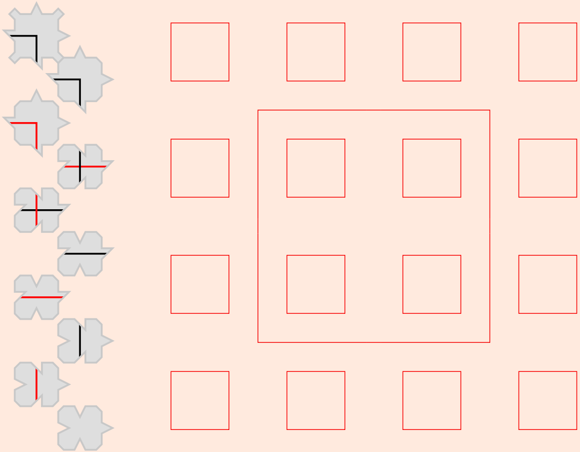


Figure 11: Alternating Red-Black structure,

Simulating Tilesets



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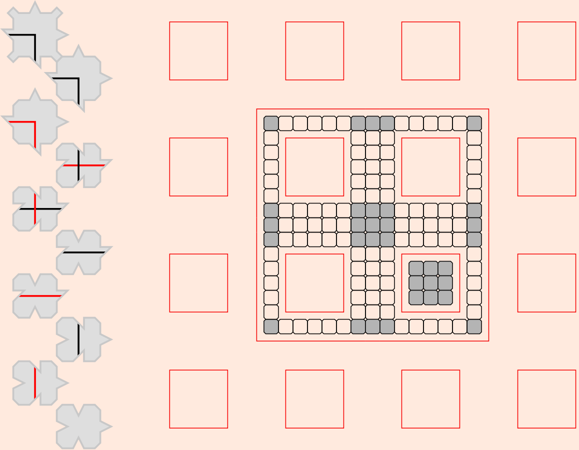

Figure 11: Alternating Red-Black structure, with a sparse computation area.

The Domino Problem

Given a set of forbidden patterns $\mathcal{F}$, can we tile the space (i.e. $X_{\mathcal{F}} \neq \emptyset$)?

Lemma

By reduction from the halting problem, the domino problem is undecidable.

This will more broadly be true of most tiling problems,
and likewise for thermodynamic properties.

Effective Simulation Results [Hoc09; DRS10; AS13]

We say that $Y \subset \mathcal{B}^{\mathbb{Z}^{d'}}$ simulates $X \subset \mathcal{A}^{\mathbb{Z}^{d}}$ if there is $\theta : \mathcal{B} \to \mathcal{A}$ (that extends to $\mathbb{Z}^d$) s.t.:

$$\theta(Y) := \{\theta(\omega'), \omega' \in Y\} = \left\{\omega^{\mathbb{Z}^{d'-d}}, \omega \in X\right\}.$$

$X_{\mathcal{F}}$ is effective if $\mathcal{F}$ (not necessarily finite) can be enumerated by a Turing machine.

Theorem

For any 1D effective subshift X and any dimension $d \geq 2$, there is a d-dimensional SFT Y that simulates X.

De Facto Simulation of Probability Distributions [GST23]

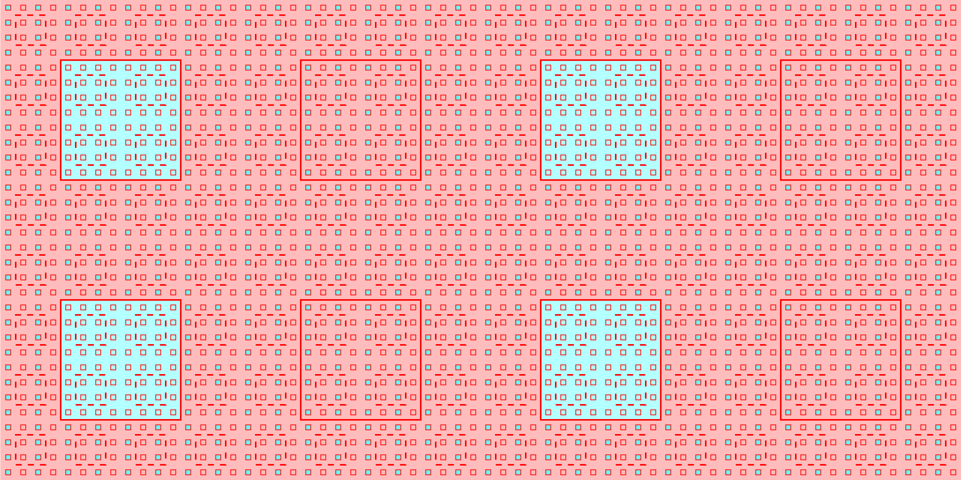

Figure 12: Multi-scale structure that “realises” probability distributions.

De Facto Simulation of Probability Distributions [GST23]

- Red lines encode bits, synchronised between neighbouring squares in blue areas.
- Each outermost blue square runs the same deterministic Turing machine.
- As the input, the machine gets the scale of simulation n,
  and a binary sequence b of length l_n.
- The machine's output must be the prefix of the encoded sequence.
- Each input b corresponds to 1 admissible tiling of the corresponding blue square,
  thus it is uniformly distributed in the thermodynamic setting,
  and we say the square "computes" the output distribution.
- Consequently, in the appropriate temperature range,
  the Gibbs states "simulate" a measure μ_n on $\{\pm 1\}^{\mathbb{N}}$,
  that averages the distributions "computed" at all the scales lower than n.

De Facto Simulation of Probability Distributions [GST23]

This simulation argument allows us to relate families of translational-invariant 2D Gibbs measures to non-invariant 1D measures, up to the aforementioned computable affine bijections.

Which sequences (μ_n) *can be simulated?*

Computability is Everywhere

Hierarchy of Complexities

Arithmetical Hierarchy

Definition

The countable set $X \subset \mathbb{N}$ is Π_k-computable *iff* there is a computable φ such that:

$$x \in X \Leftrightarrow \forall y_1, \exists y_2, \forall y_3, \ldots, \varphi(x, y_1, \ldots, y_k)$$

Likewise, we define Σ_k problems starting with an $\exists$ quantifier.
The family (Σ_k, Π_k) gives an increasing hierarchy of undecidable complexity.

The halting problem is Σ_1-complete (it's Σ_1 and any Σ_1 problem reduces to it).
The domino problem is Π_1-complete.

Computational Complexity of Uncountable Sets

Let (X, d) a metric space with a countable dense basis $\mathcal{B}$.

Definition

Let $Y \subset X$ be a closed set and $\mathcal{N}(Y) := \{(x, r) \in \mathcal{B} \times \mathbb{Q}^{+*}, \overline{B(x, r)} \cap Y \neq \emptyset\}$.

The set Y is said to be Π_k-computable *iff* the countable set $\mathcal{N}(Y)$ is.

Here, for invariant measures $\mathcal{M}_\sigma(\Omega_\mathcal{A})$ with the weak-* topology,
we use the periodic measures $\widehat{\delta_w}$, with $w \in \mathcal{A}^{[\![0,n-1]\!]^d}$, as a basis $\mathcal{B}$.

Upper Bound on the Complexity of Uniform Accumulation Sets

Let φ a computable potential, inducing a uniform model.

Proposition ([GST23, Proposition 3])

There is a sequence $\beta_k \to \infty$ such that $\operatorname{diam}(\mathcal{G}_\sigma(\beta_k)) \to 0$ and $\mathcal{G}_\sigma(\infty) = \operatorname{Acc}(\mathcal{G}_\sigma(\beta_k))$.

Without loss of generality, we can use rational parameters $\beta_k \in \mathbb{Q}$.

Theorem ([GST23, Theorem 17])

We have $\overline{B(x,r)} \cap \mathcal{G}_\sigma(\infty) \neq \emptyset$ iff:

$$\forall \varepsilon \in \mathbb{Q}^{+*}, \forall \beta_0 \in \mathbb{Q}^{+*}, \qquad \exists \beta \in \mathbb{Q}^{+*}_{>\beta_0}, \exists y \in \mathcal{B},$$
$$\mathcal{G}_\sigma(\beta) \subset B(y,\varepsilon) \text{ and } B(y,\varepsilon) \cap \overline{B(x,r)} \neq \emptyset.$$

Consequently, we have a Π_2 upper bound on the complexity of $\mathcal{G}_\sigma(\infty)$.

Equivalent Characterisation of Π_2 as Accumulation Sets

Proposition ([GST23, Proposition 5])

There is a characterisation of Π_2-computable sets through accumulation points:

$$\begin{aligned} Y \in \Pi_2 \quad &\Leftrightarrow \quad Y = \mathrm{Acc}\,(x_n) \textit{ with } (x_n) \in \mathcal{B}^{\mathbb{N}} \textit{ computable.} \\ Y \in \Pi_2 \textit{ and connected} \quad &\Leftrightarrow \quad Y = \mathrm{Acc}\,(x_n) \textit{ with } (x_n) \in \mathcal{B}^{\mathbb{N}} \textit{ computable,} \\ &\qquad \textit{and } d\,(x_n, x_{n+1}) \to 0. \end{aligned}$$

Complexity of Stability

Proposition

For any computable sequence (x_n) *associated to a connected* Π_2 *set* $Y \subset \mathcal{M}\left(\{\pm 1\}^{\mathbb{N}}\right)$, *there is a related simulated sequence* μ_n *such that* $Y = \mathrm{Acc}\,(\mu_n)$.

Theorem ([GST23, Theorem 62])

The problem of chaoticity (with a computable φ *for the input) is* Σ_3*-complete.*

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What's Next?

WIP:

- Non-robustness of the accumulation set to perturbations of the potential.

Open:

- Is robustness to perturbations of the (computable) potentials (a)typical?
- Can we realise Π_3 sets in the general (non-uniform) case?

THE END OF
PRESENTATION
ONE MORE SLIDE:

Thank you.