An Introduction to Cluster Swapping

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TIFR-CAM

Many thanks to the organisers for inviting me to present this work.

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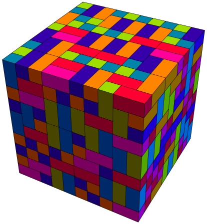
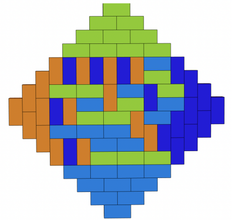
The model that I will talk about!

Dimers in 2D are *dominoes*, e.g. 1×2 or 2×1 blocks.

Dimers in 3D are *bricks*, e.g. $2 \times 1 \times 1$ or $1 \times 2 \times 1$ or $1 \times 1 \times 2$ blocks.

A dimer tiling of a region $R \subset \mathbb{Z}^2$ or $\mathbb{Z}^3$ is a collection of dimer tiles such that every square/cube is covered by exactly one tile such that the vertices of the tiles are subsets of $\mathbb{Z}^2/\mathbb{Z}^3$.





The dimer model

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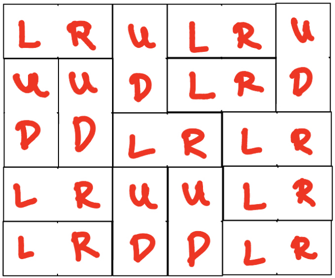
One could think of this as a shift space by placing symbols on various parts of the dimer and setting up appropriate adjacency rules but we will ignore this for the purpose of the talk.



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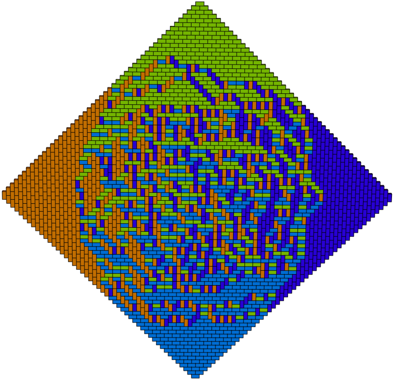
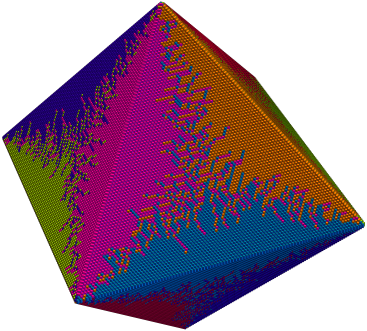
However sometimes Gibbs measures are far from unique. You may have an uncountably many of them for the same model varying according to a set parameter. This can be very useful in establishing a kind of variational principle.

Why does one want to talk about cluster swapping: Artic Circle Phenomena





The images are from simulations of a uniform dimer tilings of the Aztec diamond (Jockusch, Propp and Shor (1998) and later Cohn, Kenyon and Propp(2001)) and the Aztecohedron (C. , Sheffield, Wolfram 2023). You can clearly see certain patterns appear. One of the possible routes to explain such phenomena uses cluster swapping.

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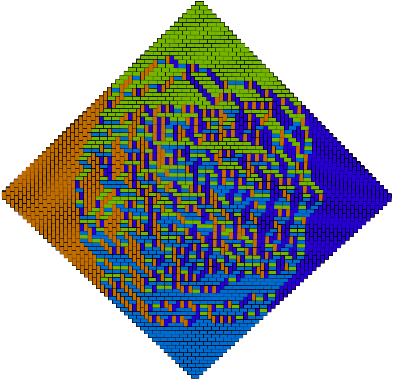

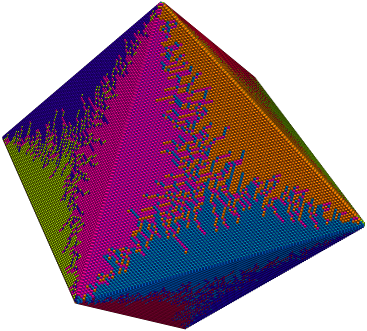

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By the way, it is still an open question to completely explain the picture on the right.

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This is a general technique which applies to many different models (a lot of which remains to be fully explored).

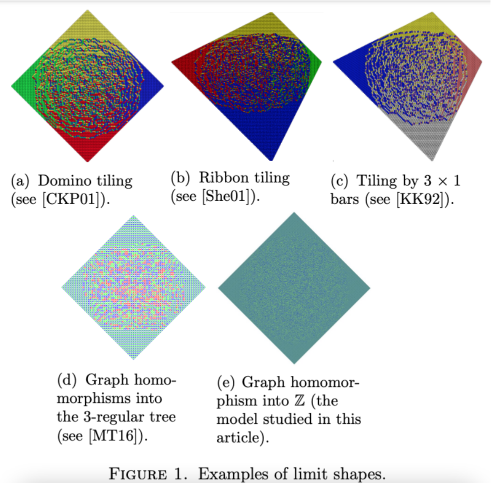

Figure 1: From the paper by Andrew Krieger, Georg Menz and Martin Tassy (2020)

“The effect of boundary conditions is, however, not entirely trivial . . . ”-
Kastelyn, 1962.

Outline

1. The basic toolbox
   1.1 What are uniform Gibbs measures?
   1.2 Lanford-Ruelle theorem.
   1.3 Dimers, discrete vector fields and the $\mathbb{Z}^3_{even}$ action
2. Mean-current and entropy maximisers
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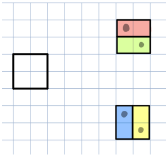
   2.2 Concavity of the entropy function.
   2.3 Statement of the main result.
   2.4 The baby case: Entropy maximisers of a given mean-current are uniform Gibbs measures.
3. The patching argument
   3.1 Dobrushin's theorem
   3.2 Ergodic Gibbs measures of the same mean-current have the same entropy
4. Cluster Swapping
   4.1 Swapping of paths in the double dimer model and its effect on mean-current and entropy
   4.2 The main result: Strict concavity of entropy as a function of the mean-current

The basic toolbox

A probability measure on the space of dimer tilings of $\mathbb{Z}^3$ is called a uniform Gibbs measure if for all finite sets $R \subset \mathbb{Z}^3$, conditioned on the appearance of R, the probability distribution is uniform on all possible extensions of the tiling to R.



On the left is a region R with two possible tilings (as given on the right). Thus for any uniform Gibbs measure on the space of dimer tilings, the conditional probabilities of seeing the tilings given on right are $1/2$ each (assuming that R appears with positive probability).

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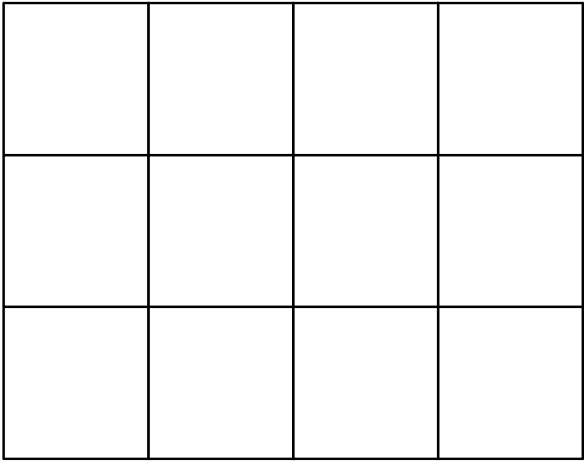

Figure 2: Division into grids

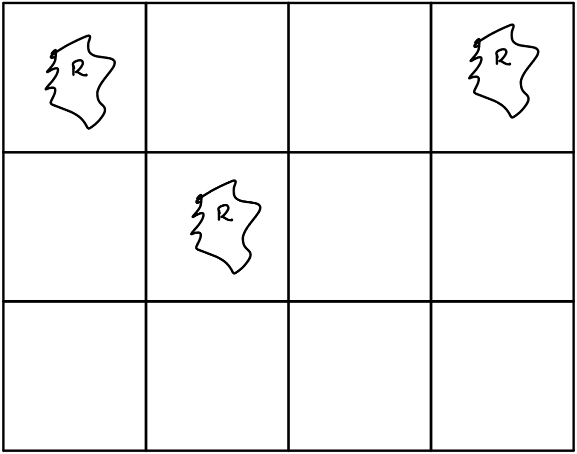

Figure 3: Resampling on translates of R in the grid regions

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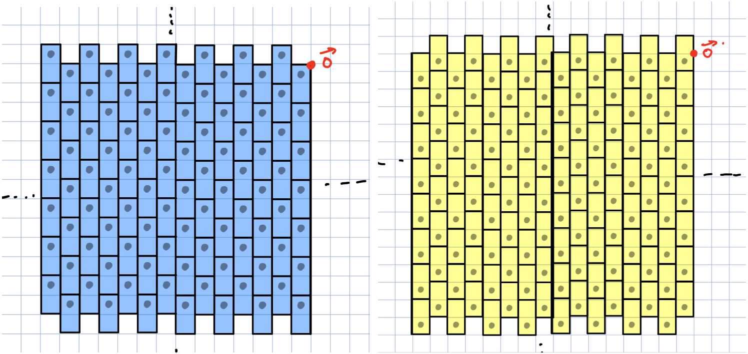

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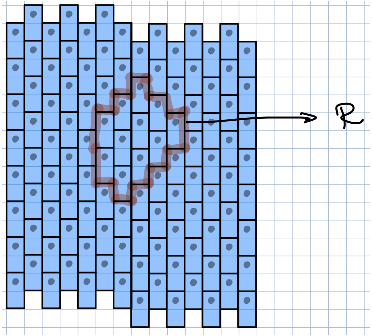

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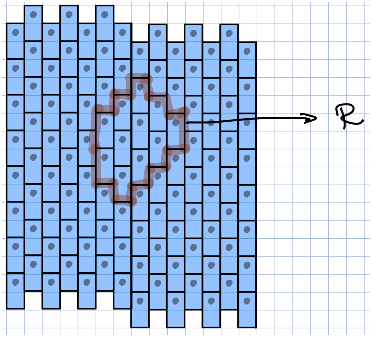

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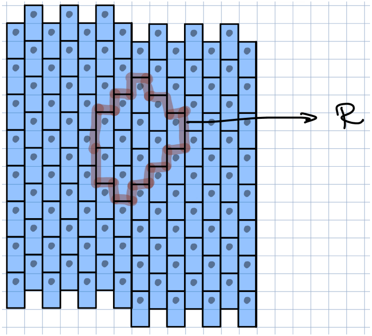
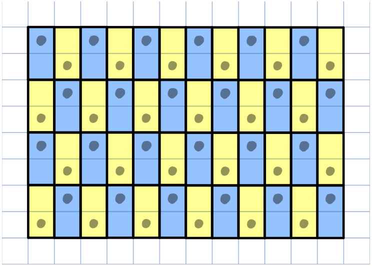
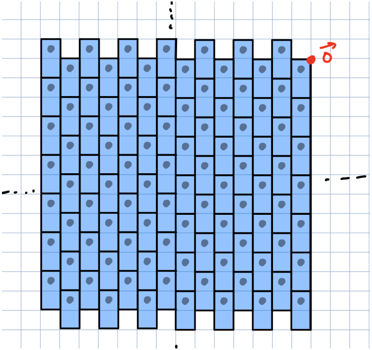


If you look at any region R appearing in the support of the measure, there is a unique way to fill it in. Thus it is a uniform Gibbs measure. However it has zero entropy. It is as far from being a measure of maximal entropy as you would imagine.

Are uniform Gibbs measures measure of maximal entropy?

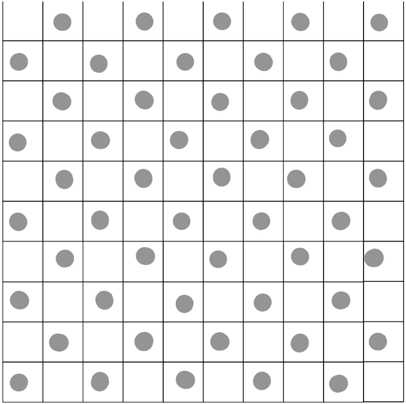
Note that though both the tilings given here use only vertical tiles, there is a clear difference between the two: The one on the left can be modified significantly to get other tilings while the one on the right cannot be modified at all.





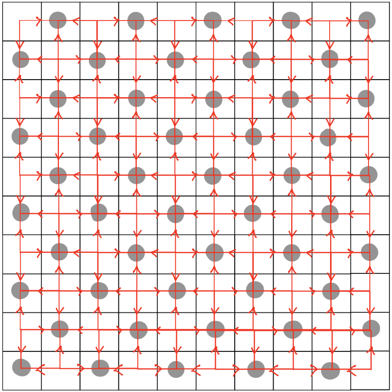
One way to distinguish these two different tilings, is to introduce a notion of parity. This can be done in various ways, we will associate a discrete vector fields with the tilings.

Label the even vertices of $\mathbb{Z}^3$ white and the odd ones black.

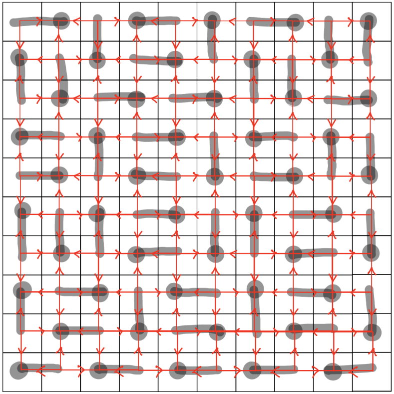


Discrete vector fields associated with dimer tilings

Now consider the flow of unit strength from white to adjacent black vertices.



For a given a domino tiling keep the flow along those edges which are part of the tiling.



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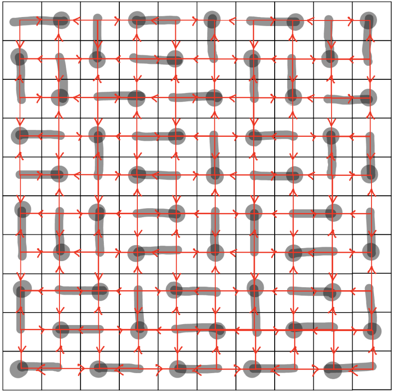

This gives a correspondence between 1) a dimer tiling τ of $\mathbb{Z}^3$ and 2) a *discrete vector field* v_τ: For each edge e of $\mathbb{Z}^3$ oriented from white to black,

$$v_\tau(e) = \begin{cases} 1 & e \in \tau \\ 0 & e \notin \tau. \end{cases}$$

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The take-away message is that the net flux through a box of even size is equal to 0.

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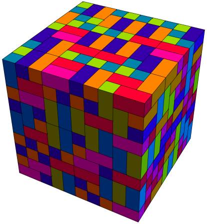
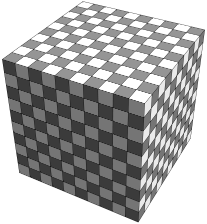
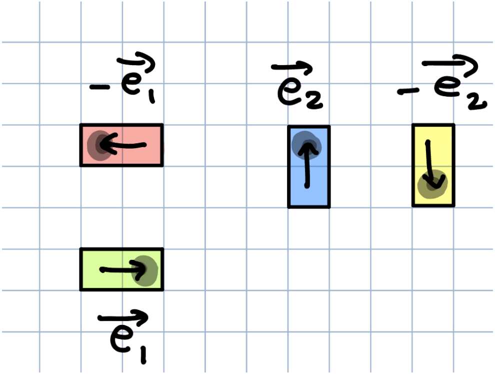


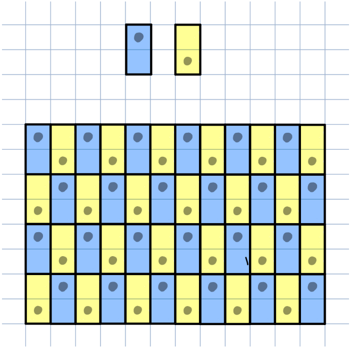


Indeed in $d = 3$, we thereby get 6 different colours. We label the tiles according to the direction of flow.

Direction of flow



The discrete vector field associated with the tilings: Zero flow and a lot of flexibility



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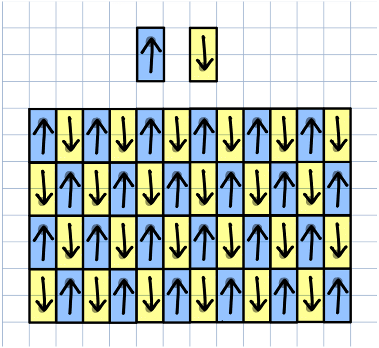

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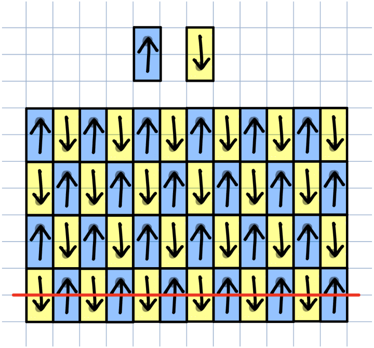
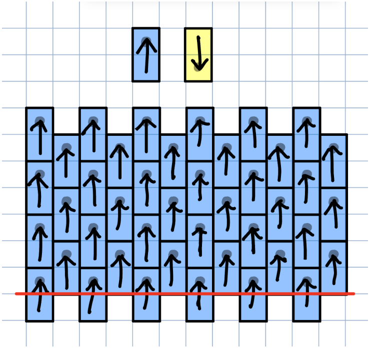


Note that this tiling is easy to modify. Now consider the vector field associated to the tiling. The net flux through the red hyperplane is equal to 0.

The discrete vector field associated with the tilings: Maximum flow and lack of flexibility



Here the tiling can't be modified and it corresponds to the maximum possible flow across the hyperplane.

The parity

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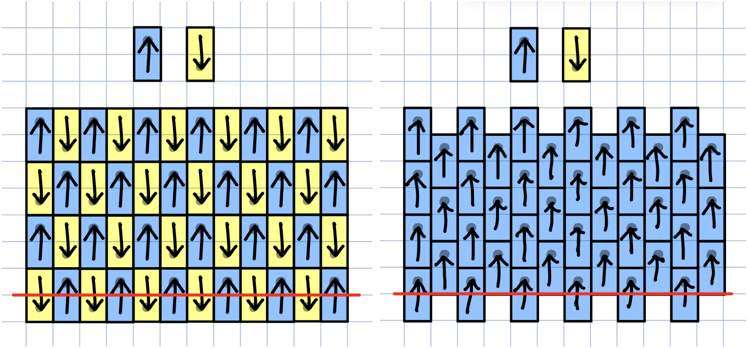
$$\mathbb{Z}^3_{even} = \{(i, j, k) \in \mathbb{Z}^3 \ : \ i + j + k \text{ is even}\}.$$

So far

1. We associate a discrete vector field with the space of dimer tilings.
2. The new flux across the boundary of an even sized box is 0.
3. We will look at parity preserving translations and hence restrict ourselves to the $\mathbb{Z}^3_{even}$ subaction.



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    1.3 Dimers, discrete vector fields and the $\mathbb{Z}^3_{even}$ action
2. Mean-current and entropy maximisers
    2.1 Mean-current for ergodic measures.
    2.2 Concavity of the entropy function.
    2.3 Statement of the main result.
    2.4 The baby case: Entropy maximisers of a given mean-current are uniform Gibbs measures.
3. The patching argument
    3.1 Dobrushin's theorem
    3.2 Ergodic Gibbs measures of the same mean-current have the same entropy
4. Cluster Swapping
    4.1 Swapping of paths in the double dimer model and its effect on mean-current and entropy
    4.2 The main result: Strict concavity of entropy as a function of the mean-current

Mean-current and entropy maximisers

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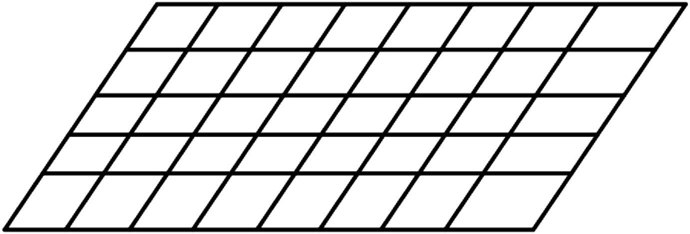
Now suppose μ is a $\mathbb{Z}^3_{even}$ invariant ergodic measure on the space of dimer tilings. Given any coordinate hyperplane, by applying the ergodic theorem along the hyperplane, we can measure the average flow across the surface.



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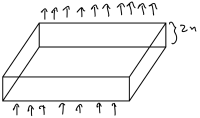

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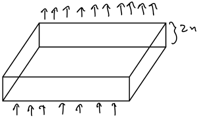

Thus the average flux passing through the hyperplane does not depend on the choice of the even translate of the hyperplane.

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We write the mean-current through the three hyperplanes xy, yz and zx as a vector $\vec{s}(\mu) = (s_1(\mu), s_2(\mu), s_3(\mu))$ which we will call the mean-current. It measures the average direction of the flow. There is another interpretation though.

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$$\begin{aligned} s_i(\mu) &= 1/2(\ \mu(\text{the dimer at the origin is in the } \vec{e}_i \text{ direction}) \\ &- \quad \mu(\text{the dimer at the origin is in the } -\vec{e}_i \text{ direction})\ ). \end{aligned}$$

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Since the sum of the probabilities of each kind of dimer is 1, we have

$$|s_1(\mu)| + |s_2(\mu)| + |s_3(\mu)| \leq 1/2.$$

Properties of the mean-current function

$$
\begin{aligned}
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Denote by $\mathcal{O} = \{(s_1, s_2, s_3)\ :\ |s_1| + |s_2| + |s_3| \leq 1/2\}$ the mean-current octahedron.

Notice that the mean current is a continuous and affine function of the probability measure. Indeed

$$\vec{s}\left(\frac{\mu_1 + \mu_2}{2}\right) = \frac{\vec{s}(\mu_1) + \vec{s}(\mu_2)}{2}.$$

The mean-current octahedron $\mathcal{O}$

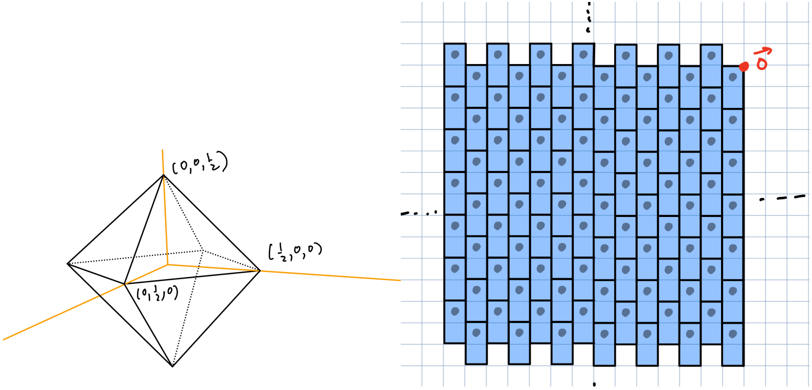

Figure 4: The mean-current octahedron and a tiling such that its mean current is a corner in $\mathcal{O}$

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If $\vec{s}(\mu)$ is one of the corners then only one kind of tile can be seen almost surely. We get only brick-work tilings.

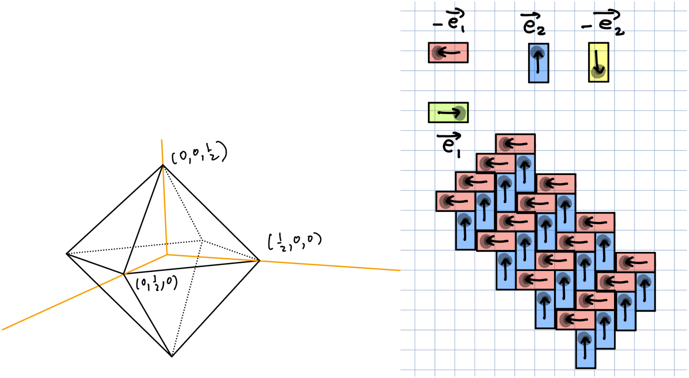

Figure 5: The mean-current octahedron $\mathcal{O}$ and a tiling such that its mean current is on an edge in $\mathcal{O}$

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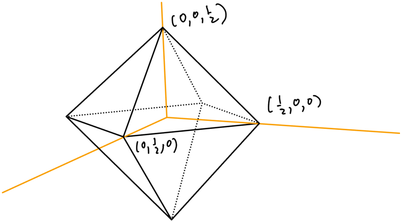
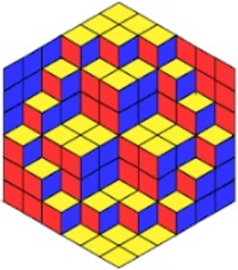
If $\vec{s}(\mu)$ is on one of the edges then only two kinds of tiles can be seen almost surely moving in perpendicular directions. These have zero-entropy as well.

The mean-current octahedron $\mathcal{O}$





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The measure for which the mean-current is on one of the faces of $\mathcal{O}$ uses three different kind of tiles and is much more interesting. It is related to lozenge tilings.

The entropy function

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Let $\mathcal{P}$ denote the space of $\mathbb{Z}^3_{even}$ invariant probability measures on the space of dimer tilings. Let $\mathcal{P}_{\vec{s}} \subset \mathcal{P}$ denote the measures with mean-current $\vec{s}$.

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Define $ent : \mathcal{O} \to [0, \infty)$ by

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ent(\vec{s}) = \sup_{\mu \in \mathcal{P}_{\vec{s}}} h_\mu .
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It is nothing but the maximal entropy with a given mean-current. Since entropy is upper semicontinuous, the equation above has a maximiser.

In fact, it is not difficult to see that *ent* is a concave, continuous function of the mean-current $\vec{s}$.

Main results

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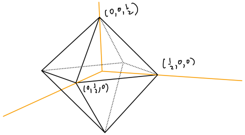

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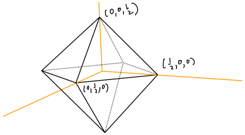

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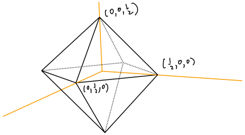

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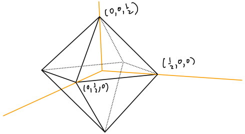

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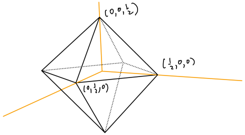

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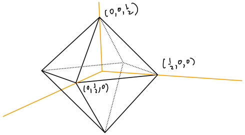

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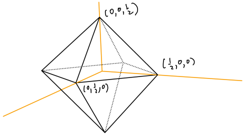

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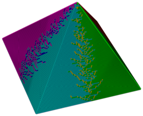
We prove a lot more. Our real main result establishes a large deviation principle for the dimer model in 3 dimensions.



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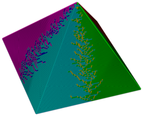
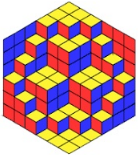


By Cohn, Kenyon and Propp (2001),
it follows from our result that if $\vec{s} \in \partial\mathcal{O}$ then $ent(\vec{s})$ has an explicit expression in terms of the Lobachevsky function.



If $\vec{s} \in \partial\mathcal{O}$ then

$$ent(\vec{s}) = \frac{1}{\pi}(L(2\pi|s_1|) + L(2\pi|s_2|) + L(2\pi|s_3|))$$

where $L(\theta) = -\int_0^\theta \ln(2\sin(x))dx$.

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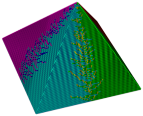
By symmetry a $\mathbb{Z}^3_{even}$ ergodic Gibbs measure
μ is a measure of maximal entropy if and only if $\vec{s}(\mu) = 0$.



Entropy maximisers of a given mean-current are uniform Gibbs measures

The baby case: Entropy maximisers of a given mean-current are uniform Gibbs measures.

The proof of this fact follows the proof of Burton and Steif. Here is a proof sketch:

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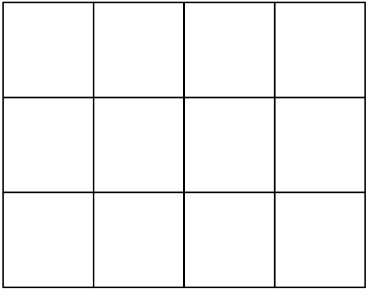

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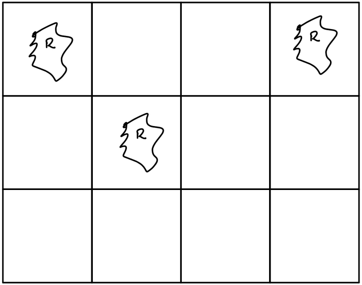

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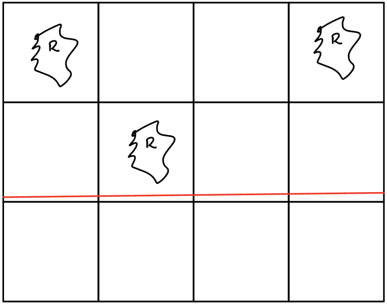
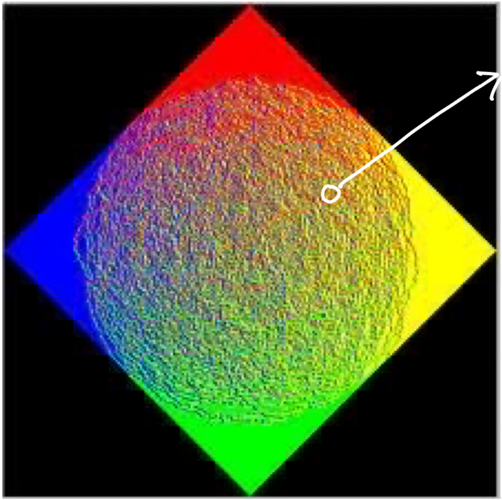


Notice that the flux along any hyperplane close to boundary of the boxes has remain unchanged. Thus the mean-current has remained unchanged. This completes the proof.



All of this relates to the Arctic circle phenomena in the following way: If one were to expand on that little white dot, one would find a sample of an ergodic Gibbs measure of a certain mean-current.

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The Patching Argument

Dobrushin's theorem

Dobrushin (1968) showed that if dimer tilings satisfied a certain mixing condition, then all $\mathbb{Z}^3_{even}$ invariant Gibbs measures are measures of maximal entropy. Ruelle calls it the *D*-condition.

Here is what is needed: Given two tilings τ_1 and τ_2 of $\mathbb{Z}^3$, can we patch in a part of τ_1 into τ_2 (say a box of size n) without wasting too much space (at most o(n))?

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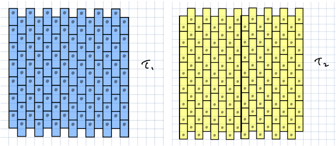

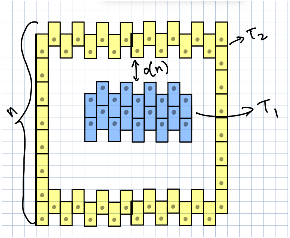

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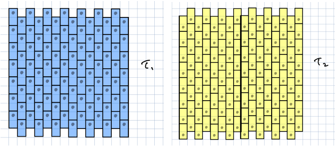

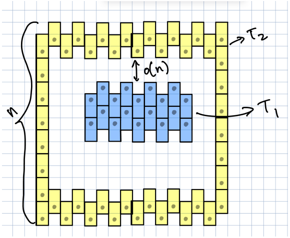

If we could do this for 'most' samples τ_1, τ_2 from uniform $\mathbb{Z}^3_{even}$ ergodic Gibbs measures μ_1, μ_2 and vice versa, we would have $h_{\mu_1} = h_{\mu_2}$.

Patching theorem

Theorem (Patching theorem - C., Sheffield, Wolfram, 2023)

Let $\vec{s} \in int(\mathcal{O})$ *and* τ_1, τ_2 *be samples from* $\mathbb{Z}^3_{even}$ *ergodic measures* $\mu_1, \mu_2 \in \mathcal{P}_{\vec{s}}$*. Then for all* n *there exists an integer* $b_n = o(n)$ *such that with a high probability there is a tiling* τ *for which*

$$\tau = \begin{cases} \tau_1 & \textit{outside } [-n, n]^3 \\ \tau_2 & \textit{in } [-n + b_n, n - b_n]^3 \end{cases}$$

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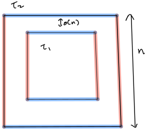

Figure 6: If the average flux along line with the same colour are close to each other then the two tilings can be patched.

Ronnie's talk

Recall that in Ronnie's talk earlier he wanted to be able to patch any possible pattern inside.

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Corollary

Let $\vec{s} \in int(\mathcal{O})$ and $\mu_1, \mu_2 \in \mathcal{P}_{\vec{s}}$ be $\mathbb{Z}^3_{even}$ ergodic Gibbs measures. Then

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We will now show that the function is strictly concave on $\mathcal{O} \setminus$ edges. This requires cluster swapping.

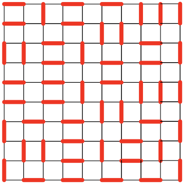
Cluster Swapping

Cluster swapping

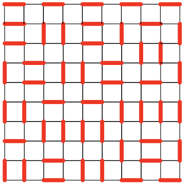
We will prove that $ent(\frac{\vec{s}_1+\vec{s}_2}{2}) > \frac{ent(\vec{s}_1)+ent(\vec{s}_2)}{2}$. This done by

1. Superimposing samples from ergodic Gibbs measures with mean-current $\vec{s}_1$ and $\vec{s}_2$ respectively.
2. Swapping dimers between the two samples.
3. Realising that
   3.1 We haven't increased entropy in the process.
   3.2 After the swaps the tilings each have mean-current $\frac{\vec{s}_1+\vec{s}_2}{2}$.

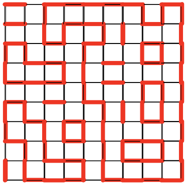
If we superimpose two dimer configurations then the edges either match up, form loops or infinite paths.



TILING 1



TILING 2



The Superimposition.

The starting point is van den Berg's disagreement percolation.:

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Suppose that we sample τ_1 and τ_2 from $\mathbb{Z}^3_{even}$ ergodic uniform Gibbs measures μ_1 and μ_2 (with respect to some invariant uniform Gibbs joint distribution). If only finite cycles are formed upon their superimposition, then $\mu_1 = \mu_2$.

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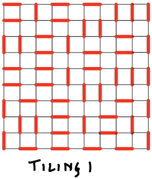

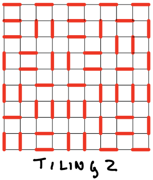

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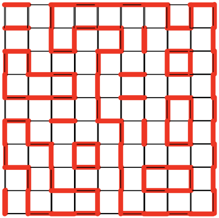

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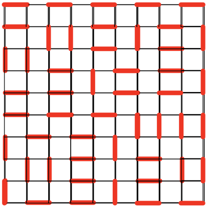

After resampling

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For this it is enough to produce $\tilde{\mu} \in \mathcal{P}_{\frac{\vec{s}_1+\vec{s}_2}{2}}$ such that it is not a $\mathbb{Z}^3_{even}$ invariant Gibbs measure and

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Since entropy maximisers are uniform Gibbs measures this is enough to complete the proof.

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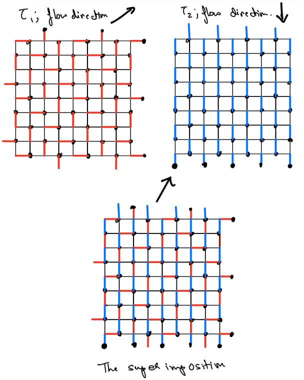

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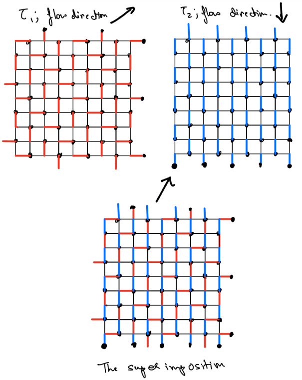

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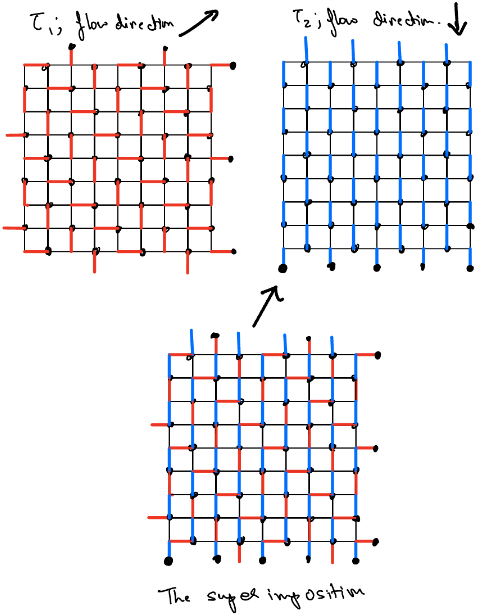

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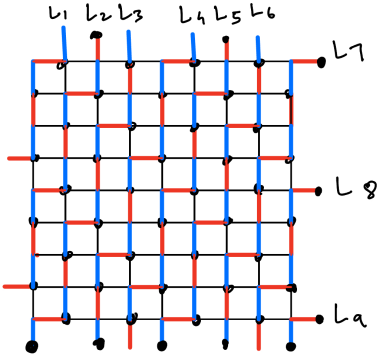

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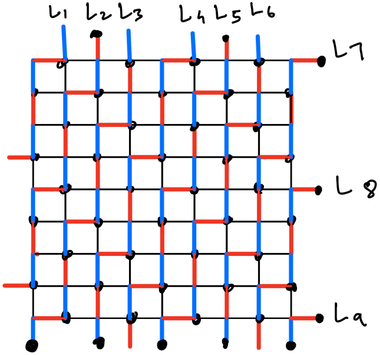

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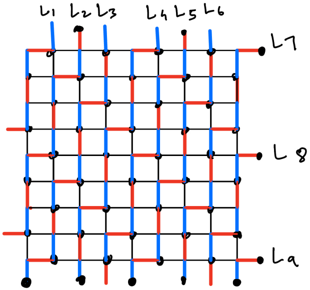

Switching L1, L3, L5, L8

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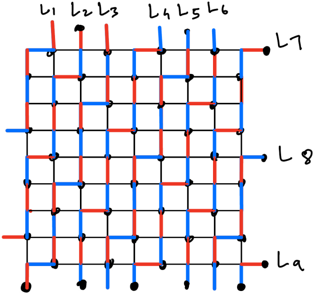

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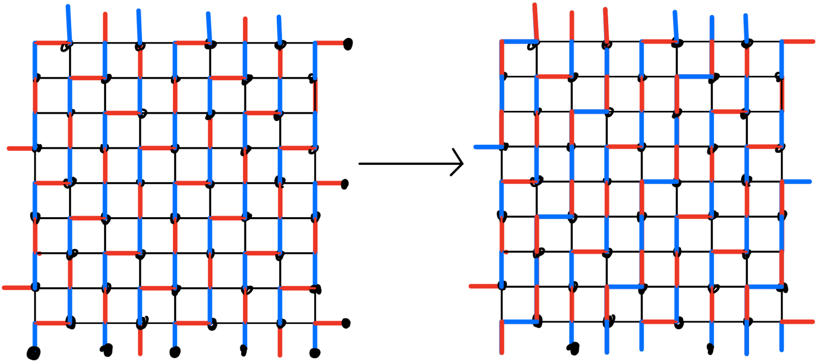

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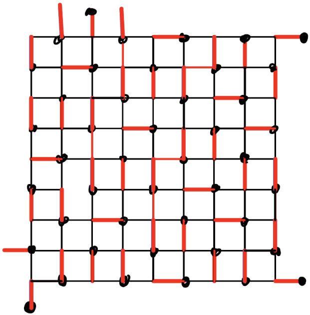
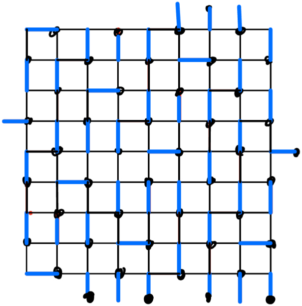


τ_1'



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Suppose τ_1', τ_2' are samples from measures μ_1' and μ_2'. Since we have switched with equal probabilities between tiles in τ_1 and tiles in τ_2 one can conclude that

$$\vec{s}(\mu_1') = \vec{s}(\mu_2') = \frac{\vec{s_1} + \vec{s_2}}{2}.$$

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Further since (τ_1, τ_2) differ from (τ_1', τ_2') only on infinite lines one can conclude that

$$h_{\mu_1} + h_{\mu_2} = h_{(\mu_1 \times \mu_2)} = h(\mu_1', \mu_2').$$

Cluster Swapping: How does it effect slope and entropy?

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This gives rise to new tilings τ_1', τ_2'. They correspond to measures μ_1', μ_2' each with slope $\frac{\vec{s}_1+\vec{s}_2}{2}$ and which have total entropy $h_{\mu_1} + h_{\mu_2}$.

Cluster Swapping: Destroying the Gibbs property

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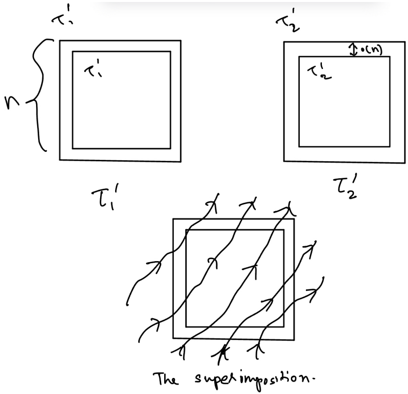
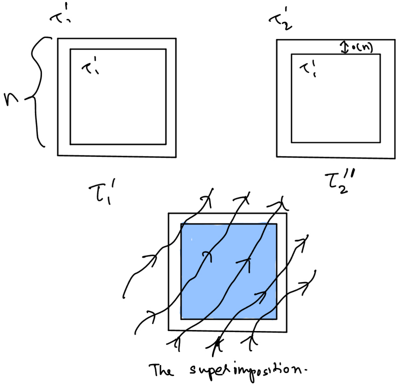
When we superimpose τ_1' and τ_2'' we will get many infinite lines (about the size of the boundary of the box) of disagreement outside $[-n, n]^3$ and complete agreement in $[-n + o(n), n + o(n)]^3$.



These were the samples (τ_1', τ_2').

Cluster Swapping: Destroying the Gibbs property



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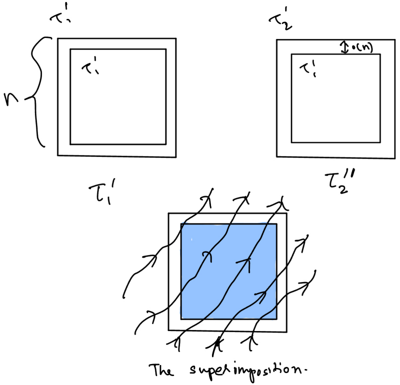

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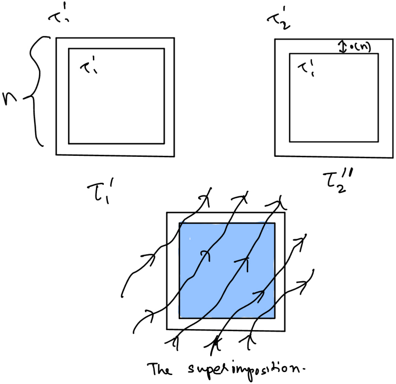

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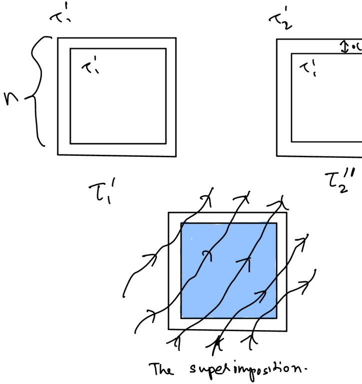

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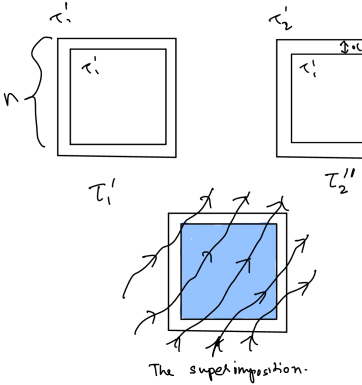

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was wrong. This completes the idea of the proof.

This is one of the important steps in proving a large deviations principle for the space of dimer tilings of $\mathbb{Z}^3$.

Open question: Proper 4 colourings in 3 dimension

This question relates to the space of proper 4 colourings of the $\mathbb{Z}^3$ lattice.

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One cannot define a height function for this model (Schmidt, 1995) and neither can one define a natural non-trivial vector field.

Thanks for listening to the talk.