Substitutive structures on general countable groups

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Multidimensional symbolic dynamics and lattice models of quasicrystals

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In this framework, substitutive subshifts are the simplest ones.

Substitutions in 1-D

Thue-Morse substitution (constant-length)

$$\zeta_{TM} : \begin{cases} 0 & \mapsto 01 \\ 1 & \mapsto 10 \end{cases}$$

A fixed point: $x = \ldots 1001.0110 \ldots$

Fibonacci substitution (non constant-length)

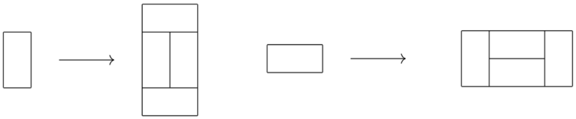
$$\zeta_{F} : \begin{cases} 0 & \mapsto 01 \\ 1 & \mapsto 0 \end{cases}$$

A fixed point: $y = \ldots 01001.01001010 \ldots$

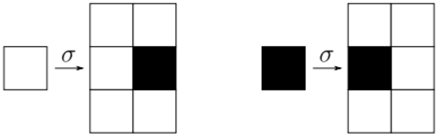
What about beyond the one-dimensional case?

The multidimensional case (square and block substitutions)

The table substitution



Rectangular substitutions



Example given by T. Fernique and V. Lutfalla

Constant-shape substitutions (Introduced in 2023 by C.).

- Let $L \in \mathcal{M}(d, \mathbb{Z})$ be an expansion matrix, i.e. L is invertible, $\|L\| > 1$, $\|L^{-1}\| < 1$.
- Let $F \subset \mathbb{Z}^d$ be a fundamental domain of $L(\mathbb{Z}^d)$ in $\mathbb{Z}^d$.
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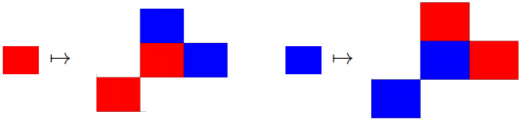
Ex.: $L = 2\mathrm{Id}_{\mathbb{R}^2}$, $F = \{(0,0),(1,0),(0,1),(-1,-1)\}$.



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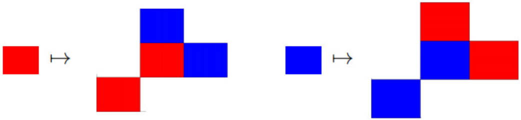

Still a lot of things to do!

Although there has been some studied examples for (non-uniform) multidimensional substitutions (such as yesterday's example of S. Labbé), there still no general framework for these type of morphisms.

What about general countable groups?

Previous works:

N. Bédaride and A. Hilion (2012): Geometric realizations of two-dimensional substitutive tilings.

S. Beckus, T. Hartnick, F. Pogorzelski (2021): Symbolic substitution systems beyond abelian groups.

A. Baraviera, R. Leplaideur (2021) and (2023): Substreetutions.

L. Bartholdi, V. Salo (2024): Substitution on the lamplighter group.

A first definition (inspired on the multidimensional case)

- Let G be a countable group.
- Let $\varphi : G \to G$ be an endomorphism (such that $\varphi(G)$ is of finite-index).
- Let F be a set of representatives of right cosets of $G/\varphi(G)$.

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This create a nested sequence of finite-index subgroups

$$G \geq \varphi(G) \geq \varphi^2(G) \geq \dots$$

and a sequence $(F_n)_{n\in\mathbb{N}}$ of set of representatives of right cosets: $F_{n+1} = \varphi(F_n)F_1$

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Remark

We require that $\bigcup_{n\in\mathbb{N}} F_n = G$. *This implies that*

$$\bigcap_{n\geq 0} \varphi^n(G) = \{1_G\}.$$

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(Possible) excluded group: $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

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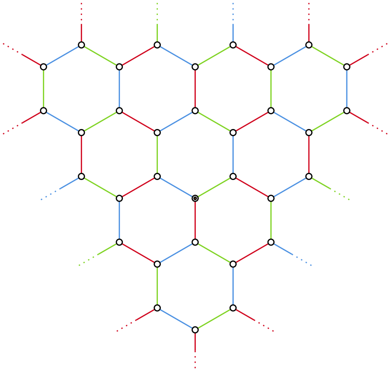
The *Honeycomb Coxeter group*, given by the presentation

$$W = \langle s, t, r \mid s^2 = t^2 = r^2 = (st)^3 = (tr)^2 = (sr)^3 = 1 \rangle.$$



admits the endomorphism defined as $\phi(s) = sts$, $\phi(t) = rsr$ and $\phi(r) = trt$. A set of representatives is $F_\varphi = \{1_W, s, t, r\}$.

A first definition (inspired on the multidimensional case)

The previous one is an example of an expanding endomorphism

Definition

A finitely generated group G *admits an expanding endomorphism* if there exists a finite generating set S, an endomorphism $\varphi : G \to G$ and $\lambda > 1$ such that $[G : \varphi(G)] < +\infty$ and for all $g \in G$

$$d_S(1_G, \varphi(g)) \geq \lambda \cdot d_S(1_G, g).$$

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The *discrete Heisenberg group* of upper triangular 3×3 matrices with 1s in the diagonal, $\mathcal{H}$, given by the presentation

$$\mathcal{H} = \langle x, y, z \mid [x, z], [y, z], [x, y]z^{-1} \rangle,$$

admits an expansive endomorphism: $\phi(x) = x^2$, $\phi(y) = y^2$ and $\phi(z) = z^4$.

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As a consequence of multiple results only finitely generated virtually nilpotent groups admit expanding endomorphisms.

J. Franks (1970): Anosov diffeomorphisms.
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(Possible) excluded groups: Free groups.

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List of (possible) excluded groups:

- The Prüfer p-groups.
- Some groups with torsion: $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
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Are they excluded to define a substitution?

A general definition: Monotileable groups

Let G be a countable group.

- If A, B are two subsets of a group G, we say that A is a (left) *monotile* for B if there exists a subset $C \subseteq G$ such that $\{cA \colon c \in C\}$ is partition of B.

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- C_n denote the set of translates that partition F_{n+1} into translated copies of F_n: $F_{n+1} = \coprod_{c \in C_n} cF_n$.
- Such sequence is *exhaustive* if $G = \bigcup_{n \in \mathbb{N}} F_n$.

A general definition: Monoform groups

Definition

A countable group G is *monoform with* φ *and* $F_1 \Subset G$ (called support) if $\varphi : G \to G$ is an injective map (called localization map) with $\varphi(1_G) = 1_G$ and $1_G \in F_1$ such that

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The second condition establishes that $|F_n| = |F_1|^n$, hence we do not consider finite groups for our purposes.

Lemma

The Prüfer p-groups, $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and the free groups are monoform.

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Proposition

The class of monoform groups is closed under direct product.

If $\varphi : G \to G$ is an expanding endomorphism with $|G/\varphi(G)| < \infty$, and F_1 is a set of representatives of right cosets of $G/\varphi(G)$, then the sequence $F_{n+1} = \varphi(F_n)F_1$ is not necessarily exhaustive.

Proposition

Let $\varphi : G \to G$ be an expanding endomorphism on a countable group and F_1 a set of representatives of right cosets of $G/\varphi(G)$. Then

- *There exists a finite set $K \Subset G$ such that*

$$G = \bigcup_{n\in\mathbb{N}} \varphi^n(K)F_n$$

- *There exists a power φ^k together with a set of representatives right cosets $H_1 \Subset G$ such that its recursive sequence $(H_n)_{n\in\mathbb{N}}$ is exhaustive (and Følner).*

Constant-shape substitutions

Let $\mathcal{A}$ be a finite alphabet and G a monoform group with localization map φ and support F_1. A *constant-shape* or *uniform substitution* is a map $\zeta : \mathcal{A} \to \mathcal{A}^{F_1}$.

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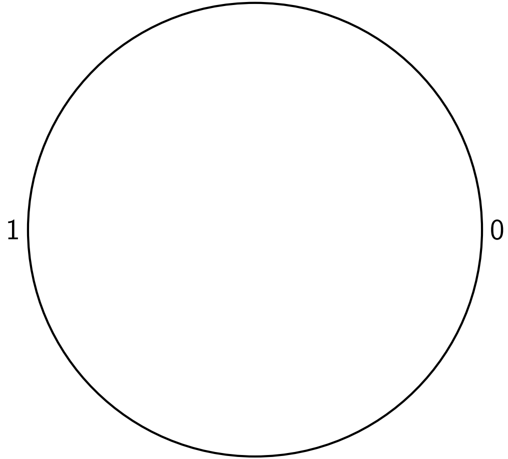
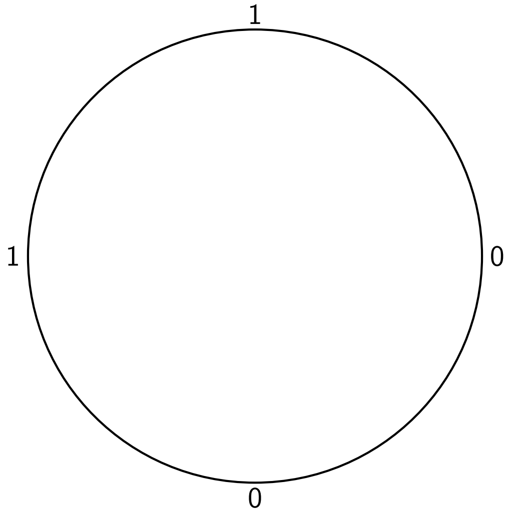
Ex.: Thue-Morse in the Prüfer 2-group: Let $G = \mathbb{Z}[1/2]/\mathbb{Z}$ be the Prüfer 2-group, with $\varphi(g) = g/2$ and $F_1 = \{0, 1/2\}$.

$$\zeta_{TM} : \begin{cases} 0 & \mapsto 01 \\ 1 & \mapsto 10 \end{cases}$$

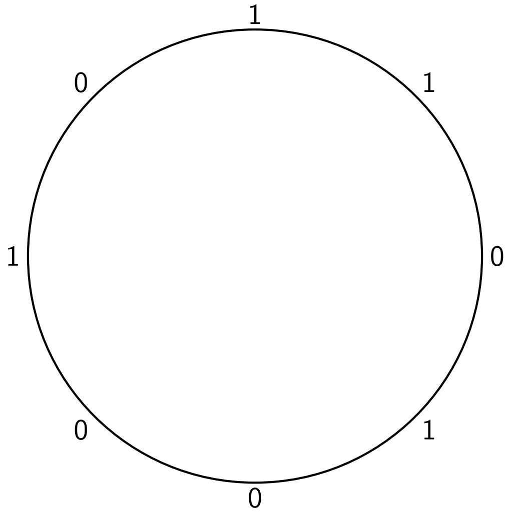
First iteration of the Thue-Morse in the Prüfer 2-group



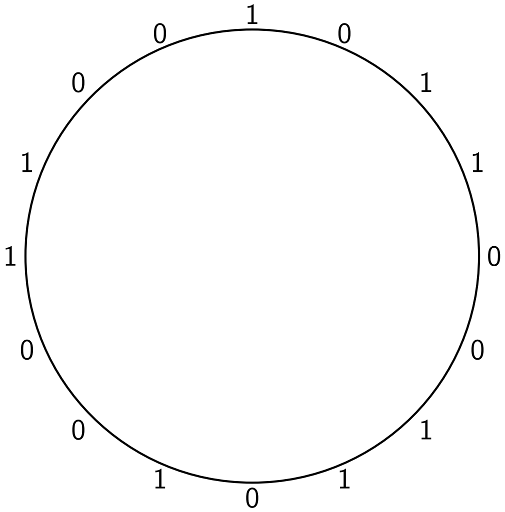
Second iteration of the Thue-Morse in the Prüfer 2-group



Third iteration of the Thue-Morse in the Prüfer 2-group



Fourth iteration of the Thue-Morse in the Prüfer 2-group



The substitutive subshift associated to ξ is

$$X_\xi = \{x \in \mathcal{A}^G; \forall F \Subset G, g \in G,\ x|_{gF} \text{ occurs in some } \xi^n(a), n > 0, a \in \mathcal{A}\}$$

with primitivity assumption:

$$\exists n > 0, \quad \forall a, b \in \mathcal{A},\ b \text{ occurs in } \xi^n(a).$$

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Proposition

The substitutive subshift is minimal if and only if the substitution is primitive.

Proposition

If the group is amenable, then the substitutive subshift is uniquely ergodic

Theorem (Bitar, C., Guillon (2024))

If a group G is monoform. There exists a minimal strongly aperiodic G-substitutive subshift.

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We also have a formalism to define nonconstant-length substitutions and even $\mathcal{S}$-adic representations on countable groups.

THANKS