Dynamical and spectral properties of some shift spaces of number-theoretic origin

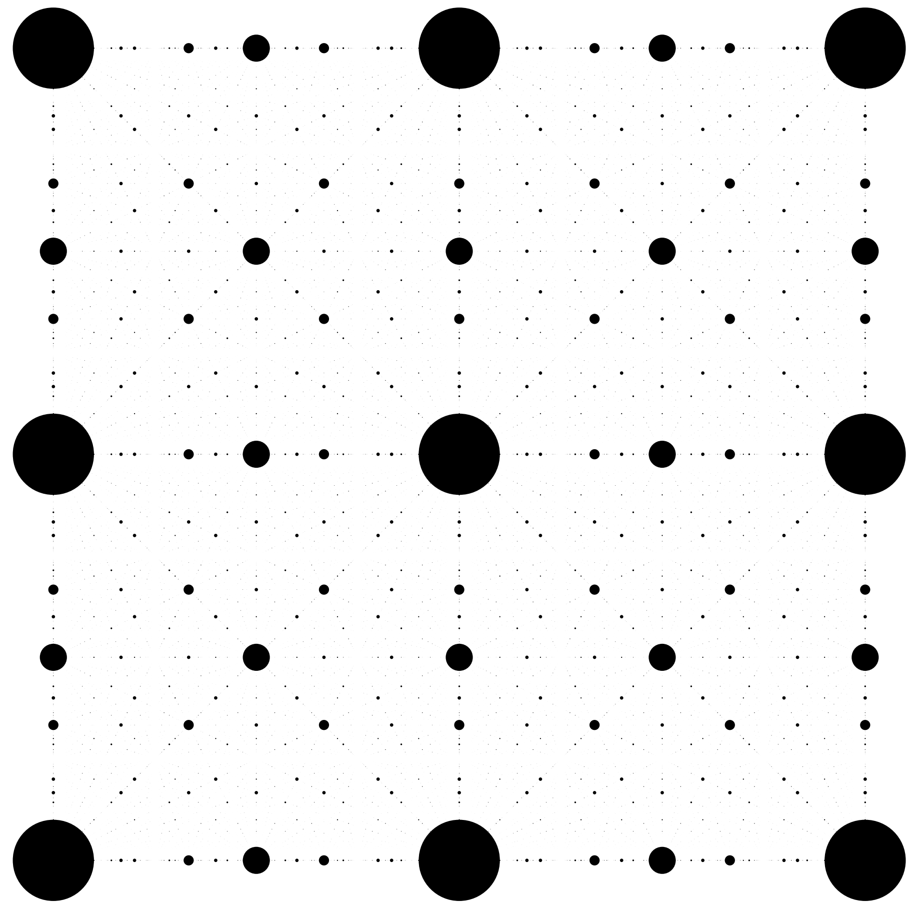
Michael Baake

Bielefeld

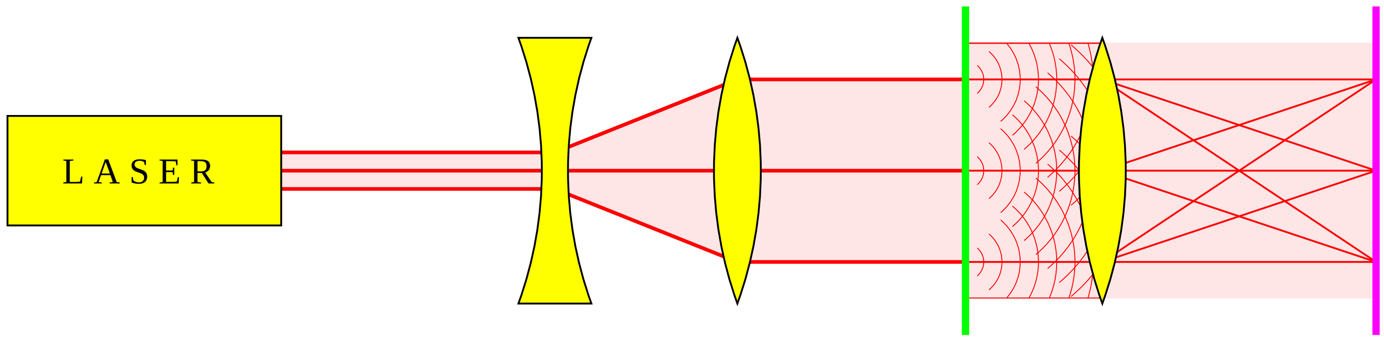
(joint work with A. Bustos, U. Grimm†, A. Nickel et al.)

Menu

- Diffraction and spectra
- Pure point spectra
- Visible lattice points
- Weak model sets
- Topological invariants
- Outlook



Diffraction and spectra



Wiener's diagram obstacle $f(x)$, with $\widetilde{f}(x) := \overline{f(-x)}$

$$\begin{array}{ccc} f & \xrightarrow{\;*\;} & f * \widetilde{f} \\ \mathcal{F}\Big\downarrow & & \Big\downarrow\mathcal{F} \\ \widehat{f} & \xrightarrow{\;|.|^2\;} & |\widehat{f}\,|^2 \end{array}$$

Diffraction and spectra

Structure translation-bounded measure ω

assumed 'self-amenable' (Hof 1995)

Autocorrelation $\gamma = \gamma_\omega = \omega \circledast \widetilde{\omega} := \lim_{R\to\infty} \frac{\omega|_R * \widetilde{\omega|_R}}{\mathrm{vol}(B_R)}$

Diffraction $\boxed{\widehat{\gamma} = (\widehat{\gamma})_{\mathsf{pp}} + (\widehat{\gamma})_{\mathsf{sc}} + (\widehat{\gamma})_{\mathsf{ac}}}$ (relative to λ_{L})

- pp: Bragg peaks
- ac: diffuse scattering (with RN density)
- sc: whatever remains ...

Diffraction and spectra

Dynamical system

$$(\mathbb{X}, \mathbb{Z}, \mu) \quad \text{with} \quad \mathbb{Z} \simeq \{T^n \mid n \in \mathbb{Z}\}$$

$\curvearrowright$ Hilbert space $\mathcal{H} = L^2(\mathbb{X}, \mu)$

$\curvearrowright$ unitary operator on $\mathcal{H}$, $\big(U_T f\big)(x) := f(Tx)$

$\curvearrowright$ spectrum of U_T (Koopman, von Neumann, Halmos)

Extension analogous definition for other groups, e.g. $\mathbb{R}^d$

Spaces shifts, tilings, Delone sets, measures, ...

(Host 1986, Queffélec 1987, Pytheas Fogg 2002)

(Radin/Wolff 1992, Robinson 1996, Solomyak 1997)

Diffraction and spectra

Theorem Let $(\mathbb{X}, \mathbb{R}^d, \mu)$ be an (ergodic) point set dynamical system with diffraction $\widehat{\gamma}$. Then, $\widehat{\gamma}$ is pure point iff $(\mathbb{X}, \mathbb{R}^d, \mu)$ has pure point dynamical spectrum. The latter then is the group generated by the support of $\widehat{\gamma}$, the so-called Fourier–Bohr spectrum of γ.

(Dworkin 1993, Hof 1995, Schlottmann 2000, Lee/Moody/Solomyak 2002, B/Lenz 2004, Lenz/Strungaru 2009, Lenz/Moody 2012, ...)

Connection $\varLambda \subset \mathbb{R}^d$, $\mathbb{X} = \overline{\{t + \varLambda : t \in \mathbb{R}^d\}}$, $(\mathbb{X}, \mathbb{R}^d, \mu)$

FB coefficients $a_{\varLambda}(k) := \lim\limits_{r\to\infty} \frac{1}{\mathrm{vol}(B_r)} \sum_{x \in \varLambda_r} \mathrm{e}^{-2\pi \mathrm{i} k x}$

Eigenfunctions $\boxed{a_{t+\varLambda}(k) = \mathrm{e}^{-2\pi \mathrm{i} k t} a_{\varLambda}(k)}$ ($\neq 0$ for $k \in L^{\circledast}$)

Pure point spectra

Point measures δ_x , $\delta_S := \sum_{x \in S} \delta_x$

Poisson's summation formula $$\boxed{\widehat{\delta_\Gamma} = \mathrm{dens}(\Gamma)\, \delta_{\Gamma^*}}$$

for lattice Γ, dual lattice Γ^*

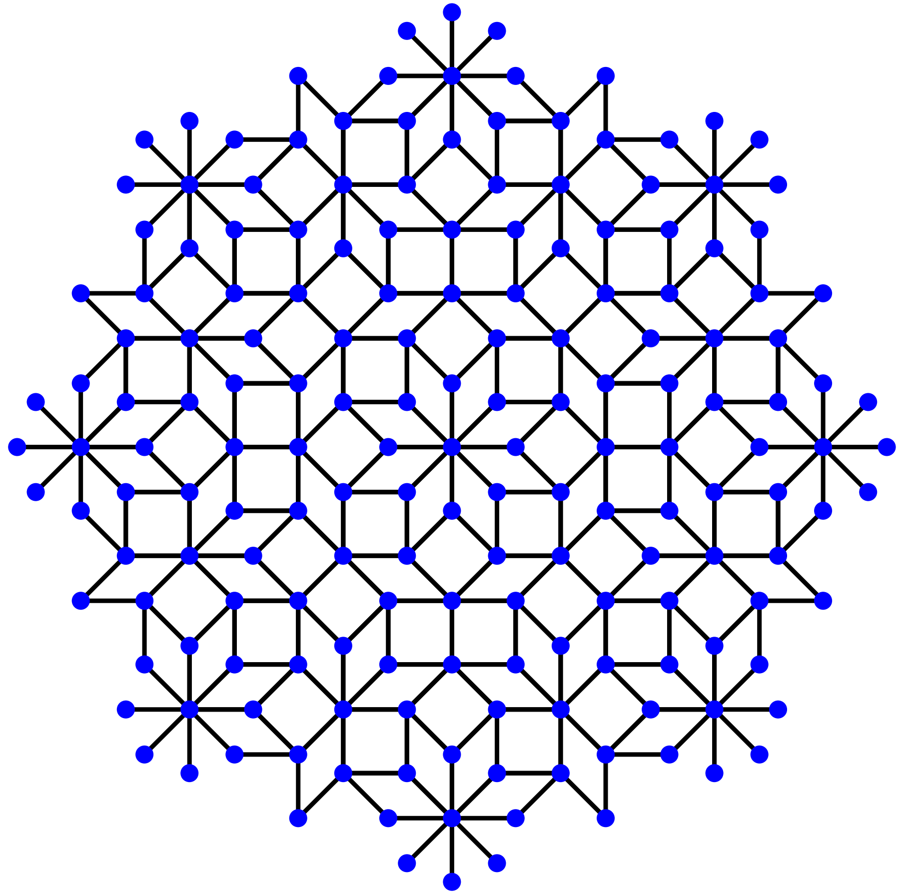
Perfect crystals $\omega = \mu * \delta_\Gamma$ (μ finite, Γ maximal)

⟹ $\gamma = \mathrm{dens}(\Gamma)\, (\mu * \widetilde{\mu}) * \delta_\Gamma$

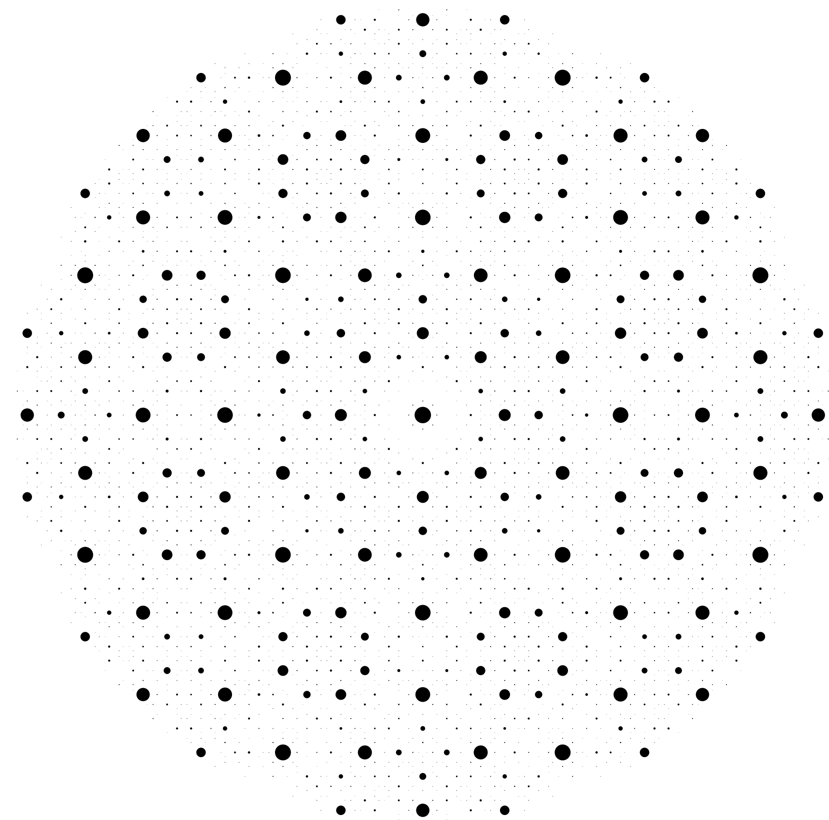
⟹ $$\boxed{\widehat{\gamma} = \big(\mathrm{dens}(\Gamma)\big)^2 \, \big|\widehat{\mu}\big|^2 \, \delta_{\Gamma^*}}$$ pure point !!

⟹ dynamical spectrum Γ^* , also pure point

Pure point spectra



AB point set



diffraction

Pure point spectra

CPS

$$\begin{array}{ccccc} \mathbb{R}^d & \xleftarrow{\;\pi\;} & \mathbb{R}^d \times \mathbb{R}^m & \xrightarrow{\;\pi_{\text{int}}\;} & \mathbb{R}^m \\ \cup & & \cup & & \cup \;\; \text{dense} \\ \pi(\mathcal{L}) & \xleftarrow{\;1-1\;} & \mathcal{L} & \longrightarrow & \pi_{\text{int}}(\mathcal{L}) \\ \| & & & & \| \\ L & & \xrightarrow{\qquad\qquad\star\qquad\qquad} & & L^{\star} \end{array}$$

(Meyer 1972)

(Moody 1997)

Model set $\boxed{\varLambda = \{ x \in L : x^{\star} \in W \}}$ (assumed regular)

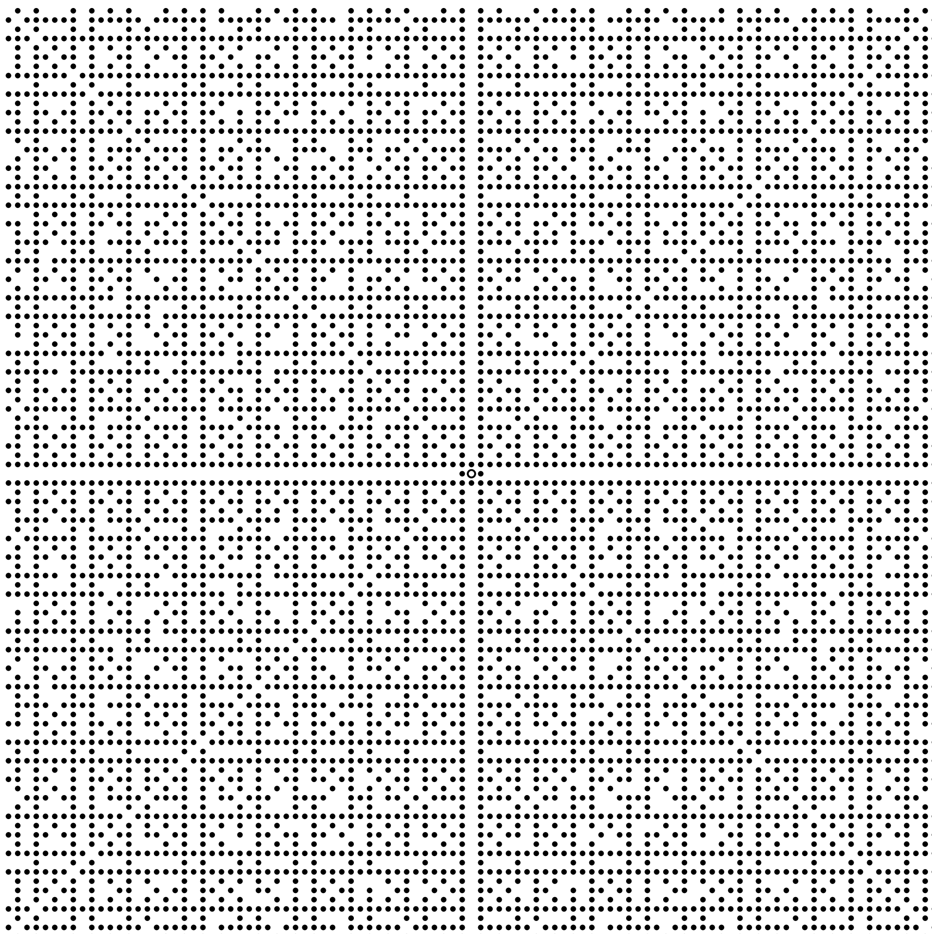
with $W \subset \mathbb{R}^m$ compact, $\lambda_{\mathrm{L}}(\partial W) = 0$

Diffraction $\boxed{\widehat{\gamma} = \sum_{k \in L^{\circledast}} \lvert A(k) \rvert^2 \, \delta_k}$ pure point !! $(\omega = \delta_{\varLambda})$

with $L^{\circledast} = \pi(\mathcal{L}^{*})$ (Fourier module of $\varLambda$: spectrum)

and amplitude $A(k) = a_{\varLambda}(k) = \frac{\mathrm{dens}(\varLambda)}{\mathrm{vol}(W)} \, \widehat{1_W}(-k^{\star})$

Visible lattice points



$$V = \{x \in \mathbb{Z}^2 \mid \gcd(x) = 1\}$$

Properties

- $\mathrm{dens}(V) = 6/\pi^2$
- V not Delone
- $V - V = \mathbb{Z}^2$
- pure point diffraction
- weak model set
- $h_{\mathrm{top}}(V) > h_{\mathrm{m}}(V) = 0$

Theorem PP dynamical spectrum, trivial top. point spectrum

Visible lattice points

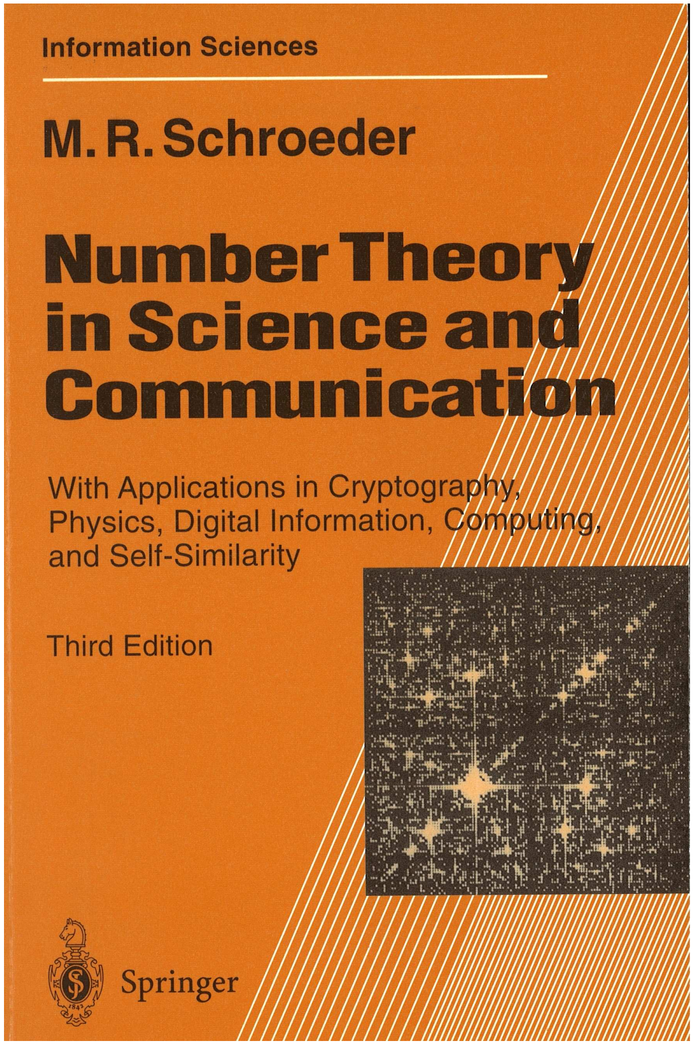
Shift space $\quad \mathbb{X} = \overline{\{t + V : t \in \mathbb{Z}^2\}} \overset{(!)}{\subset} \{0,1\}^{\mathbb{Z}^2}$

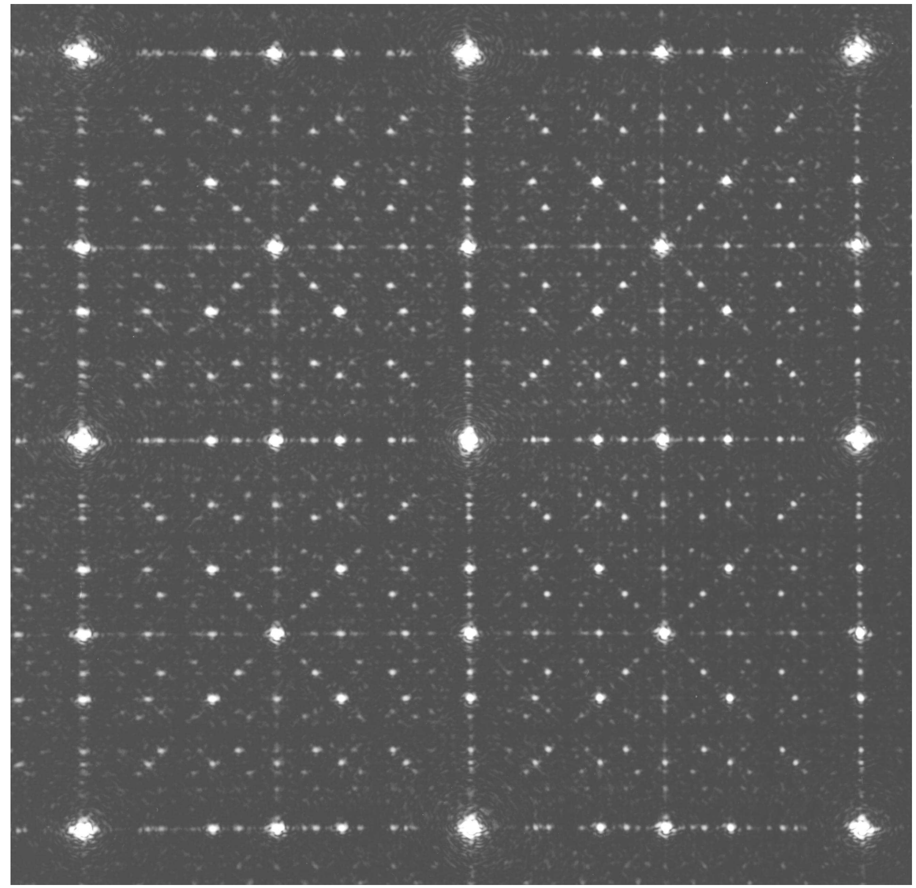
Theorem $(\mathbb{X}, \mathbb{Z}^2)$ is a TDS, and $\mathbb{X}$ is hereditary, so every subset of V is an element of $\mathbb{X}$. Also, $\mathbb{X} = \mathbb{A}$, where $\mathbb{A}$ consists of all $U \subset \mathbb{Z}^2$ that miss at least one coset modulo $p\mathbb{Z}^2$ for every prime p.

Entropy

$$\boxed{h_{\text{top}}(\mathbb{X}) = h_{\text{pc}}(V) = \text{dens}(V)\log(2) = \frac{\log(2)}{\zeta(2)}}$$

Visible lattice points

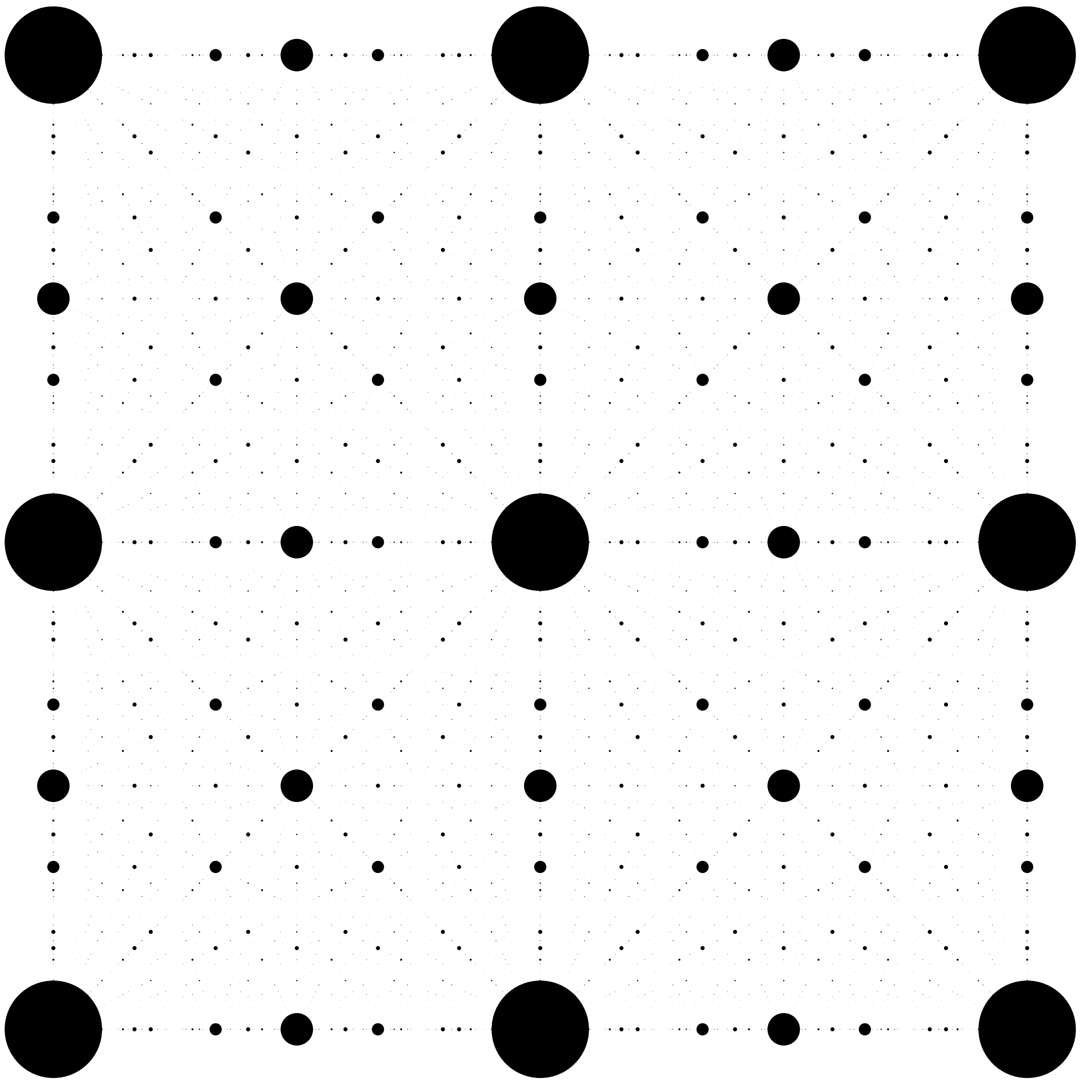




B/Grimm/Warrington 1994

Schroeder 1982, Mosseri 1992, B/Moody/Pleasants 2000, B/Huck 2013

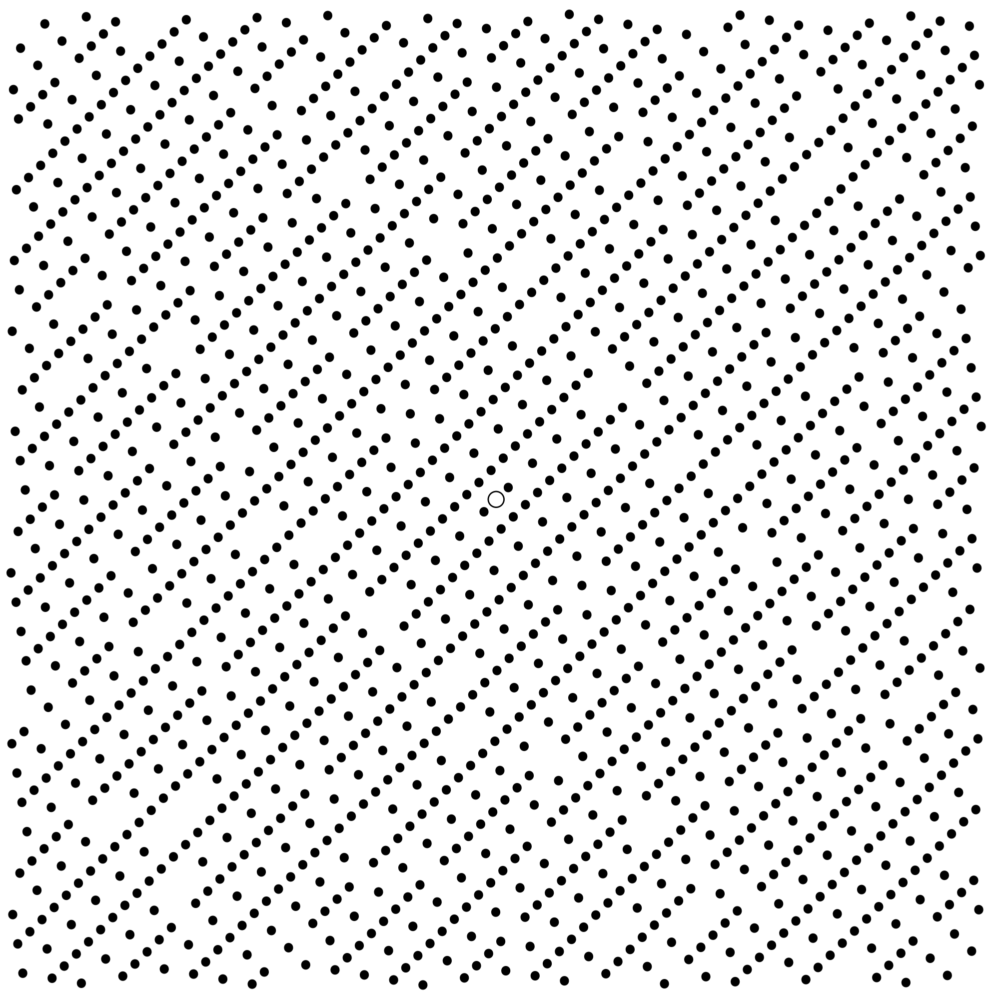
Visible lattice points



Properties

- $\mathbb{Z}^2$-periodic
- D_4-symmetric
- $\mathrm{GL}(2,\mathbb{Z})$-invariant
- support of $\widehat{\gamma}$:
  $S = \{k \in \mathbb{Q}^2$ with
  $\mathrm{den}(k)$ square-free$\}$:
  FB (dyn.) spectrum
- intensity for $k \in S$
  with $\mathrm{den}(k) = q$
  $\left(\frac{6}{\pi^2}\right)^2 \prod_{p|q} \frac{1}{(p^2-1)^2}$

Squarefree integers in $\mathbb{Z}[\sqrt{2}\,]$



$V = \{(x, x') \mid x \text{ sq.-free}\}$

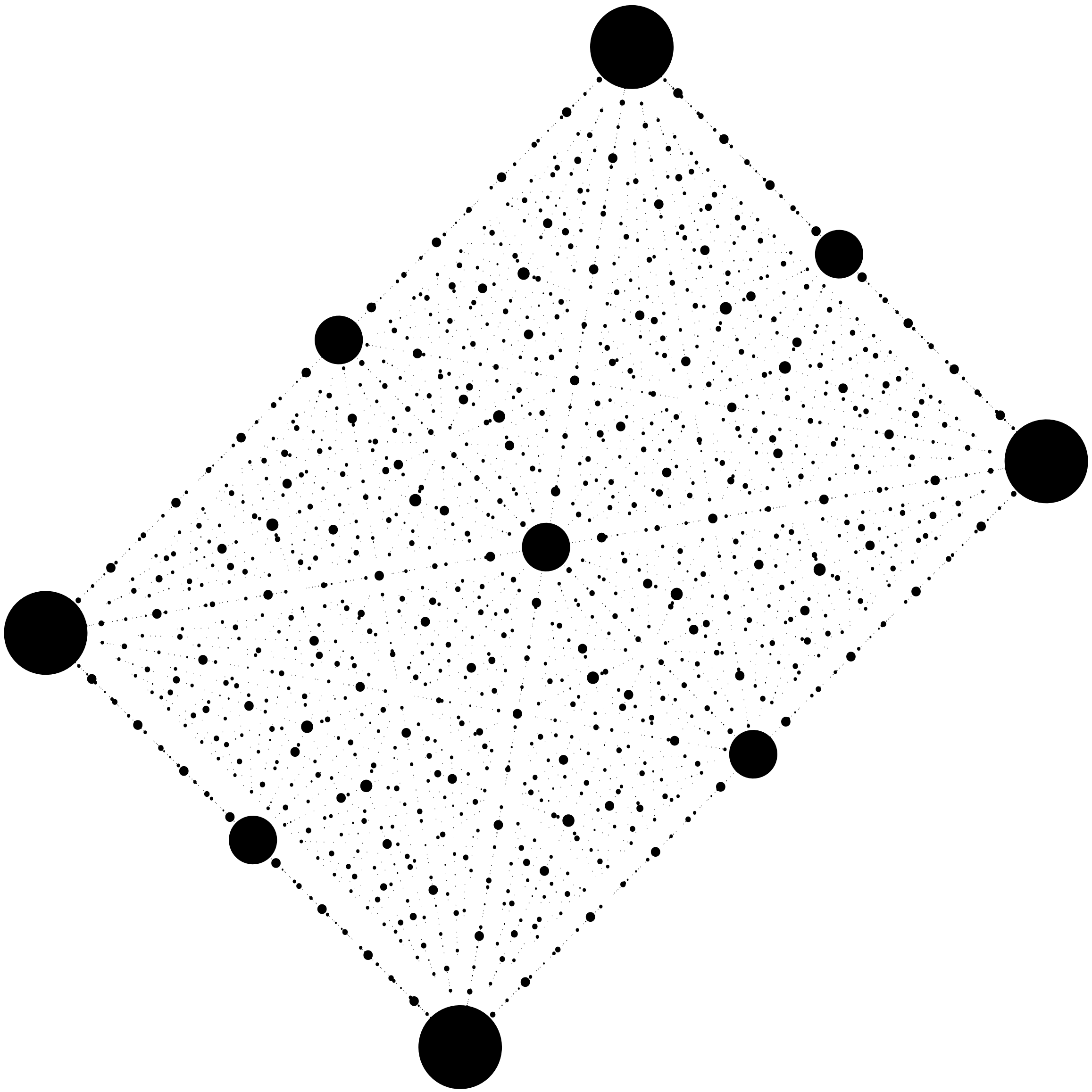
Properties

- $\mathrm{dens}(V) = \frac{24}{\pi^4} = \frac{\mathrm{dens}(\mathcal{L})}{\zeta_K(2)}$
- V not Delone
- $V - V = \langle V \rangle = \mathcal{L}$
- pure point diffraction
- weak model set
- $h_{\mathrm{top}}(V) > h_{\mathrm{m}}(V) = 0$

Theorem PP dynamical spectrum, trivial top. point spectrum

(Cellarosi/Vinogradov 2013, B/Huck 2013)

Squarefree integers in $\mathbb{Z}[\sqrt{2}\,]$



Properties

- $\mathcal{L}^*$-periodic
- $C_2 \times C_2$-symmetric
- $\mathrm{GL}(2,\mathbb{Z})$-invariant
- support of $\widehat{\gamma}$:
  $S = \{k \in \mathbb{Q}\mathcal{L}^*$ with $\mathrm{den}(k)$ cube-free$\}$:
  FB (dyn.) spectrum
- intensity for $k \in S$ with $\mathrm{den}(k) = q$:
  $\left(\frac{24}{\pi^4}\right)^2 \prod_{p|q} \frac{1}{(N(p)^2-1)^2}$

(B/Huck 2013)

Weak model sets

CPS

$$\begin{array}{ccccc} G & \xleftarrow{\pi} & G \times H & \xrightarrow{\pi_{\mathrm{int}}} & H \\ \cup & & \cup & & \cup \ \text{dense} \\ \pi(\mathcal{L}) & \xleftarrow{1-1} & \mathcal{L} & \longrightarrow & \pi_{\mathrm{int}}(\mathcal{L}) \\ \| & & & & \| \\ L & & \xrightarrow{\quad\quad\star\quad\quad} & & L^{\star} \end{array}$$

G: σ-compact

H: comp. gen.

$\mathcal{A}$: van Hove in G

WMS

$$\boxed{\varLambda = \{ x \in L \mid x^{\star} \in W \}}$$

with $W \subset H$ compact, $\theta_H(W) > 0$

max. density: $\mathrm{dens}(\varLambda) = \mathrm{dens}(\mathcal{L})\, \theta_H(W)$

$$\gamma_{\varLambda} := \lim_{n\to\infty} \frac{\delta_{\varLambda\cap A_n} * \delta_{-(\varLambda\cap A_n)}}{\theta_G(A_n)}$$ (exists !!)

(B/Huck/Strungaru 2015, Lenz/Spindeler/Strungaru 2020)

Weak model sets

Diffraction $\widehat{\gamma} = \sum_{k\in L^0} |A(k)|^2 \, \delta_k$ pure point !! $(\omega = \delta_{\varLambda})$

with $L^0 = \pi(\mathcal{L}^0)$ (annihilator of $\mathcal{L}$ in dual CPS)

amplitude $A(k) = a_{\varLambda}(k) = \frac{\mathrm{dens}(\varLambda)}{\theta_H(W)} \, \widehat{1_W}(-k^{\star})$

Hull $\mathbb{X}_{\varLambda} = \overline{G + \varLambda}$

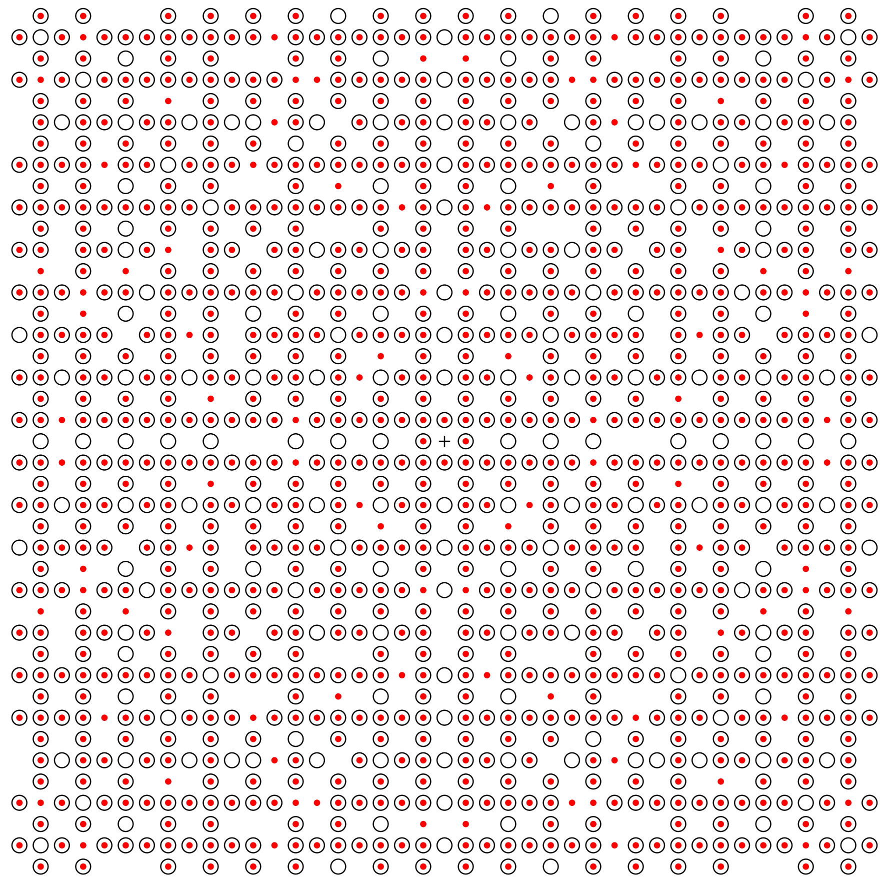
with patch frequency measure μ

μ is ergodic, $\varLambda$ is generic for μ

Theorem $(\mathbb{X}_{\varLambda}, G, \mu)$ has pure point dynamical spectrum: L^0

(Keller/Richard 2015, B/Huck/Strungaru 2015)

Topological invariants



Visible lattice points (red dots) versus

square-free Gaussian integers (circles)

Topological invariants

Systems $(\mathbb{X}_{\mathrm{G}}, \mathbb{Z}^2)$ versus $(\mathbb{X}_{\mathrm{V}}, \mathbb{Z}^2)$

not measure-theor. isomorphic, by Halmos-von Neumann theorem

Top. invariants $\mathcal{S} = \mathrm{cent}_{\mathrm{Aut}(\mathbb{X})}(\mathbb{Z}^2)$ $\quad \mathrm{Aut}(\mathbb{X}) = \mathrm{Homeo}(\mathbb{X})$

$$\mathcal{R} = \mathrm{norm}_{\mathrm{Aut}(\mathbb{X})}(\mathbb{Z}^2)$$

Theorem $\mathcal{S}_{\mathrm{V}} = \mathcal{S}_{\mathrm{G}} = \mathbb{Z}^2$ together with

$$\boxed{\mathcal{R}_{\mathrm{G}} = \mathbb{Z}^2 \rtimes D_4} \quad \text{and} \quad \boxed{\mathcal{R}_{\mathrm{V}} = \mathbb{Z}^2 \rtimes \mathrm{GL}(2, \mathbb{Z})}$$

Question General structure ?

Topological invariants

System $(\mathbb{X}, \mathbb{Z}^2)$ with $\mathbb{X} = \overline{\mathbb{Z}^2 + V_2}$ and

$$V_2 = \{m + n\sqrt{2} \text{ is } k\text{-free} \mid m, n \in \mathbb{Z}\} \qquad \text{any } k \geqslant 2$$

Theorem $\boxed{\mathcal{S} = \mathbb{Z}^2}$ and $\boxed{\mathcal{R} = \mathcal{S} \rtimes \mathcal{H}}$ with

$$\mathcal{H} \simeq \mathcal{O}^{\times} \rtimes \mathrm{Gal}\big(\mathbb{Q}(\sqrt{2}\,)/\mathbb{Q}\big) \simeq C_2 \times D_{\infty}$$

(B/Bustos/Huck/Lemańczyk/Nickel 2020)

Extensions

Gen. alg. number fields (Gundlach/Klüners 2024)

Topological entropy $\log(2)/\zeta_K(k)$

Further invariants (B/B/N 2023, Gundlach/Klüners 2024)

Outlook

- Connections with almost periodicity
- Compute dynamical spectra
- General algebraic number fields
- General $\mathcal{B}$-free systems for $d > 1$
- Efficient topological invariants
- Systems with mixed spectrum
- Consider all invariant measures

References

- M. B., U. Grimm, *Aperiodic Order. Vol. 1: A Mathematical Invitation*, CUP (2013).
- D. Lenz, T. Spindeler, N. Strungaru, Pure point spectrum for dynamical systems and mean almost periodicity, *preprint*, `arXiv:2006.10825`.
- R.V. Moody, Model sets: A survey, `arXiv:math.MG/0002020`.
- M.-B., R.V. Moody, P.A.B. Pleasants, Diffraction from visible lattice points and k-th power free integers, *Discr. Math.* **221** (2000) 3–42; `arXiv:math/9906132`.
- M. B., A. Bustos, C. Huck, M. Lemańczyk, A. Nickel, Number-theoretic positive entropy shifts with small centraliser and large normaliser, *ETDS* **41** (2021) 3201–3226; `arXiv:1910.13876`.
- M. B., A. Bustos, A. Nickel, Power-free points in quadratic number fields: Stabiliser, dynamics and entropy, *Israel J. Math.*, in press; `arXiv:2110.02520`.
- M. B., C. Huck, N. Strungaru, On weak model sets of extremal density, *Indag. Math.* **28** (2017) 3–31; `arXiv:1512.07129`.
- M. B., D. Lenz, Dynamical systems on translation bounded measures: Pure point dynamical and diffraction spectra, *ETDS* **24** (2004) 1867–1893; `arXiv:0302061`.
- F. Gundlach, J. Klüners, in preparation (2024).