Wang tiles, Kari tiles, and computation in tilings

Multidimensional symbolic dynamics and lattice models of quasicrystals

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Wang tiles and tilesets

Proving theorems by pattern recognition II

"In connection with the $\forall x\ \exists y\ \forall z$ case, an amusing combinatorial problem is suggested in Section 4.1."

— Hao Wang.

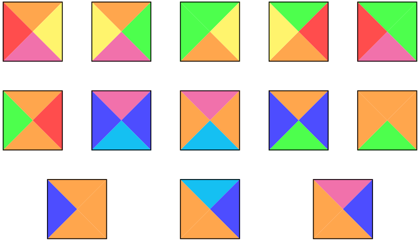
Proving theorems by pattern recognition II.

Bell Systems Technical Journal, XL(1), 1961.

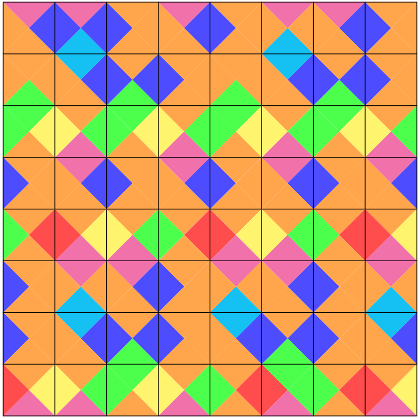
Wang tiles

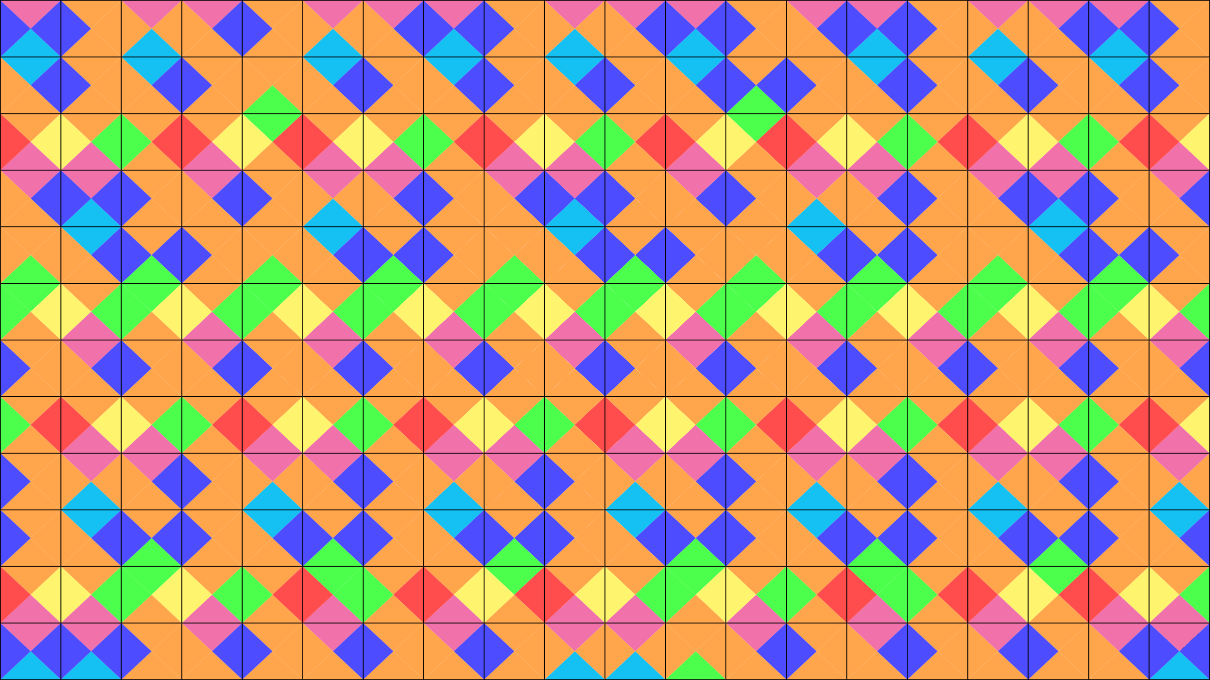
Wang tile:

Tileset:



Tiling of a square:





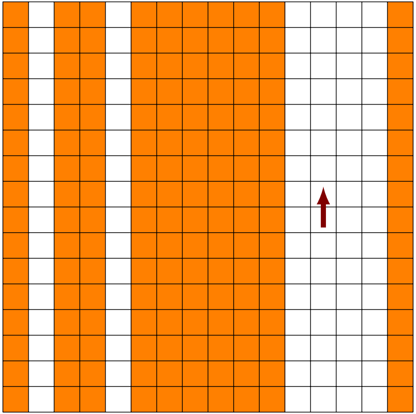
Periodicity in 2D

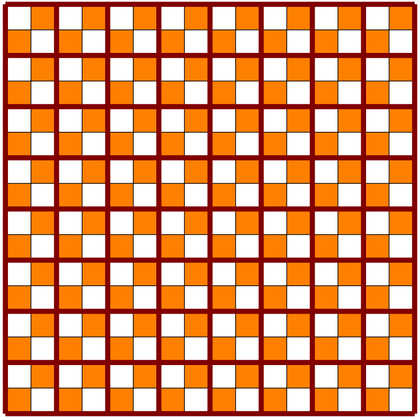
Definition

A 2D tiling is **weakly periodic** if it has **one** direction of periodicity.

Definition

A 2D tiling is **strongly periodic** if it has **two** directions of periodicity.





Wang's algorithm

Definition

The **Domino problem** is the following:

INPUT: a tileset T

OUTPUT: does T have tiling?

Remark

Wang's algorithm finds **periodic** tilings.

Conjecture (1961)

Wang's algorithm is correct.

Algorithm (Wang, 1961)

1. $n \leftarrow 1$
2. Try all tilings of the $n \times n$ square
3. If there is a repeatable tiled square
   - Return `true`
4. If there is no tiled square
   - Return `false`
5. Else
   - $n \leftarrow n + 1$
   - Go to 2

Plot twist

Theorem (Berger, 1966)

The Domino problem is undecidable.

Corollary

Wang's algorithm doesn't stop on some inputs.

Corollary

There exist **aperiodic tilesets**.

(Tilesets with at least one tiling, but only aperiodic tilings.)

The long list of aperiodic tilesets

Tilings:

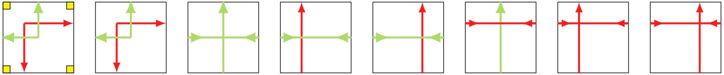
- Berger, 1964
- Knuth, 1968
- Robinson, 1971
- Penrose, 1974
- Ammann, 1977
- Kari, 1996
- Kari-Culik, 1996
- Jeandel-Rao, 2015
- Labbé, 2018, 2024

Methods:

- **Self-similarity**
- **Cut-and-project**
- **Computation on reals**

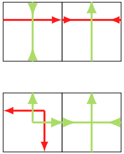
Robinson tiles

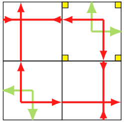
Tiles:

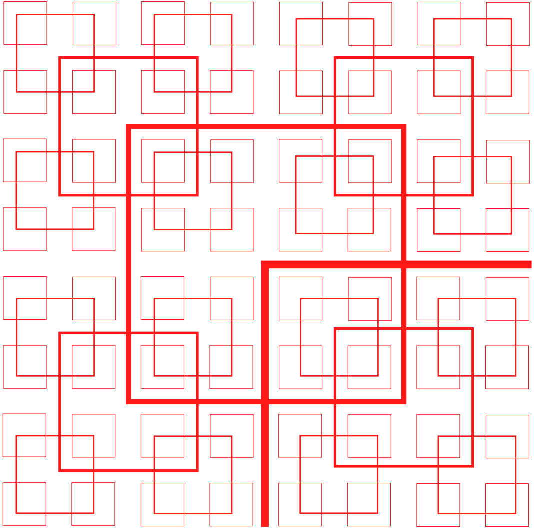


+ Rotations

Rules (can be encoded with colors)**:**







Robinson tilings

Theorem (Robinson, 1971)

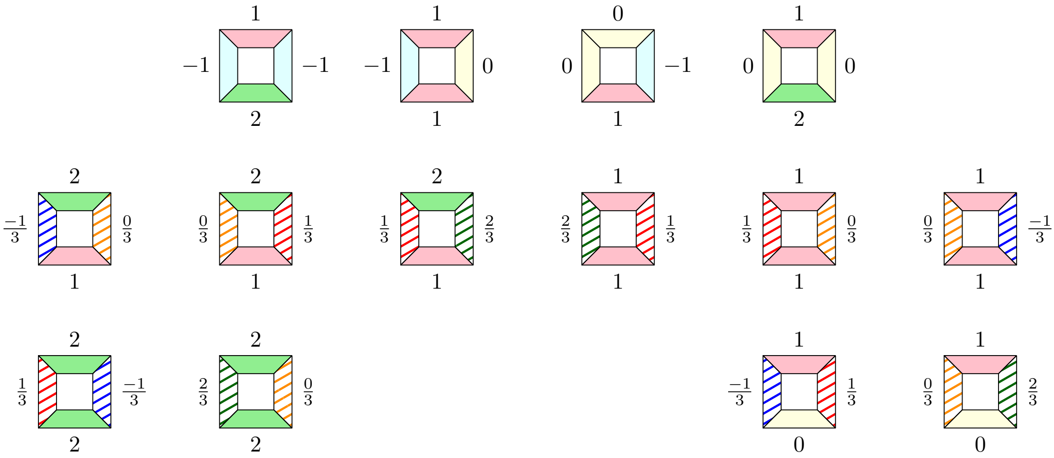
Robinson tiles do tile the plane, and all tilings are fixed-points of the substitution.

Claim

Any Robinson tiling is aperiodic.

- Any periodicity vector would send each red square to another
- There are arbitrarily large squares
- The periodicity vector would have to be infinite

Another aperiodic tileset



Computing on averages

Averages

Let $\mathbf{w} = \dots \mathbf{w}_{-2}\mathbf{w}_{-1}\,\mathbf{w}_0\,\mathbf{w}_1\mathbf{w}_2\dots$ be a biinfinite word.

Definition

The **average** of $\mathbf{w}$ is the following limit:

$$\mathrm{avg}(\mathbf{w}) = \lim_{n\to\infty} \frac{\mathbf{w}_{-n} + \cdots + \mathbf{w}_0 + \cdots + \mathbf{w}_n}{2n+1}$$

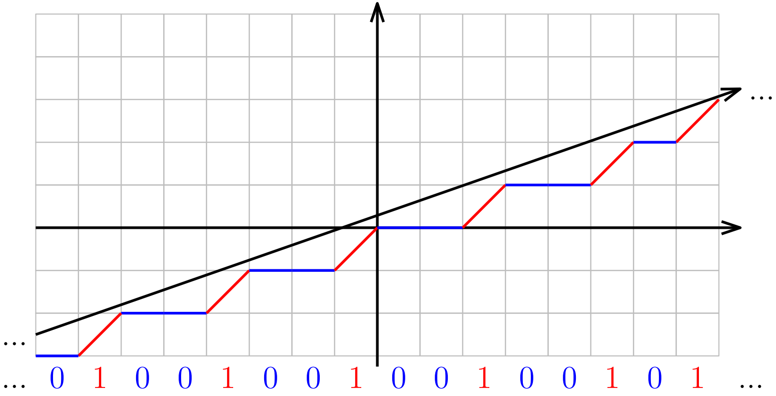
if it exists.

$$\mathrm{avg}(\dots 01\,01\,01\,01\,01\,01\dots) = 1/2$$

$$\mathrm{avg}(\dots 722\,722\,722\,722\,722\,722\dots) = 11/3$$

$$\mathrm{avg}(\dots 0\,11\,0000\,11111111\,0000000000000000\,\dots) = \text{ not well-defined}$$

Line encodings



We can encode a line as its $(-, /)$-approximation on the unit grid.

If the line has slope α, the alphabet is $\{\lfloor\alpha\rfloor, \lceil\alpha\rceil\}$.

Line encodings have averages

Claim

The average of a line encoding exists and equals the slope of the line.

Proof ideas

If $\mathbf{w}$ encodes the line $y = \alpha x + \beta$, then:

$$\mathbf{w}_i = \lfloor (i+1)\alpha + \beta \rfloor - \lfloor i\alpha + \beta \rfloor.$$

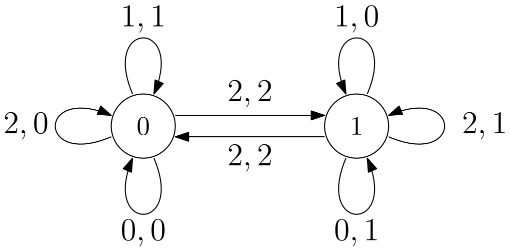
This yields a telescopic sum:

$$\sum_{i=-n}^{+n} \mathbf{w}_i = \lfloor (n+1)\alpha + \beta \rfloor - \lfloor -(n+1)\alpha + \beta \rfloor.$$

Now use $r - 1 \leq \lfloor r \rfloor \leq r$ to bound the average above and below by α.

Line encodings have many amazing properties! But that's out of scope for this talk.

Transducers



Definition

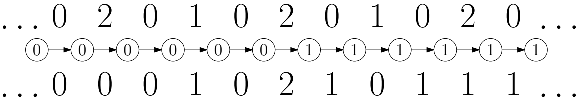
A **transducer** T is a digraph where each arc is labeled with a couple of symbols.

Vertices are called **states** and arcs **transitions**.

If $\mathbf{u}$ and $\mathbf{w}$ are words, we say that T **accepts** output $\mathbf{w}$ on input $\mathbf{u}$ iff

$$\ldots (\mathbf{u}_{-1}, \mathbf{w}_{-1})\ (\mathbf{u}_0, \mathbf{w}_0)\ (\mathbf{u}_1, \mathbf{w}_1)\ \ldots$$

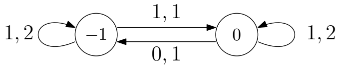
is a biinfinite path in T.



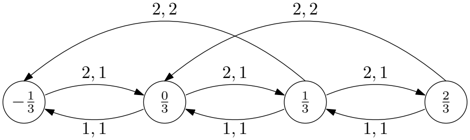
One input may have several possible outputs!

T_2 and $T^{2/3}$ and $T_{2/3}$

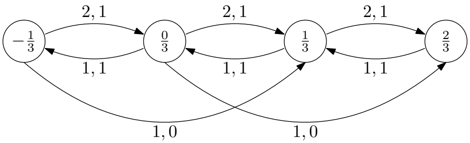
$T_2 =$



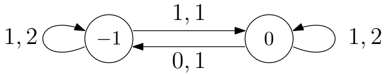
$T^{2/3} =$



$T_{2/3} =$



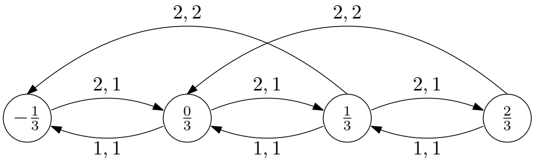
Examples of T_2-runs



$$\ldots\ 1\ \ 1\ \ 1\ \ 0\ \ 1\ \ 1\ \ 1\ \ 0\ \ 1\ \ 1\ \ 1\ \ 1\ \ 0\ \ 1\ \ 1\ \ldots$$

$$\ldots\ 2\ \ 1\ \ 2\ \ 1\ \ 2\ \ 1\ \ 2\ \ 1\ \ 2\ \ 2\ \ 1\ \ 2\ \ 1\ \ 2\ \ 2\ \ldots$$

Examples of $T^{2/3}$-runs



$$\ldots\ 2\ \ 1\ \ 2\ \ 1\ \ 2\ \ 1\ \ 2\ \ 1\ \ 2\ \ 1\ \ 2\ \ 1\ \ 2\ \ 1\ \ 2\ \ldots$$

$$\ldots\ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ldots$$

α-multiplicative transitions

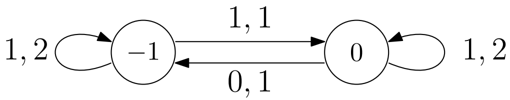
Let $\alpha \in \mathbb{R}$ and T be a transducer.

Definition

Let $p \xrightarrow{(u,w)} q$ denote a transition. If:

$$p + \alpha u = w + q$$

then it is an **α-multiplicative** transition.



Theorem

Assume all transitions of T are α-multiplicative.

If T accepts $\mathbf{w}$ on input $\mathbf{u}$ and $\text{avg}(\mathbf{u}) = x$, then $\text{avg}(\mathbf{w}) = \alpha x$.

α-multiplicative transducers compute on averages

Let $p \xrightarrow{(u,w)} q$ denote a transition. If $p + \alpha u = w + q$ then it is α-multiplicative.

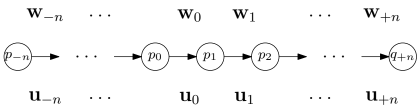
Theorem

If T (α-multiplicative) accepts $\mathbf{w}$ on input $\mathbf{u}$ and $\text{avg}(\mathbf{u}) = x$, then $\text{avg}(\mathbf{w}) = \alpha x$.

Let $p_n \xrightarrow{(\mathbf{u}_n, \mathbf{w}_n)} q_n$ be the n^{th} transition of the path. We have:

$$\sum_{i=-n}^{+n} p_i + \alpha \mathbf{u}_i = \sum_{i=-n}^{+n} \mathbf{w}_i + q_i$$

$$\sum_{i=-n}^{+n} \alpha \mathbf{u}_i - \mathbf{w}_i = \sum_{i=-n}^{+n} q_i - p_i$$



… but $q_i = p_{i+1}$! (Calculation on the blackboard: replace q_i and divide by $2n+1$.)

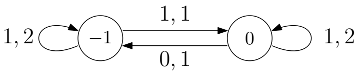
Properties of T_2, $T^{2/3}$ and $T_{2/3}$

Let $\mathcal{L}(\alpha, \beta)$ denote the line encoding with slope α and offset β.

Claims about T_2

All transitions are 2-multiplicative.

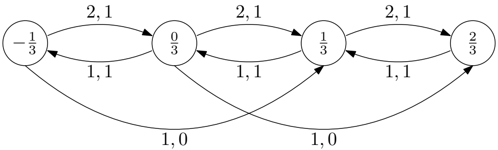
Accepts $\mathcal{L}(2\alpha, \beta)$ on $\mathcal{L}(\alpha, \beta)$ for all $\alpha \in [\frac{1}{2}, 1]$.



Claims about $T_{2/3}$

All transitions are (2/3)-multiplicative.

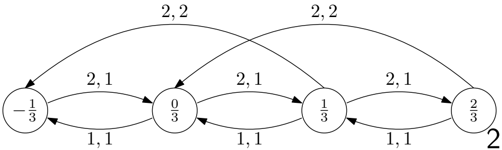
Accepts $\mathcal{L}(\frac{2}{3}\alpha, \beta)$ on $\mathcal{L}(\alpha, \beta)$ for all $\alpha \in [1, \frac{3}{2}]$.



Claims about $T^{2/3}$

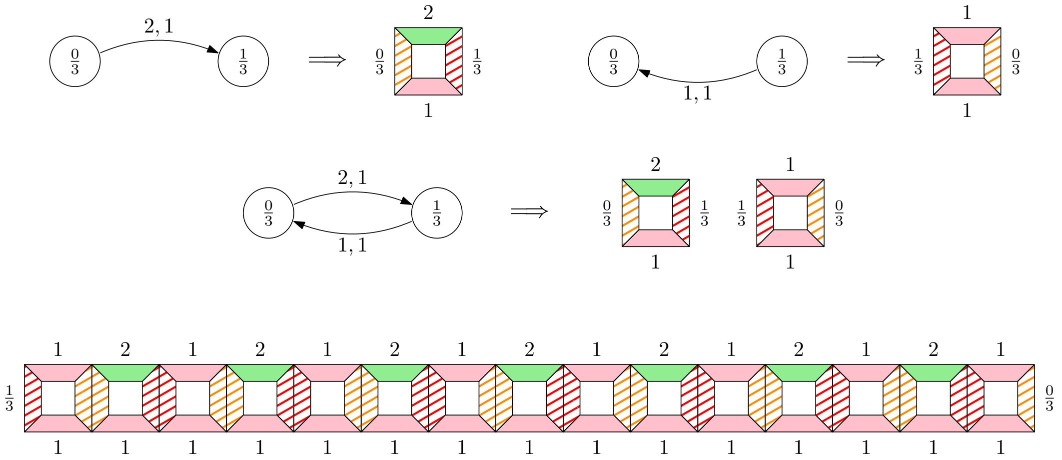
All transitions are (2/3)-multiplicative.

Accepts $\mathcal{L}(\frac{2}{3}\alpha, \beta)$ on $\mathcal{L}(\alpha, \beta)$ for all $\alpha \in [\frac{3}{2}, 2]$.

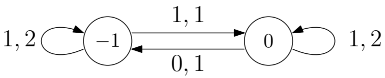


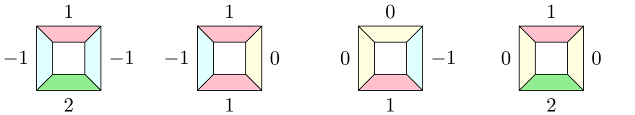
Kari's tileset

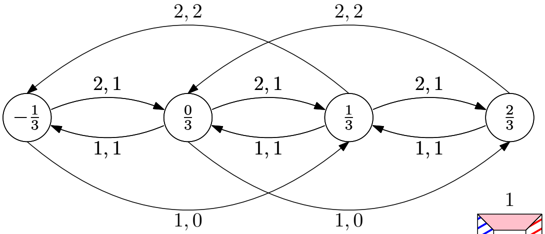
Transducers to tilesets

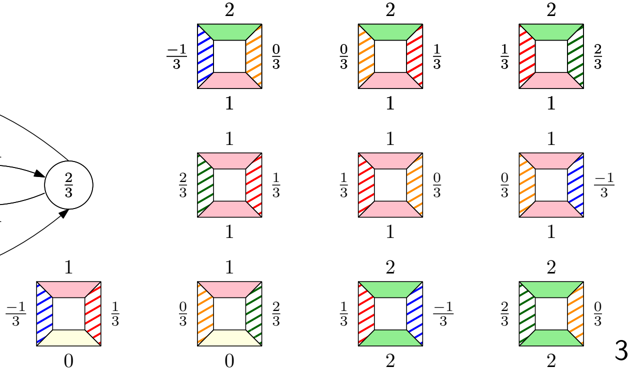


The tiles



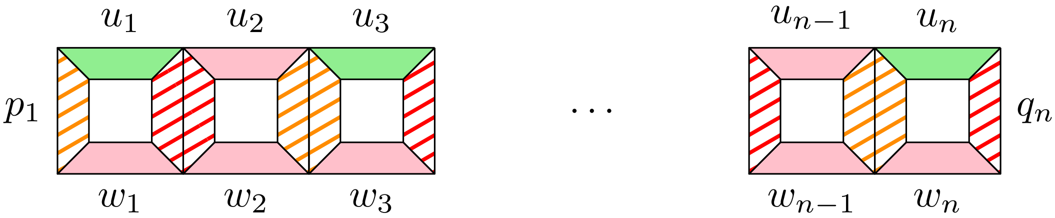






Multiplicative tiles

$p + \alpha u = q + w$



$$p_1 + \alpha \sum_i u_i = q_n + \sum_i w_i \quad \frac{p_1}{n} + \alpha \operatorname{avg}(u_i) = \operatorname{avg}(w_i) + \frac{q_n}{n}$$

The dynamical system

Recall that: T_2 is 2-multiplicative and accepts inputs over $\{0, 1\}$
$T_{2/3}$ is $(2/3)$-multiplicative and accepts inputs over $\{1, 2\}$.

Claim

Suppose we have a tiling where the n^{th} line has an existing average x_n. Then:

$$x_{n+1} = f(x_n) = \begin{cases} 2x_n & \text{if } x_n \leq 1, \\ 2x_n/3 & \text{if } x_n \geq 1. \end{cases}$$

Observe that (x_n) is an aperiodic sequence (otherwise $2^{k+\ell} = 3^\ell$ for some k, ℓ).

In a periodic tiling, every line has an average.

Lemma (Kari, 1996)

There is no periodic tiling of the plane.

At least one tiling

Recall that: T_2 accepts $\mathcal{L}(2\alpha, \beta)$ on $\mathcal{L}(\alpha, \beta)$ for all $\alpha \in [\frac{1}{2}, 1]$
$T_{2/3}$ accepts $\mathcal{L}(\frac{2}{3}\alpha, \beta)$ on $\mathcal{L}(\alpha, \beta)$ for all $\alpha \in [1, 2]$.

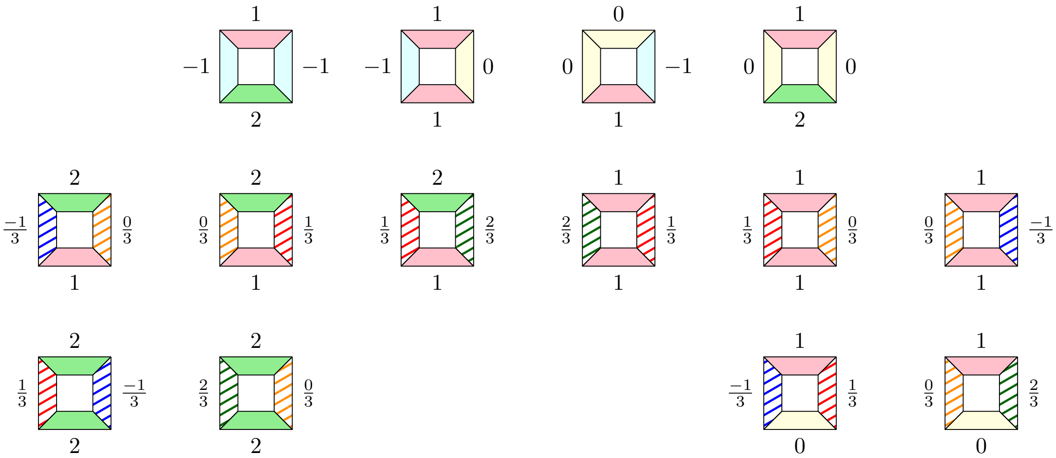
Lemma (Kari, 1996)

There is at least one periodic tiling of the plane (with line encodings).

Theorem (Kari, 1996)

There is an aperiodic tileset of 14 tiles.

14 tiles



1
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−1
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0
−1
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$\frac{0}{3}$
2
1
$\frac{-1}{3}$
$\frac{1}{3}$
0
1
$\frac{0}{3}$
$\frac{2}{3}$
0

Properties

Line averages always exist

We have tilings but only aperiodic tilings.

Nothing guarantees that line averages exist!
We could have "emergent behavior" — but that behavior has to be aperiodic.

Proposition (Durand, Γ, Grandjean, 2014)

In any tiling, the line averages exist.

Lemma

The dynamical system f has dense orbits.

Up to a change of variable, the dynamical system reduces to:

$$y \mapsto y + \frac{\log(2)}{\log(2) - \log(2/3)} \mod 1,$$

which has dense orbits. □

Line averages always exist

Lemma

The dynamical system f has dense orbits.

Proposition (Durand, Γ, Grandjean, 2014)

In any tiling, the line averages exist.

- Suppose the line 0 has no average.
- We have large blocks of average ρ and large blocks of average $\rho' < \rho$.
- Because the orbits of f are dense, we have $f^n(\rho') < 1 < f^n(\rho)$ for some n.
- The line n is both over alphabet $\{0, 1\}$ and $\{1, 2\}$, which is impossible.

□

Factor complexity

Fix a tileset T.

Definition

The **complexity** $P_T(n)$ is the number of distinct $n \times n$-blocks that appear in T-tilings.

The growth of P_T encodes the "chaos level" of T-tilings.

Definition

The **topological entropy** h_T is the following limit (that always exists):

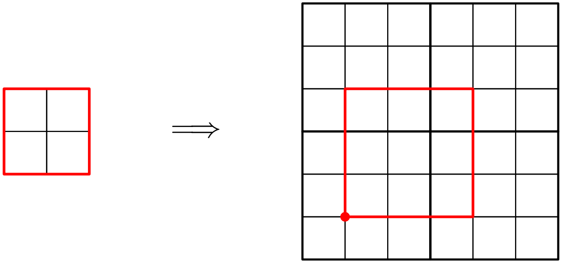
$$\lim_{n \to \infty} \frac{\log_2 P_T(n)}{n^2}.$$

Entropy $=$ how many bits of information in each cell of a T-tiling?

Topological entropy of substitutive tilesets

Theorem (folklore)

If T is a substitutive tileset, then $h_T = 0$.



$$P_T(n) \leq P_T(n/k) + k^2$$

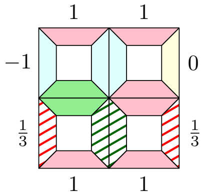
Interchangeable pairs

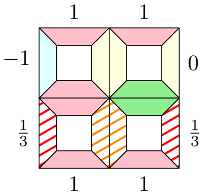
Theorem (Durand, Γ, Grandjean, 2014)

If K is Kari's tileset, then $h_K > 0$.

Definition

An **interchangeable pair** is a pair of distinct blocks with the same colors on the borders.





Lemma

The above interchangeable pair appears "often" in any tiling.

This comes from the denseness of f.

Conclusion

Recap and thank you

- Domino problem undecidable $\implies$ aperiodic tilesets exist
- Substitutive tilesets (e.g. Robinson) are aperiodic
- Averages and line encodings; T_2 and $T_{2/3}$ multiply averages
- $f : x \mapsto 2x$ or $2x/3$ such that $f(x) \in [\frac{2}{3}, 2]$ is aperiodic + has dense orbits
- Kari's tileset is aperiodic because f is, and has tilings by line encodings
- Line averages always exist because f has dense orbits
- Interchangeable pairs occur often $\implies$ doubly-exponentially many blocks of size $n \times n$

Thank you for your attention!