# Multidimensional Contours à la Fröhlich-Spencer for Long-Range Ising Models

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# Group

Pictures and part of the slides thanks to the (former) students Lucas Affonso, João Maia, Kelvyn Welsh, and Thiago Raszeja.



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# The models

- $\Omega = \{-1,+1\}^{\mathbb{Z}^d}$, $\Lambda \Subset \mathbb{Z}^d$ (finite), $\Omega_\Lambda = \{-1,+1\}^\Lambda$.
- $H^\omega_\Lambda : \Omega_\Lambda \to \mathbb{R}$, $\omega \in \Omega$ (boundary condition).

$$H^\omega_\Lambda(\sigma) = -\sum_{\substack{\{x,y\}\subset\Lambda \\ x\neq y}} J_{x,y}\sigma_x\sigma_y - \sum_{\substack{x\in\Lambda \\ y\in\Lambda^c}} J_{x,y}\sigma_x\omega_y,$$

where $J_{x,y} \geq 0$, $\sum_{y\in\mathbb{Z}^d} J_{x,y} < \infty$.

Sometimes we add a magnetic field $(h_x \in \mathbb{R}, x \in \mathbb{Z}^d)$.

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**Ferromagnetic Ising models**

- *Nearest-neighbour Ising model*

$$J_{xy} = \begin{cases} J & x \neq y, |x-y| = 1 \\ 0 & otherwise. \end{cases}$$

- *Long-Range Ising model*

$$J_{xy} = \begin{cases} \frac{J}{|x-y|^\alpha}, & x \neq y \\ 0 & otherwise. \end{cases}$$

for $\alpha > d, J > 0$.

# Gibbs Measures

The *Gibbs measure* in $\Lambda$ with $\omega$ boundary condition is defined (on $\Omega = \{-1,+1\}^{\mathbb{Z}^d}$) as

$$\mu^{\omega}_{\Lambda,\beta}(\{\sigma\}) = \frac{e^{-\beta H^{\omega}_{\Lambda}(\sigma)}}{Z^{\omega}_{\Lambda}}$$

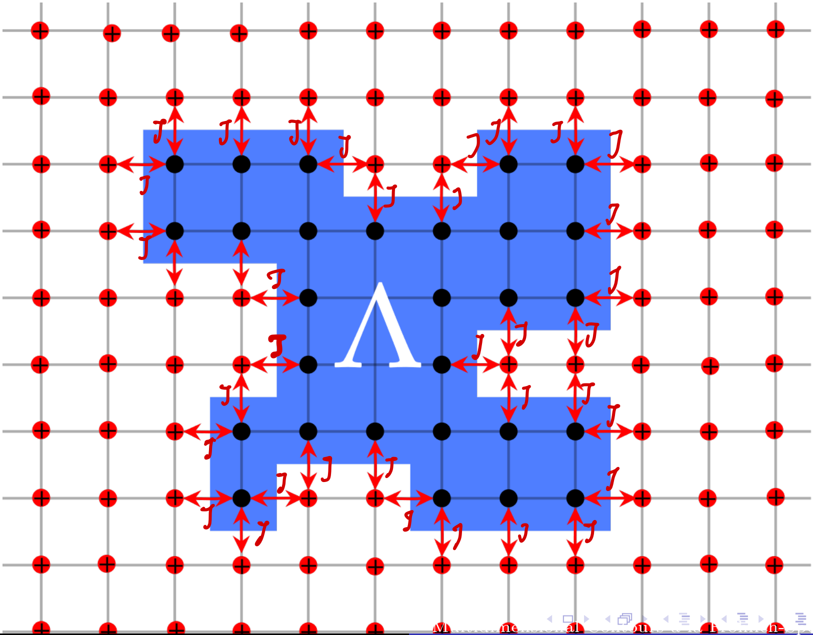
if $\sigma_{\Lambda^c} = \omega_{\Lambda^c}$ and zero otherwise.

**Important boundary conditions:**

$\omega_x = +1, \forall\ x \in \mathbb{Z}^d$. (plus boundary condition)

$\omega_x = -1, \forall\ x \in \mathbb{Z}^d$. (minus boundary condition)

# + boundary condition - Nearest-neighbour Ising model



# How to prove the phase transition?

**Phase transition = more than one Gibbs measure** (at $\beta$).

Take $\Lambda_n \nearrow \mathbb{Z}^d$. Try to show that, when $\beta$ is large enough we have different limit measures:

$$\mu^+_{\Lambda_n,\beta} \longrightarrow \mu^+_\beta \quad \neq \quad \mu^-_{\Lambda_n,\beta} \longrightarrow \mu^-_\beta$$

Warning! There exist examples where the **Phase transition (or the uniqueness) IS NOT related to the lack of analyticity of the pressure function.**

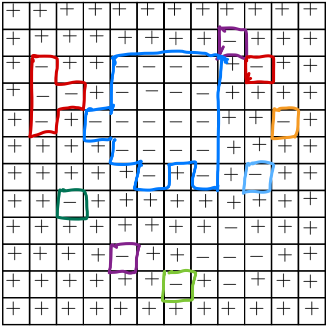
**Reason:** The pressure for models with decaying fields coincides with the model with zero field. Example: $h_x = \frac{1}{|x|^\delta}$, with $0 < \delta < 1$. *We have uniqueness for every* $\beta > 0$.

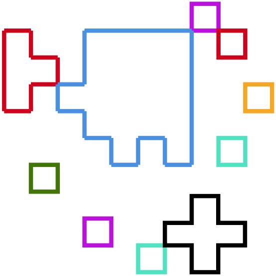
# Gibbs Measures - Equivalent defintions

- S. Muir. *A new characterization of Gibbs measures on* $\mathbb{N}^{\mathbb{Z}^d}$. Nonlinearity. **24**, 2933-2952, (2011).
- E. Beltrán, __, L. Borsato and R. Briceño. *Thermodynamic formalism for amenable groups and countable state spaces.* Journal of the Institute of Mathematics of Jussieu, (2024). (Here the configuration space is $\mathbb{N}^{\mathcal{G}}$)
- __, B.H. Fukushima-Kimura, R. Pereira Lima and T. Raszeja. *Gibbs Measures on Multidimensional Spaces. Equivalences and a Groupoid Approach.* arXiv:2308.16641, (2023).
- C.-E. Pfister. *Gibbs measures on compact ultra metric spaces*. arXiv:2202.06802, (2022).
- T. Meyerovitch. *Gibbs and equilibrium measures for some families of subshifts*. (ETDS), **33**, (2013).

# Peierls contours

- Separate the spin flips into connected components





**The case $d = 2$, ferromagnetic and nearest-neighbor.**

$\Omega = \{-1, +1\}^{\mathbb{Z}^d} : \sigma \longrightarrow \sigma_0$ (value of the configuration at 0).

- $\mathbb{E}^{+}_{\Lambda,\beta}(\sigma_0) = 1 - 2\mu^{+}_{\Lambda,\beta}(\sigma_0 = -1)$
- $\mathbb{E}^{-}_{\Lambda,\beta}(\sigma_0) = 2\mu^{-}_{\Lambda,\beta}(\sigma_0 = +1) - 1$

$$\mu^{+}_{\Lambda,\beta}(\sigma_0 = -1) \leq \mu^{+}_{\Lambda,\beta}(\exists \gamma \circledcirc 0) \leq \sum_{\gamma \circledcirc 0} e^{-2J\beta|\gamma|}$$

$$\Downarrow$$

$$\lesssim e^{c(\alpha,d).n}$$

$$\mu^{+}_{\Lambda,\beta}(\sigma_0 = -1) \leq \sum_{n \geq 4} \#\{\gamma \circledcirc 0; |\gamma| = n\} e^{-2J\beta n}$$

# Peierls contours

Contours are **connected** $(d-1)$-dimensional objects.

$$\Downarrow$$

It works for $\alpha > d+1$.

- J. Ginibre, A. Grossman, D. Ruelle *Condensation of Lattice Gases*. Commun. Math. Phys. **3**, 187-193, (1966).
- Y.M. Park. *Extension of Pirogov-Sinai Theory of Phase Transition to infinite range interactions. I. Cluster Expansion*, Comm. Math. Phys. **114**, 187-218, (1988) - Contour argument for finite state space, but for $\alpha > 3d+1$.

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## KEY QUANTITY:

$\Lambda \Subset \mathbb{Z}^d$

FINITE

$$F_\Lambda = \sum_{\substack{x \in \Lambda \\ y \in \Lambda^c}} J_{xy}$$

WHEN $J_{xy} = \frac{1}{|x-y|^\alpha}$, $\alpha > d$

BALLS

$$F_{B_n(0)} \approx \begin{cases} n^{2d-\alpha}, & d < \alpha < d+1 \\ n^{d-1} \cdot \log n, & \alpha = d+1 \\ n^{d-1}, & \alpha > d+1 \end{cases}$$

ASYMPTOTICALLY $n \to +\infty$

# One-dimensional results - Inspiration


- J. Fröhlich, T. Spencer, *The phase transition in the one-dimensional Ising Model with $1/r^2$ interaction energy*. Commun. Math. Phys. **84**, 87-101, (1982).
- J. Imbrie. *"Decay of Correlations in the One Dimensional Ising Model With $J_{xy} = |x-y|^{-2}$"*. Commun. Math. Phys. **85**, 491-515 (1982).
- M. Cassandro, P. Ferrari, I. Merola, E. Presutti, *Geometry of contours and Peierls estimates in $d = 1$ Ising models with long-range interaction.* J. Math. Phys. **46**, (2005).
- M. Cassandro, P. Picco, T. Merola, U. Rozikov. *One-Dimensional Ising Models with Long Range Interactions: Cluster Expansion, Phase-Separating Point*. Commun. Math. Phys. **327**, 951-991, (2014).



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# One-dimensional results - Inspiration

Fix $M > 0$.

**Proposition (Imbrie - 1982)**

There is a unique way of partitioning the spin flips into irreducible subsets $\gamma_1, \ldots, \gamma_n$ satisfying

$$d(\gamma_i, \gamma_j) > M \min\{\text{diam}(\gamma_i), \text{diam}(\gamma_j)\}^{3/2}$$


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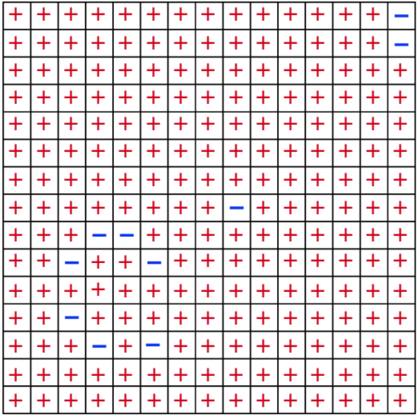
# Incorrect Points

A point $x \in \mathbb{Z}^d$ is *incorrect* for $\sigma$ when $\sigma_y \neq \sigma_x$ for some $y$ with $|x - y| = 1$.



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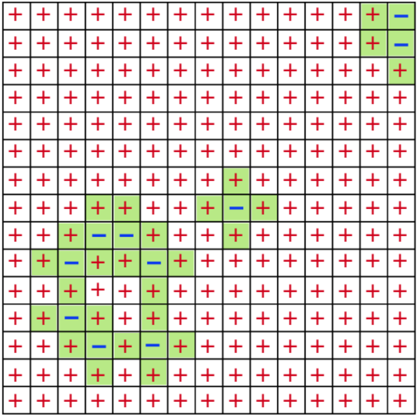


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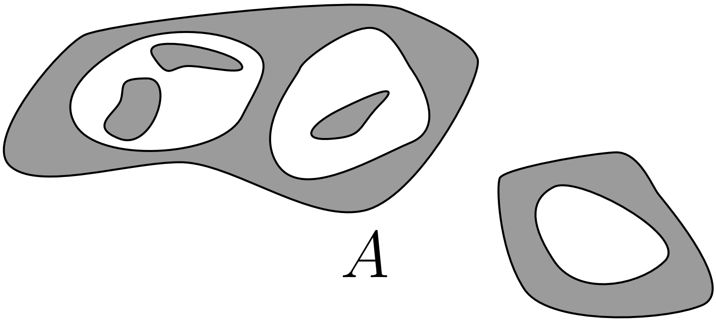
**Grey area = Incorrect points.**





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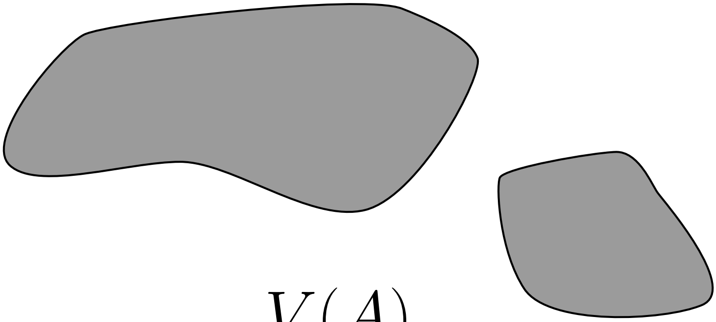
# Volume



$$V(A)$$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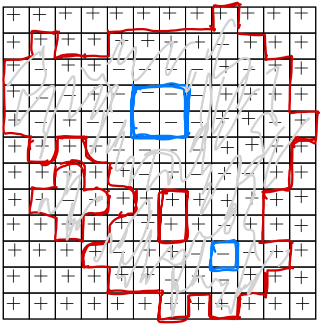

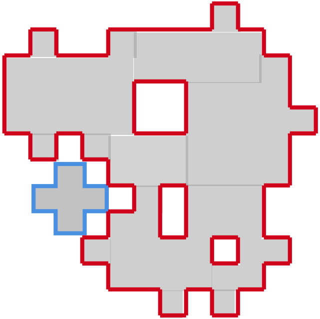


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# $(M, a)$-partition

- Fix real numbers $M, a > 0$. Given $A \Subset \mathbb{Z}^d$, a set $\Gamma(A) := \{\overline{\gamma} : \overline{\gamma} \subset A\}$ is a *$(M, a)$-partition* when

**(A)** They form a partition of $A$, i.e., $\bigcup_{\overline{\gamma} \in \Gamma(A)} \overline{\gamma} = A$ and $\overline{\gamma} \cap \overline{\gamma}' = \emptyset$ for distinct elements of $\Gamma(A)$.

**(B)** For all $\overline{\gamma}, \overline{\gamma}' \in \Gamma(A)$

$$\text{dist}(\overline{\gamma}, \overline{\gamma}') > M \min \left\{ |V(\overline{\gamma})|, |V(\overline{\gamma}')| \right\}^{\frac{a}{d+1}}.$$

- The exists the finest one. (possibly non-trivial).
- $a := a(\alpha, d) = \frac{3(d+1)}{\min\{\alpha - d, 1\}}$ (for the phase transition)

# References for Multi-scaled Contours


- L.Affonso, __, E.O. Endo, S. Handa. *Long-Range Ising models: Contours, Phase Transitions, and Decaying Fields*, (2021). ArXiv:2105.06103.
- L.Affonso. *Multidimensional Contours à la Fröhlich-Spencer and Boundary Conditions for Quantum Spin Systems*, Ph.D. thesis - University of São Paulo (USP), (2023). ArXiv:2105.06103.
- L.Affonso, __, J. Maia. *Phase Transitions in Long-Range Random Field Ising Models in Higher Dimensions*. ArXiv:2307.14150, (2023).
- J. Maia. *Phase Transitions in Ising models: the Semi-infinite with decaying field and the Random Field Long-range.* arXiv:2403.04921.
  Ph.D. thesis - University of São Paulo (USP), (2024).

# Main results with these contours

- A direct proof (via contours) for the phase transition for the ferromagnetic long-range Ising model $\alpha > d$. (with or without decaying fields). ($d \geq 2$).
- Proof for the phase transition for the random field ferromagnetic long-range Ising model $\alpha > d$. ($d \geq 3$).
- Proof of the convergence of the cluster expansion for the pressure at low temperatures for the ferromagnetic long-range Ising model $\alpha > d$. ($d \geq 2$).
- Proof of the decay of correlations at low temperatures for the ferromagnetic long-range Ising model $\alpha > d$. ($d \geq 2$).

$$|\langle \sigma_x \sigma_y \rangle_\beta^+ - \langle \sigma_x \rangle_\beta^+ \langle \sigma_y \rangle_\beta^+| \leq \frac{C(\alpha, d, \beta)}{|x-y|^\alpha} \quad \text{(sharp result)}.$$

*The correlations can not decay faster than the interactions.*


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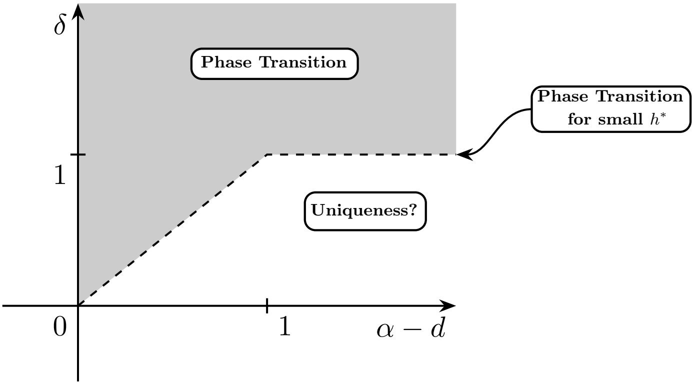


Figure: The phase diagram for the long-range Ising model at low temperatures depends on $\alpha$ and $\delta$.

# Other references


- J. Ding and Z. Zhuang. *Long range order for random field Ising and Potts models*. Communications on Pure and Applied Mathematics, **77**(1):37–51, (2024).
- D. Iagolnitzer and B. Souillard.*Decay of correlations for slowly decreasing potentials*. Phys. Rev. A **16**, (1977).
- L. Affonso, __, J. Maia, J. F. Rodrigues, K. Welsch. *Cluster Expansion and Decay of Correlations for Multidimensional Long-Range Ising Models.* Preprint, (2024).

Thank You!

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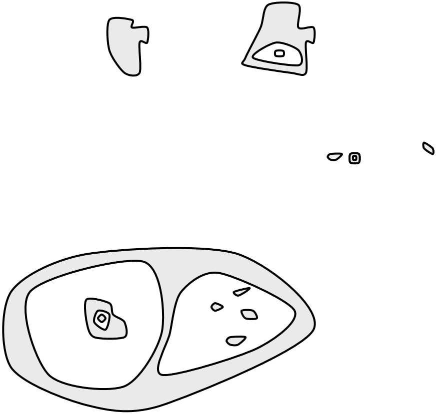

$d(x, y) \leq M d^a$

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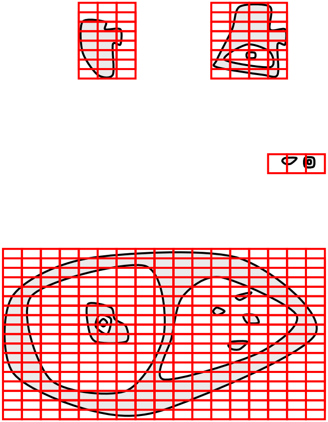
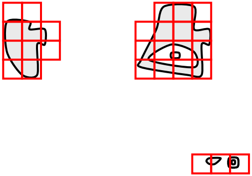


$side = 2^r, center \in \mathbb{Z}^d$



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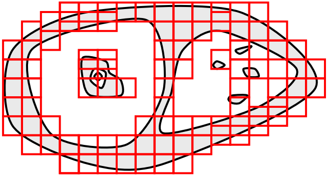

$$d(C_x, C_y) \leq M d^a 2^{ar}$$

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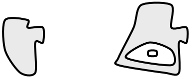

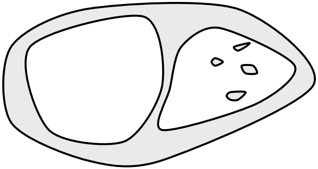

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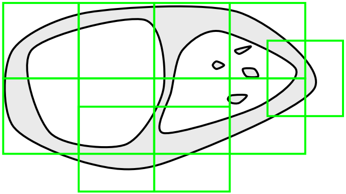

$$d(C_x, C_y) \leq M d^a 2^{a3r}$$

$$side = 2^{3r}, center \in 2^{2r}\mathbb{Z}^d$$

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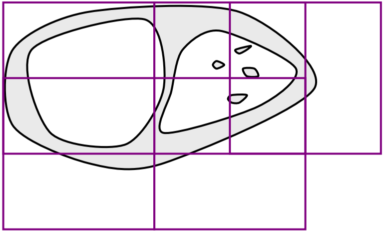

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