Entropy, pressure, and densities for $\mathbb{Z}^d$-SOFT

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Overview

- $\mathbb{Z}$-SOFT
- Entropies of $\mathbb{Z}$-SOFT
- $\mathbb{Z}^d$-SOFT, NNSOFT and Wang tiles
- Entropies of $\mathbb{Z}^d$-SOFT
- A symmetricity condition
- Pressure and its conjugate

Figure: Uri Natan Peled, Photo - December 2006

Uri N. Peled

Uri was born in Haifa, Israel, in 1944.
Education:
Hebrew University, Mathematics-Physics, B.Sc., 1965.
Weizmann Institute of Science, Physics, M.Sc., 1967
University of Waterloo, Mathematics, Ph.D., 1976
University of Toronto, Postdoc in Mathematics, 1976–78
Appointments:
1978–82, Assistant Professor, Columbia University
1982–91, Associate Professor, University of Illinois at Chicago
1991–2009, Professor, University of Illinois at Chicago
Areas of research: Graphs, combinatorial optimization, boolean functions.
Uri published about 57 papers and one influential book jointly with Mahadev
THRESHOLD GRAPHS and related topics
Uri died September 4, 2009 after a long battle with brain tumor.

$\mathbb{Z}$-Subshift

View $[n] = \{1, \ldots, n\}$, as $n \geq 2$ particles (alphabet)

Hamming distance on $[n]$: $d_h(p, q) = \delta_{pq}$

$\phi : \mathbb{Z} \to [n]$ configuration $\phi = \{\phi(i), i \in \mathbb{Z}\}$, $\pi_i(\phi) = \phi(i)$

$[n]^{\mathbb{Z}}$ configurations space, maps from $\mathbb{Z}$ to $[n]$

metric on $[n]^{\mathbb{Z}} : d(\phi, \psi) = \sum_{i \in \mathbb{Z}} 2^{-|i|} d_h(\phi(i), \psi(i))$

$[n]^{\mathbb{Z}}$-complete metric space, with diameter 3

The shift map: $\sigma(\phi)(i) = \phi(i + 1), i \in \mathbb{Z}$

$\mathcal{S} \subset [n]^{\mathbb{Z}}$ subshift: $\mathcal{S}$-closed, $\sigma(\mathcal{S}) = \mathcal{S}$

$\mathbb{Z}$-Subshift Of Finite type (SOFT)

$\exists$ finite window $W \subset \mathbb{Z}$, admissible configurations $A \subset [n]^W$

s. t. $\pi_W(\phi) \in A \iff \phi \in \mathcal{S}$

Near Neighbor SOFT (NNSOFT): $\exists$ digraph $\Gamma = ([n], \vec{E})$:

$\pi_{[2]}(\phi) \in \mathcal{S} \iff (\phi(1), \phi(2)) \in \vec{E})$

Equivalently: $\phi \in \mathcal{S} \iff (\phi(i), \phi(i+1)) \in \vec{E}$ for all $i \in \mathbb{Z}, \phi \in \mathcal{S}$

Every NNSOFT is Wang tiling: every diedge $\vec{pq}$

corresponds to an interval with left and right painted in colors p and q

Every Wang tiling is NNSOFT: corresponds to a diedge $\vec{pq}$

NNSOFT is more efficient presentation than Wang tiling for $\mathbb{Z}$-SOFT

Every $\mathbb{Z}$-SOFT can be coded as NNSOFT

Can assume $W = [M], M \geq 2$

Code each element of allowable configuration as a particle $a_1, \cdots, a_{|A|}$

view a and b as configurations on $\{1, \ldots, M\}$ and $\{2, \ldots, M+1\}$

$\vec{ab}$ allowable iff a and b have the same projections on $\{2, \ldots, M\}$

Equivalently $\pi_{\{2,\ldots,M\}}(a) = \pi_{\{1,\ldots,M-1\}}(b)$ if $a, b \in A$.

$\mathbb{Z}$-SOFT is a binfinite walk on a digraph Γ

Γ symmetric, (undirected graph): $\vec{pq} \in \Gamma \iff \vec{qp} \in \Gamma$

$\mathbb{Z}$-SOFT empty set iff Γ has no dicycle

Entropies of $\mathbb{Z}$-SOFT I

$W^m(\Gamma)$ -all allowable Γ words of length m, all walks on Γ of length m

$|W^{k+m}(\Gamma)| \le |W^k(\Gamma)||W^m(\Gamma)| \Rightarrow$

subadditivity: $\log|W^{k+m}(\Gamma)| \le \log|W^k(\Gamma)| + \log|W^m(\Gamma)|$

Fekete lemma: $h_{com}(\mathcal{S}) = \lim_{m\to\infty} \frac{\log|W^m(\Gamma)|}{m} \le \frac{\log|W^k(\Gamma)|}{k}, l \in \mathbb{N}$

Combinatorial entropy, or capacity

Theorem: $h_{com}(\Gamma) = h_{top}(\Gamma)$-topological entropy of $\mathcal{S}$

Proof outline: Topological entropy is the growth of ε-separated points

equivalent to combinatorial entropy

Entropies of $\mathbb{Z}$-SOFT II

$W^m(\Gamma) = \mathbf{1}^\top A^{m-1}\mathbf{1}$, $A = A(\Gamma)$ adjacency matrix of Γ

$h_{com}(\Gamma) = \lim_{m\to\infty} \frac{\log \mathbf{1}^\top A^{m-1}\mathbf{1}}{m} = \log \rho(A)$

$W^m_{per}(\Gamma)$-words of length $m+1$: first letter=last letter

$|W^m_{per}(\Gamma)| = \mathrm{Tr}\, A^m$, $A\mathbf{u} = \rho(A)\mathbf{u}, A^\top\mathbf{v} = \rho(A)\mathbf{v}, \mathbf{v}^\top\mathbf{u} = 1$

$h_{per}(\Gamma) = \limsup_{m\to\infty} \frac{\log |W^m_{per}(\Gamma)|}{m} = \log \rho(A)$

Theorem: $h_{com}(\Gamma) = h_{top}(\Gamma) = h_{per}(\Gamma)$

Expl. $A^m = [a_{ij}^{(m)}] \in Z_+^{n\times n}$, $a_{ij}^{(m)}$-$\#$-config: $\underbrace{i\ldots j}_{m+1}$

A-primitive: $A^m = \rho(A)^m\mathbf{u}\mathbf{v}^\top(1+o(1))$, else use Frobenius normal form

The measure of maximal entropy

$\mathrm{C}(i_1,\ldots,i_m)=\{\phi\in\mathcal{S},(\phi(1),\ldots,\phi(m))=(i_1,\ldots,i_m)\}$-cylinder of $\mathcal{S}$

Assume Γ strongly connected $\iff A=A(\Gamma)$-irreduble:

$A=[a_{ij}]\in\{0,1\}^{n\times n}, A\mathbf{u}=\rho\mathbf{u}, \mathbf{v}^\top A=\rho\mathbf{v}^\top, \mathbf{u},\mathbf{v}>\mathbf{0}, \mathbf{v}^\top\mathbf{u}=1$

Measure of maximal entropy, Parry measure:

$\mu(\mathrm{C}(i_1,\ldots,i_m)))=\rho(A)^{-m+1}v_{i_1}a_{i_1i_2}a_{i_2i_3}\cdots a_{i_{m-1}i_m}u_{i_m}$

If A nor irreducible, $\mathcal{S}(\Gamma)\neq\emptyset$ same formulas apply, where $\mathbf{u},\mathbf{v}\gneq\mathbf{0}$

$\mathbf{u},\mathbf{v}$ may not be unique:

Frobenius nomal (upper triangular) form

$\mathbb{Z}^d$-subshift

$[n]^{\mathbb{Z}^d}$-all configurations ϕ of n-particles on $\mathbb{Z}^d, d \in \mathbb{N}$

$d(\phi, \psi) = \sum_{\mathbf{i} \in \mathbb{Z}^d} 2^{-|\mathbf{i}|_1} h_d(\phi(\mathbf{i}), \psi(\mathbf{i}))$.

$\pi_X : [n]^{\mathbb{Z}^d} \to [n]^X$ projection on $X \subset \mathbb{Z}^d$

$\mathbf{e}_j = (\delta_{j1}, \ldots, \delta_{jd})^\top, j \in [d], \mathbf{1} = (1, \ldots, 1)^\top \in \mathbb{Z}^d$

j-shift: $\sigma_j(\phi)(\mathbf{i}) = \phi(\mathbf{i} + \mathbf{e}_j), \mathbf{i} \in \mathbb{Z}^d, j \in [d]$

$\mathcal{S} \subset [n]^{\mathbb{Z}^d}$-subshift, if

$\mathcal{S}$ closed

invariant: $\sigma_j(\mathcal{S}) = \mathcal{S}, j \in [d]$

$\mathbb{Z}^d$-SOFT I

$\mathbf{m} = (m_1, \ldots, m_d)^\top \in \mathbb{N}^d$, $[\mathbf{m}] = [m_1] \times \cdots [m_d], vol(\mathbf{m}) = m_1 \cdots m_d$

$\mathbf{m} + \mathbb{Z}\mathbf{e}_j$: $\mathbb{Z}$-line in direction of axis j through $\mathbf{m}$

A subshift $\mathcal{S} \subset \mathbb{Z}^d$-is $\mathbb{Z}^d$-SOFT:

$\exists$ finite window $W \subset \mathbb{Z}^d$, admissible configurations $A \subset [n]^W$

s. t. $\pi_W(\phi) \in A \iff \phi \in \mathcal{S}$

$\mathbb{Z}^d$-SOFT II

$\mathbb{Z}^d$-NNSOFT: given by $\Gamma = (\Gamma_1, \ldots, \Gamma_d)$, digraph $\Gamma_j = ([n], \vec{E_j}), j \in [d]$

Projection of $\phi \in \mathcal{S}(\Gamma)$ on $\mathbf{m} + \mathbb{Z}\mathbf{e}_j$ is $\mathbb{Z}$-NNSOFT given by Γ_j

$d = 2$: Every $\mathbb{Z}^2$-NNSOFT is Wang tiling

(Γ_1, Γ_2)-allowable filling of $[2] \times [2]$ is a Wang tile

square with the colors corresponding di-edges of Γ_1, Γ_2 respec.

colors of diedges of Γ_1 and Γ_2 are different

Every Wang tiling is $\mathbb{Z}^2$-NNSOFT on n-number of Wang tiles

$\vec{pq} \in \vec{E_1}$ if tile q can be to the right of tile p

$\vec{pq} \in \vec{E_2}$ if tile q can be on top of tile p

Every $\mathbb{Z}^d$-SOFT can be coded as NNSOFT

Can assume $W = [\mathbf{M}], \mathbf{M} = (M_1, \ldots, M_d) \geq 2 \cdot \mathbf{1}$

Code each element of allowable configuration as a particle $a_1, \cdots, a_{|A|}$

Assume a and b allowable configurations on $[\mathbf{M}]$

$\overrightarrow{ab} \in \overrightarrow{E_1}$ if

$\pi_{\{2,\ldots,M_1\}\times[M_2]\times\cdots\times[M_d]}(a) = \pi_{\{1,\ldots,M_1-1\}\times[M_2]\times\cdots\times[M_d]}(b)$ if $a, b \in A$

similar conditions for $\overrightarrow{ab} \in \overrightarrow{E_j}, j > 1$

$d = 2$: $\overrightarrow{ab} \in \overrightarrow{E_2}$ if

$\pi_{[M_1]\times\{2,\ldots,M_2\}}(a) = \pi_{[M_1]\times\{1,\ldots,M_2-1\}}(b)$ if $a, b \in A$

Decidability of $\mathbb{Z}^2$-SOFT

Köning 1927$\Rightarrow$ $\mathbb{Z}^2$-NNSOFT nonempty iff $W^{(n,n)}(\Gamma) \neq \emptyset \quad \forall n \in \mathbb{N}$

Outline of proof: $T_k = [-2^k, 2^k]^2 \cap \mathbb{Z}^2, k \in \mathbb{N}$, Θ_k-admissible

configuration of T_k. Choose infinite subseq. $\{\Theta_{k_i^1}\}_{i=1}^{\infty}$ with same

projection on T_1. Choose infinite subsequence $\{\Theta_{k_i^2}\}_{i=1}^{\infty}$ of $\{\Theta_{k_i^1}\}_{i=1}^{\infty}$

same projection on T_2, and so on. Take Cantor diagonal subsequence.

Wang: $\mathcal{S}(\Gamma)$-decidable if either $\exists$ periodic configuration or $\mathcal{S}(\Gamma) = \emptyset$

Berger 1966: $\exists$ nondecidable Wang tiling$=$nonperiodic (Gamard lec.)

Shahar Mozes 1989: $\mathbb{Z}^2$-ergodic theory yields nonperiodic Wang tilings

Jeandel-Rao 2021: 11 Wang tiles with 4 colors -minimal example

Combinatorial entropy of $\mathbb{Z}^d$-SOFT I

$\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{N}^d, d > 1, p \in [d],$

$\mathbf{m}^{\hat{p}} = (m_1, \ldots, m_{p-1}, m_{p+1}, \ldots, m_d), \mathbf{m} = (\mathbf{m}^{\hat{p}}, m_p)$

$W^{\mathbf{m}}(\Gamma)$ -allowable Γ configuration on box $[\mathbf{m}]$

$|W^{(\mathbf{m}^{\hat{p}}, j+k)}(\Gamma)| \le |W^{(\mathbf{m}^{\hat{p}}, j)}(\Gamma)||W^{(\mathbf{m}^{\hat{p}}, k)}(\Gamma)| \Rightarrow$

p-subadditivity: $\log|W^{(\mathbf{m}^{\hat{p}}, j+k)}(\Gamma)| \le \log|W^{(\mathbf{m}^{\hat{p}}, j)}(\Gamma)| + \log|W^{(\mathbf{m}^{\hat{p}}, k)}(\Gamma)|$

(1) $\log \rho(p, \mathbf{m}^{\hat{p}}) := \lim_{j \to \infty} \frac{\log|W^{(\mathbf{m}^{\hat{p}}, j)}(\Gamma)|}{j} \le \frac{\log|W^{(\mathbf{m}^{\hat{p}}, j)}(\Gamma)|}{j}$

$\rho(p, \mathbf{m}^{\hat{p}})$-spectral radius $\mathbb{Z}$-NNSOFT in direction p on states $W^{\mathbf{m}^{\hat{p}}}(\Gamma^{\hat{p}})$

Combinatorial entropy of $\mathbb{Z}^d$-SOFT II

FIx j, observe $\frac{\log|W^{(\mathbf{m}^{\hat{p}},j)}(\Gamma)|}{j}$ subadditive in variable $q \in [d] \setminus \{p\}$

$\log \rho(p, \mathbf{m}^{\hat{p}})$-subbaddive in each variable in $[d] \setminus \{j\}$

(2) $h_{com}(\Gamma) = \lim_{\mathbf{m}\to\infty} \frac{\log|W^{\mathbf{m}}(\Gamma)|}{vol(\mathbf{m})} \le \frac{\log \rho(p,\mathbf{m}^{\hat{p}})}{vol(\mathbf{m}^{\hat{p}})} \le \frac{\log|W^{\mathbf{m}}(\Gamma)|}{vol(\mathbf{m})}, \mathbf{k} \in \mathbb{N}^d$

$\log 0 = -\infty \Rightarrow h_{com}(\Gamma) \in \{-\infty\} \cup \mathbb{R}_+$

Theorem: $h_{com}(\Gamma) = h_{top}(\Gamma)$-topological entropy of $\mathcal{S}(\Gamma)$

No periodic solution for nonempty $\mathbb{Z}^d$-NNSOFT

periodic state: $\phi \in \mathcal{S}_{per,\mathbf{m}}(\Gamma), \quad \phi(\mathbf{i}+m_j\mathbf{e}_j) = \phi(\mathbf{i}), \mathbf{i} \in \mathbb{Z}^d, m_j \in \mathbb{N}, j \in [d]$

$W^{\mathbf{m}}_{per}(\Gamma) = \{\pi_{[\mathbf{m}+\mathbf{1}]}(\phi), \phi \in \mathcal{S}_{per,\mathbf{m}}(\Gamma)\}$

all states in $W^{\mathbf{m}}(\Gamma)$ extending to periodic states in $\mathcal{S}_{per,\mathbf{m}}(\Gamma)$

Lemma: Assume $d > 1, p \in [d]$ and $W^{\mathbf{m}^{\hat{p}}}(\Gamma^{\hat{p}})$ nonempty.

$\mathcal{S}_{(\mathbf{m}^{\hat{p}})}(\Gamma) = \emptyset$, iff $\mathbb{Z}$-SOFT induced by Γ_p on $W^{\mathbf{m}^{\hat{p}}}(\Gamma^{\hat{p}})$ is empty

equivalently $A^{|W^{\mathbf{m}^{\hat{p}}}|}(\Gamma^{\hat{p}})(p, W^{\mathbf{m}^{\hat{p}}}(\Gamma^{\hat{p}}) = 0$

Corollary $\mathcal{S}((\Gamma_1, \Gamma_2))$ has no periodic states iff for each $i \in \mathbb{N}$ s.t.

$W^i_{per}(\Gamma_1) \neq \emptyset$, $\mathbb{Z}$ -SOFT induced by Γ_2 on $W^i_{per}(\Gamma_1)$ is empty

Periodic entropies

$\log 0 = -\infty$

$h_{per}(\Gamma) = \limsup_{\mathbf{m}\to\infty} \frac{\log |W^{\mathbf{m}}_{prod}(\Gamma)|}{vol(\mathbf{m})} \in \{-\infty\} \cup \mathbb{R}_+$

Theorem (Friedland 1997) If $d-1$ digraphs in $(\Gamma_1, \ldots, \Gamma_d)$ are symmetric then $h_{top}(\Gamma) = h_{per}(\Gamma)$, and the entropy is computable

The case $d = 2$ and Γ_1 symmetric I

Claim: $\mathcal{S}(\Gamma) \neq \emptyset \iff \rho(2,2) > 0$

Proof: $\rho(2,2) > 0 \iff W^{(2,i)}(\Gamma) \neq \emptyset \; \forall i \geq 2$

Assume $W^{(2,i)}(\Gamma) \neq \emptyset$, $A(1,i) \in \{0,1\}^{N \times N}, N = |W^i(\Gamma_2)|$

nonzero symmetric transfer matrix on states in $W^i(\Gamma_2) \Rightarrow$

$W^{i,i}(\Gamma) \neq \emptyset \Rightarrow \mathcal{S}(\Gamma) \neq \emptyset$

The case $d = 2$ and Γ_1 symmetric II

$W_{per,1}^{(k,i)}(\Gamma)$-$i$-states induced by Γ_2 on $W_{per}^{k}(\Gamma_1)$

$A_{per}(2,k)$-transfer matrix in direction 2 on states $W_{per}^{k}(\Gamma_1)$

Observe $\rho_{per}(2,k) = \rho(A_{per}(2,k)) \le \rho(2,k)$ (explained later)

$A(1,i)^{2m} \succeq 0 \Rightarrow \rho(1,i)^{2m} = \rho(A^{2m}(1,i)) \le \mathrm{Tr}\, A^{2m}(1,i) =$

$|W_{per,1}^{(2m,i)}(\Gamma)| = \mathbf{1}^\top A_{per}^{i-1}(2,2m)\mathbf{1}, \mathbf{1} \in \mathbb{R}^N, N = |W_{per}^{2m}(\Gamma_1)|$

$\frac{\log \rho(1,i)}{i} \le \frac{\log |W_{per,1}^{(2m,i)}(\Gamma)|}{(2m)i} = \frac{\log |W_{per}^{(2m,i)}(\Gamma)|}{(2m)i}$ for $i \gg 1$ fixed m

$$i \to \infty \Rightarrow \frac{\log \rho(1,i)}{i} = h(\Gamma) \le \frac{\log \rho_{per}(2,2m)}{2m} \qquad (3)$$

let $m \to \infty \Rightarrow h_{com}(\Gamma) \le h_{per}(\Gamma) \Rightarrow h_{com}(\Gamma) = h_{per}(\Gamma)$

Computability of $h(\Gamma)$ for $d = 2$ and Γ_1 symmetric

(4) $\frac{\log \rho(1,p+2q+1)}{p} - \frac{\log \rho(1,2q+1)}{p} \le h(\Gamma) \le \frac{\log \rho_{per}(2,2m)}{2m}, p \in \mathbb{N}, q \in \mathbb{Z}_+$

RHS of (5) is shown in (3)

max characterization of $\rho(1,i)^p \ge \frac{\mathbf{x}^\top A(1,i)^p \mathbf{x}}{\mathbf{x}^\top \mathbf{x}}$ choose $\mathbf{x} = A(1,i)^q \mathbf{1} \Rightarrow$

$$\frac{\log \rho(1,i)}{i} \ge \frac{\log \mathbf{1}^\top A(1,i)^{p+2q} \mathbf{1}}{pi} - \frac{\log \mathbf{1}^\top A(1,i)^{2q} \mathbf{1}}{pi} = \frac{\log W^{p+2q+1,i}(\Gamma)}{pi} - \frac{\log W^{2q+1,i}(\Gamma)}{pi}$$

let $i \to \infty$ to obtain LHS of (4)

Use (2)and $q = 0$: $\frac{\log \rho(1,p+1)}{p} - \frac{\log \rho(1,1)}{p} \le h(\Gamma) \le \frac{\log \rho(1,p+1)}{p+1}$ (5)

Markley-Paul 1981 showed (5) for primitive symmetric $A(\Gamma_1)$

Note that computation of $\rho(1, p+1)$ is exponential in p

as the number of nonzero entries of $A(1, p+1)$ is at least $O(\rho(2,2)^{p+1})$

Observations on $\rho_{per}(j,q)$ and $\rho(j,q)$

Assume$\mathcal{S}((\Gamma_1,\Gamma_2)) \neq \emptyset$

Claim $\rho_{per}(j,q) \le \rho(j,q)$ for $j \in [2], q \ge 2$, $\rho(2,q) \le \rho(2,q+1)$

Proof: Enough to assume $\rho_{per}(2,q) \ge 1$. View

$A_{per}(2,q) = [a_{st}], s,t \in W^q(\Gamma_1)$ s.t. $a_{st} = 0$, unless $s,t \in W^q_{per}(\Gamma_1)$

$A_{per}(2,q) \le A(2,q) \Rightarrow \rho_{per}(2,q) \le \rho(2,q)$

As Γ_1 is symmetric extend each k-walk to $k+1$ walk by reversing

last edge $\Rightarrow A(2,k)$ prin. subm. of $A(2,k+1) \Rightarrow \rho(2,k) \le \rho(2,k+1)$

Γ_1 symmetric $\Rightarrow \lim_{k\to\infty} \frac{\log \rho(2,k)}{k} = \limsup_{k\to\infty} \frac{\log \rho_{per}(2,k)}{k} = h(\Gamma)$

Residual entropy of ice

$\mathbb{Z}^2$-NNSOFT particles, where no two same particles are adjacent

$\Gamma_1 = \Gamma_2 = K_3$-complete graph on 3 vertices (symmetric)

E. Lieb computed periodic entropy 1967:

$h(\Gamma) = \frac{3}{2}\log\frac{4}{3} = \log\left(4/3\right)^{3/2} = 0.43152\cdots$

$h(\Gamma)$ not a log of algebraic integer as for $\mathbb{Z}$-SOFT

For $p = 2$ lower bound in (5) is 0.4122579570

Upper bound in (4) for $m = 2$: 0.462989385

Monomer-dimer model in $\mathbb{Z}^d$

A monomer is a particle that occupies a point in $\mathbb{Z}^d$,

or a d-unit cube centered at a point in $\mathbb{Z}^d$

A dimer is a domino in positioned in direction $\mathbf{e}_j, j \in \mathbb{Z}^d, j \in [d]$

or two glued unit cubes

One can consider just dimers without monomers

The dimer coding correspond to $n = 2d$ particles

The monomer-dimer model corresponds to $n = 2d + 1$ particles

$\mathbb{Z}^2$-dimer entropy is $\frac{1}{\pi} = \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)^2} = 0.29156090\ldots$

Density of the Monomer-dimer model

For a fixed $p \in [0, 1]$: $W^{\mathbf{m}}_{p_1,p_2}(\Gamma) \subset W^{\mathbf{m}}(\Gamma)$-all configurations with density of dimers $\in (p_1, p_2)$

$h((p_1, p_2), \Gamma) = \lim_{\mathbf{m}\to\infty} \frac{\log |W^{\mathbf{m}}_{p_1,p_2}(\Gamma)|}{|W^{\mathbf{m}}(\Gamma)}$, $h(p, \Gamma) = \lim_{p_1\to p} h((p_1, p), \Gamma)$

Claim $h(\Gamma) = \max_{p\in[0,1]} h(p, \Gamma)$

Pressure function for an external field

$c_i(\phi)$-# i-particles in $\phi \in W^{\mathbf{m}}(\Gamma)$, $\mathbf{c}(\phi) = (c_1(\phi), \ldots, c_n(\phi))^\top \in \mathbb{Z}_+^n$

$W^{\mathbf{m}}(\Gamma, \mathbf{c}) = \{\phi \in W^{\mathbf{m}}(\Gamma), \mathbf{c}(\phi) = \mathbf{c}\}$, $\mathbf{c} \in \mathrm{vol}(\mathbf{m})\Pi_n \cap \mathbb{Z}_+^n$

$$\mathrm{P}_\Gamma(\mathbf{u}, \mathbf{m}) = \frac{\log \sum_{\phi \in W^{\mathbf{m}}(\Gamma)} \exp(\mathbf{c}(\phi)^\top \mathbf{u})}{vol(\mathbf{m})}, \mathbf{u} = (u_1, \ldots, u_n)^\top \in \mathbb{R}^n$$

$\mathrm{P}_\Gamma(\mathbf{u}, \mathbf{m})$-convex in $\mathbf{u}$, subadditive in each m_i

$\nabla \mathrm{P}_\Gamma(\mathbf{u}, \mathbf{m}) \in \mathbb{P}_n$ - the set of probability vectors in $\mathbb{R}^n \Rightarrow$

$|\mathrm{P}_\Gamma(\mathbf{v}, \mathbf{m}) - \mathrm{P}_\Gamma(\mathbf{u}, \mathbf{m})| \leqslant \|\mathbf{v} - \mathbf{u}\|_\infty$

$\mathrm{P}_\Gamma(\mathbf{u} + t\mathbf{1}, \mathbf{m}) = t + \mathrm{P}_\Gamma(\mathbf{u}, \mathbf{m}) \Rightarrow$ can assume $u_n = 0$

$\mathrm{P}_\Gamma(\mathbf{u}) := \lim_{\mathbf{m} \to \infty} \mathrm{P}_\Gamma(\mathbf{u}, \mathbf{m}) \leq \mathrm{P}_\Gamma(\mathbf{u}, \mathbf{m})$

convex, $\quad |\mathrm{P}_\Gamma(\mathbf{v}) - \mathrm{P}_\Gamma(\mathbf{u})| \leqslant \|\mathbf{v} - \mathbf{u}\|_\infty$

subgradient $\partial \mathrm{P}_\Gamma(\mathbf{u}) \in \Pi_n$ exists $\forall \mathbf{u} \in \mathbb{R}^n$, $\quad \nabla \mathrm{P}_\Gamma)(\mathbf{u}) \exists$ a.e.

Pressure function for an external field for $\mathbb{Z}$-SOFT

$d = 1$: $\mathrm{P}_\Gamma(\mathbf{u}) = \log \rho(A(\Gamma, \mathbf{u}))$, $A(\Gamma, \mathbf{u}) = [a_{ij} \exp((u_i + u_j))/2]$

$A(\Gamma)$-irreducible$\Rightarrow$ $\mathrm{P}_\Gamma(\mathbf{u})$ analytic in $\mathbf{u}$, and

Π_Γ-a convex hull of probability vectors

corresponding to uniform distribution on cycles in Γ

Example: Γ has on cycle on $[n]$: $\Pi_\Gamma = \{\frac{1}{n}\mathbf{1}\}$

$\mathrm{P}_\Gamma(\mathbf{u}) = \frac{\sum_{i=1}^n u_i}{n}, \qquad \nabla \mathrm{P}_\Gamma(\mathbf{u}) = \frac{1}{n}\mathbf{1} \quad \forall \mathbf{u} \in \mathbb{R}^n$

$A(\Gamma)$ reducible-Γ has k-strongly connected components $\Gamma_1, \ldots, \Gamma_j$

$\mathrm{P}_\Gamma(\mathbf{u}) = \max_{j \in [k]} \log A(\Gamma_j, \mathbf{u}_j)$

It is possible that $\mathrm{P}_\Gamma(\mathbf{u})$-not differentialble at some points

Density points

$\mathbf{p} \in \Pi_n$ idensity point of $C(\mathbb{Z}^d)$: $\exists \{\mathbf{m}_q\} \subset \mathbb{N}^d, \mathbf{c}_q \in \big(vol(\mathbf{m}_q)\Pi_n\big) \cap \mathbb{Z}_+^n$,

$W^{\mathbf{m}_q}(\Gamma, \mathbf{c}_q) \neq \emptyset, q \in \mathbb{N}$, (5) $\lim_{\mathbf{m}_q \to \infty} \frac{1}{\mathrm{vol}(\mathbf{m}_q)} \mathbf{c}_q = \mathbf{p} \, (\in \Pi_n)$

Π_Γ closed nonempty set of density pts (Cantor diagonal sequence)

$h_\Gamma(\mathbf{p}) = \limsup_{\mathbf{m}_q \to \infty} \frac{\log |W^{\mathbf{m}^q}(\Gamma, \mathbf{c}_k)|}{vol(\mathbf{m}_q)}$ (≥ 0) for all $\mathbf{m}_q$ satisfying (5)

$h_\Gamma(\mathbf{p})$ - density entropy

is upper-semicontinuous on Π_Γ

Conjugate pressure function I

Legendre-Fenchel transform: $\mathrm{P}^*_\Gamma(\mathbf{v}) := \sup_{\mathbf{u}\in\mathbb{R}^n} \mathbf{v}^\top\mathbf{u} - \mathrm{P}_\Gamma(\mathbf{u}), \mathbf{v} \in \mathbb{R}^m$

convex, $\mathbf{v} \in \partial P_\Gamma(\mathbf{u}) \Rightarrow P^*_\Gamma(\mathbf{v}) = \mathbf{v}^\top\mathbf{u} - \mathrm{P}_\Gamma(\mathbf{u})$

$\{\mathbf{v}, \mathrm{P}^*_\Gamma(\mathbf{v}) < \infty\} =$dom $\mathrm{P}^*_\Gamma \supseteq \partial\mathrm{P}_\Gamma(\mathbb{R}^n) \supseteq$r.i. dom$(\mathrm{P}^*_\Gamma)$, $\quad \mathrm{P}^{**}_\Gamma = \mathrm{P}_\Gamma$

THM(F-Peled) 2011

$h_\Gamma(\mathbf{p}) \le -\mathrm{P}^*_\Gamma(\mathbf{p}), \quad \mathbf{p} \in \Pi_\Gamma, \quad$ dom $P^*_\Gamma = conv\,\Pi_\Gamma$

$P_\Gamma(\mathbf{u}) = \max_{\mathbf{p}\in\Pi_\Gamma}(\mathbf{p}^\top\mathbf{u} + h_\Gamma(\mathbf{p})), \mathbf{u} \in \mathbb{R}^n \Rightarrow$

$\mathrm{P}_\Gamma(\mathbf{0}) = h_\Gamma = \max_{\mathbf{p}\in\Pi_\Gamma} h_\Gamma(\mathbf{p})$

Let $\Pi_\Gamma(\mathbf{u}) := \arg\max_{\mathbf{p}\in\Pi_\Gamma}(\mathbf{p}^\top\mathbf{u} + h_\Gamma(\mathbf{p}))$

$h_\Gamma(\mathbf{p}) = \mathrm{P}_\Gamma(\mathbf{u}) - \mathbf{p}^\top\mathbf{u} = -P^*_\Gamma(\mathbf{p})$ for $\mathbf{p} \in \Pi_\Gamma(\mathbf{u}), \mathbf{u} \in \mathbb{R}^n$

Conjugate pressure function II

Generalization of Hammersley for the monomer-dimer entropy:

h_Γ is concave on a convex subset of Π_Γ

$h_\Gamma(\mathbf{p}) = -\mathrm{P}^*_\Gamma(\mathbf{p})$ continuous, has subdifferential on

$\Pi_\Gamma(\mathbb{R}^n) = \cup_{\mathbf{u}\in\mathbb{R}^n}\Pi_\Gamma(\mathbf{u})$,

If $[n]$ has a friendly particle, or configutation then Π_Γ is convex

For monomer-dimer $\mathbb{Z}^d$-SOFT, full d-dimers and monomer

are friendly configurations

Identification of particles

In the study of monomer-dimer models one identifies dimers in

in each direction $i \in [d]$ in pressure function

Computation of pressure function reduces to a function in one variable

In general $\mathbb{Z}^d$-SOFT identify $i \equiv j$ by letting $u_i = u_j$ in $\mathrm{P}_\Gamma(\mathbf{u})$

First order phase transition

First order phase transition (FOPT) at $\mathbf{u} \in \mathbb{R}^n$ if $\nabla \mathrm{P}_\Gamma(\mathbf{u})$-does not exist

$h_\Gamma = 0 \Rightarrow \mathrm{P}_\Gamma = \max_{\mathbf{p}\in\Pi_\Gamma} \mathbf{p}^\top \mathbf{u} = \max_{\mathbf{p}\in conv\,\Pi_\Gamma} \mathbf{p}^\top \mathbf{u}$

In this case FOPT for $\mathbf{u} \neq \mathbf{0}$ if supporting hyperplane to $conv\,\Pi_\Gamma$

orthogonal to $\mathbf{u}$ passes at least through two points in Π_Γ

Hammersley's results

Hammersley in 60's studied extensively the monomer-dimer model. He showed $\Pi_\Gamma = \Pi_{d+1}$ for d-dimensional model $\mathbf{p} = (p_1, \ldots, p_d, p_{d+1})$
p_i-the dimer density in $\mathbf{e}_i$-direction $i = 1, \ldots, d$ p_{d+1}-the monomer density Hammersley studied $p := p_1 + \ldots + p_d$-the total dimer density
$h_d(p)$-the p-dimer density in $\mathbb{Z}^d$, $p \in [0, 1]$
He showed $h_d(p)$-concave continuous function on $[0, 1]$
Heilman and Lieb 72: $h_d(p)$ analytic on $(0, 1)$
No phase transition in parameter $p \in (0, 1)$

Graph estimates for $h_2(p)$

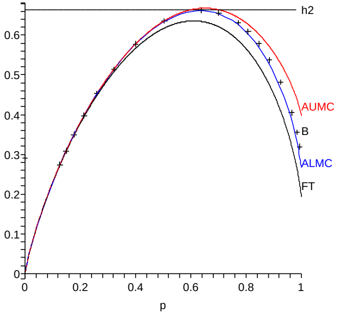


Figure 1: Monomer-dimer tiling of the 2-dimensional grid: entropy as a function of dimer density. FT is the Friedland-Tverberg lower bound, h2 is the true monomer-dimer entropy. B are Baxter's computed values. ALMC is the Asymptotic Lower Matching Conjecture. AUMC is the entropy of a countable union of $K_{4,4}$, conjectured to be an upper bound by the Asymptotic Upper Matching Conjecture.

Graphs of two dimensional pressure for MD

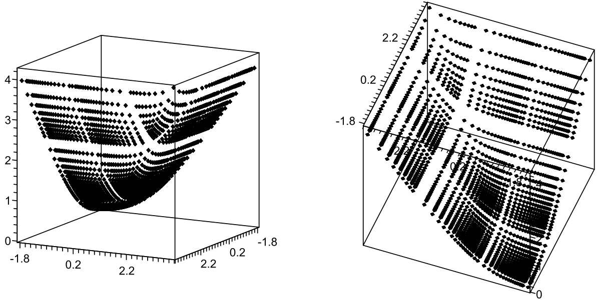
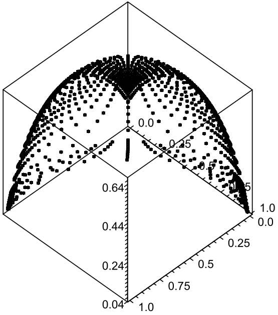


Figure 1: The graph of $\frac{\bar{P}_1(12,(v_1,v_2))}{12}$ for angles $\theta = 28^o, \varphi = 78^o$ and $\theta = -159^o, \varphi = 42^0$

Graphs of two dimensional density entropy for MD



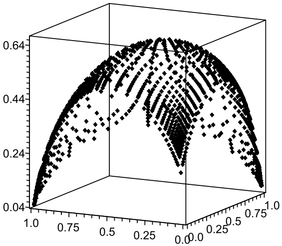


Figure 1: The graph of an approximation of $\bar{h}_2((p_1, p_2)$ for angles $\theta = 45^o, \varphi = 45^o$ and $\theta = -153^o, \varphi = 78^o$

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