Regularization by noise: a Malliavin calculus approach

Rémi Catellier
(joint work with Romain Duboscq)

Stochastic and Deterministic Analysis for Irregular Models, 12th of January 2024

Regularization by noise for differential equations

Let $b \in C_b^0(\mathbb{R}^d)$ and consider

$$\mathrm{d}x_t = b(x_t)\,\mathrm{d}t, \quad x_0 \in \mathbb{R}^d.$$

- Uniqueness for $b \in Lip$ (or Osgood condition, or BV for some notion of solutions...)
- Non uniqueness when b is not regular enough. Peano example : $d = 1$, $b(x) = 2\mathrm{sgn}\,(x)\sqrt{|x(t)|}$, $x_0 = 0$. Solutions:

$$x : t \in \mathbb{R}^+ \mapsto (t - t_0)^2 \mathbb{1}_{t \geq t_0}, \quad \forall t_0 \in (0, +\infty].$$

Regularization by noise for differential equations

Let $b \in C_b^0(\mathbb{R}^d)$ and consider

$$\mathrm{d}x_t = b(x_t)\,\mathrm{d}t, \quad x_0 \in \mathbb{R}^d.$$

- Uniqueness for $b \in Lip$ (or Osgood condition, or BV for some notion of solutions...)
- Non uniqueness when b is not regular enough. Peano example : $d = 1$, $b(x) = 2\mathrm{sgn}\,(x)\sqrt{|x(t)|}$, $x_0 = 0$. Solutions:

$$x : t \in \mathbb{R}^+ \mapsto (t - t_0)^2 \mathbb{1}_{t \geq t_0}, \quad \forall t_0 \in (0, +\infty].$$

An additive perturbation w may restore uniqueness (and even existence).

$$\mathrm{d}x_t = b(x_t)\,\mathrm{d}t + \mathrm{d}w_t, \quad x_0 \in \mathbb{R}^d.$$

- Usual perturbations for such a phenomenon are stochastic processes
- One may also consider multiplicative perturbations

$$\mathrm{d}x_t = b(x_t)\,\mathrm{d}t + \sigma(x_t)\,\mathrm{d}w_t, \quad x_0 \in \mathbb{R}^d.$$

Davie's path-by-path type uniqueness

Definition

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $x_0 \in \mathbb{R}^d$, $b : \mathbb{R}^d \mapsto \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \mapsto \mathbb{R}^{d\times d}$ be two functions and let w be a $\mathbb{R}^d$ value stochastic process. Equation

$$x_t = x_0 + \int_0^t b(x_r)\,\mathrm{d}r + \int_0^t \sigma(x_r)\,\mathrm{d}w_r$$

admits path-by-path uniqueness if there exists $\mathcal{N} = \mathcal{N}(x_0, b, \sigma) \in \mathcal{F}$ with $\mathbb{P}(\mathcal{N}) = 0$ and for all $\omega \notin \mathcal{N}$, a unique $x(\omega) : [0, T] \mapsto \mathbb{R}^d$ exists such that

$$x_t(\omega) = x_0 + \int_0^t b(x_r(\omega))\,\mathrm{d}r + \int_0^t \sigma(x_r(\omega))\,\mathrm{d}w_r(\omega) \tag{1}$$

- Conditions on b, σ and w are needed for (1) to make sense.
- Path-by-path uniqueness and pathwise (strong) uniqueness are different, thanks to examples due to [SW22, Anz22].

PATH-BY-PATH TYPE UNIQUENESS : ADDITIVE CASE

The equation reads as

$$x_t = x_0 + \int_0^t b(x_r)\,\mathrm{d}r + w_t.$$

- Davie [Dav07] : w Brownian motion, $b \in L^\infty$,
- Catellier and Gubinelli [CG16]: w fractional Brownian motion and $b \in \mathcal{C}^{\varepsilon \vee (1-\frac{1}{2H}+\varepsilon)}$ (or $b \in \mathcal{C}^{\frac{3}{2}-\frac{1}{2H}}$ for semi-flow property),
- Galeati and Gubinelli [GG21]: w almost all continuous path and $b \in H^{-\varepsilon}$,
- Priola [Pri20]: w a Lévy process
- And many others...

THE YOUNG TRANSFORMATION IN THE ADDITIVE CASE

Set $\theta = x - w$. Then θ solves

$$\theta_t = \theta_0 + \int_0^t b(\theta_r + w_r)\,\mathrm{d}r. \tag{2}$$

We reinterpret this equation with the idea that θ oscillates slowly compared to w.

The Young transformation in the additive case

Set $\theta = x - w$. Then θ solves

$$\theta_t = \theta_0 + \int_0^t b(\theta_r + w_r)\, \mathrm{d}r. \tag{2}$$

We reinterpret this equation with the idea that θ oscillates slowly compared to w. In particular, for b, θ and w continuous, the integral is approximated as

$$\int_0^t b(\theta_r + w_r)\, \mathrm{d}r = \lim_{\substack{|\pi|\to 0\\ \pi\in\Pi(0,t)}} \sum_{k=0}^{\#\pi-1} \int_{t_k}^{t_{k+1}} b(\theta_{t_k} + w_r)\, \mathrm{d}r = \lim_{\substack{|\pi|\to 0\\ \pi\in\Pi(0,t)}} \sum_{k=0}^{\#\pi-1} (T^w b)_{t_k,t_{k+1}}(\theta_{t_k})\, \mathrm{d}r$$

Well-posedness of Equation (2) is then linked to the space-time regularity of the averaged field

$$(s,t,z) \mapsto (T^w b)_{s,t}(z) = \int_s^t b(z + w_r)\, \mathrm{d}r.$$

viz the following result :

Theory of Young differential equations (YDE)

Using the space/time regularity of the averaged field and (non stochastic) sewing techniques, one has the following result :

Cauchy problem for averaged-field SDE [CG16, HL17, Gal23]

Suppose that solutions of Equation

$$\theta_t = \theta_0 + \int_0^t b(\theta_r + w_r)\,\mathrm{d}t$$

are a priori Lipschitz continuous in time.
Suppose furthermore that there exists $\gamma > \frac{1}{2}$, and $\alpha > \frac{3}{2}$ such that

$$T^w b \in \mathcal{C}^\gamma([0,T]; \mathcal{C}^\alpha_{loc}(\mathbb{R}^d)).$$

Then the solutions are unique.

Theory of Young differential equations (YDE)

Using the space/time regularity of the averaged field and (non stochastic) sewing techniques, one has the following result :

Cauchy problem for averaged-field SDE [CG16, HL17, Gal23]

Suppose that solutions of Equation

$$\theta_t = \theta_0 + \int_0^t b(\theta_r + w_r)\, \mathrm{d}t$$

are a priori Lipschitz continuous in time.
Suppose furthermore that there exists $\gamma > \frac{1}{2}$, and $\alpha > \frac{3}{2}$ such that

$$T^w b \in \mathcal{C}^\gamma([0,T]; \mathcal{C}^\alpha_{loc}(\mathbb{R}^d)).$$

Then the solutions are unique.

- The previous Lipschitz continuity condition holds as soon as $b \in L^\infty(\mathbb{R}^d)$.

Theory of Young differential equations (YDE)

Using the space/time regularity of the averaged field and (non stochastic) sewing techniques, one has the following result :

Cauchy problem for averaged-field SDE [CG16, HL17, Gal23]

Suppose that solutions of Equation

$$\theta_t = \theta_0 + \int_0^t b(\theta_r + w_r)\,\mathrm{d}t$$

are a priori Lipschitz continuous in time.
Suppose furthermore that there exists $\gamma > \frac{1}{2}$, and $\alpha > \frac{3}{2}$ such that

$$T^w b \in \mathcal{C}^\gamma([0,T];\mathcal{C}^\alpha_{loc}(\mathbb{R}^d)).$$

Then the solutions are unique.

- The previous Lipschitz continuity condition holds as soon as $b \in L^\infty(\mathbb{R}^d)$.
- Using stochastic properties of w one can deduce the well-posedness for irregular b of the following equation
$$\mathrm{d}x_t = b(x_t)\,\mathrm{d}t + \mathrm{d}w_t.$$

Theory of Young differential equations (YDE)

Using the space/time regularity of the averaged field and (non stochastic) sewing techniques, one has the following result :

Cauchy problem for averaged-field SDE [CG16, HL17, Gal23]

Suppose that solutions of Equation

$$\theta_t = \theta_0 + \int_0^t b(\theta_r + w_r)\,\mathrm{d}t$$

are a priori Lipschitz continuous in time.
Suppose furthermore that there exists $\gamma > \frac{1}{2}$, and $\alpha > \frac{3}{2}$ such that

$$T^w b \in \mathcal{C}^\gamma([0,T]; \mathcal{C}^\alpha_{loc}(\mathbb{R}^d)).$$

Then the solutions are unique.

- The previous Lipschitz continuity condition holds as soon as $b \in L^\infty(\mathbb{R}^d)$.
- Using stochastic properties of w one can deduce the well-posedness for irregular b of the following equation

$$\mathrm{d}x_t = b(x_t)\,\mathrm{d}t + \mathrm{d}w_t.$$

- Everything relate on the space-time regularity of the averaged field and of the previous Riemann sum approximation.

Path-by-path type uniqueness : multiplicative case

$$x_t = x_0 + \int_0^t b(x_r)\,\mathrm{d}r + \int_0^t \sigma(x_r)\,\mathrm{d}w_r$$

We need a pathwise meaning of the previous equation when w is a stochastic process, which can be derived using rough paths theory.

PATH-BY-PATH TYPE UNIQUENESS : MULTIPLICATIVE CASE

$$x_t = x_0 + \int_0^t b(x_r)\,\mathrm{d}r + \int_0^t \sigma(x_r)\,\mathrm{d}w_r$$

We need a pathwise meaning of the previous equation when w is a stochastic process, which can be derived using rough paths theory.

Definition (Davie/Friz-Victoir)

Let $\frac{1}{3} < \nu \leq \frac{1}{2}$. A geometric rough path is a couple $\mathbf{W} = (W, \mathbb{W})$, with $W \in \mathcal{C}^{\nu}([0,T];\mathbb{R}^d)$ and $\mathbb{W} \in \mathcal{C}^{2\nu}([0,T];\mathbb{R}^{d\times d})$ and

$$\mathbb{W}_{s,t} = \lim_{\varepsilon\to 0} \int_s^t (W_r^\varepsilon - W_s^\varepsilon) \otimes \frac{\mathrm{d}W_r^\varepsilon}{\mathrm{d}r}\,\mathrm{d}r,$$

where W^ε is a smooth approximation of W.
A path $x \in \mathcal{C}^{\nu}([0,T];\mathbb{R}^d)$ is a solution of the rough differential equation

$$\mathrm{d}x_t = b(x_t)\,\mathrm{d}t + \sigma(x_t)\,\mathrm{d}\mathbf{W}_t, \quad x_0 \in \mathbb{R}^d$$

if two constants $C > 0$ and $a > 1$ exist such that for $0 \leq s \leq t \leq T$,

$$|x_t - x_s + b(x_s)(t-s) + \sigma(x_s)(W_t - W_s) + D\sigma(x_s)\sigma(x_s)\mathbb{W}_{s,t}| \leq C|t-s|^a.$$

Path-by-path type uniqueness : multiplicative case

Theorem (Lyons, Davie, Gubinelli, Friz-Victoir...)

In the scope of the previous definition, when $b \in Lip$ and bounded and $\sigma \in C_b^3$, There is a unique solution to the previous RDE. Furthermore, it defines a flow which is continuous with respect to the initial condition and to the driving signal $\mathbf{W} = (W, \mathbb{W})$.

Path-by-path type uniqueness : multiplicative case

Theorem (Lyons, Davie, Gubinelli, Friz-Victoir...)

In the scope of the previous definition, when $b \in Lip$ and bounded and $\sigma \in C_b^3$, There is a unique solution to the previous RDE. Furthermore, it defines a flow which is continuous with respect to the initial condition and to the driving signal $\mathbf{W} = (W, \mathbb{W})$.

- The process $\mathbb{W}$ is a data of the problem.
- When $W = B$ is a standard Brownian motion, one can take $\nu < \frac{1}{2}$ and $\mathbb{B}_{s,t} = \int_s^t (B_r - B_s) \otimes \circ \, \mathrm{d}B_r$, and we retrieve standard (Stratonovitch) solutions for SDEs.

Path-by-path type uniqueness : multiplicative case

Theorem (Lyons, Davie, Gubinelli, Friz-Victoir...)

In the scope of the previous definition, when $b \in Lip$ and bounded and $\sigma \in C_b^3$, There is a unique solution to the previous RDE. Furthermore, it defines a flow which is continuous with respect to the initial condition and to the driving signal $\mathbf{W} = (W, \mathbb{W})$.

- The process $\mathbb{W}$ is a data of the problem.
- When $W = B$ is a standard Brownian motion, one can take $\nu < \frac{1}{2}$ and $\mathbb{B}_{s,t} = \int_s^t (B_r - B_s) \otimes \circ \, \mathrm{d}B_r$, and we retrieve standard (Stratonovitch) solutions for SDEs.
- One can also take $W = B^H$ a standard fractional Brownian motion of Hurst parameter $H \in (\frac{1}{3}, \frac{1}{2})$ (this is a centered continuous Gaussian process of covariance $s, t \to \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}) I_d$), and one can take $\nu < H$.

PATH-BY-PATH TYPE UNIQUENESS : MULTIPLICATIVE CASE

Theorem (Lyons, Davie, Gubinelli, Friz-Victoir...)

In the scope of the previous definition, when $b \in Lip$ and bounded and $\sigma \in C_b^3$, There is a unique solution to the previous RDE. Furthermore, it defines a flow which is continuous with respect to the initial condition and to the driving signal $\mathbf{W} = (W, \mathbb{W})$.

- The process $\mathbb{W}$ is a data of the problem.
- When $W = B$ is a standard Brownian motion, one can take $\nu < \frac{1}{2}$ and $\mathbb{B}_{s,t} = \int_s^t (B_r - B_s) \otimes \circ \, \mathrm{d}B_r$, and we retrieve standard (Stratonovitch) solutions for SDEs.
- One can also take $W = B^H$ a standard fractional Brownian motion of Hurst parameter $H \in (\frac{1}{3}, \frac{1}{2})$ (this is a centered continuous Gaussian process of covariance $s, t \to \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}) I_d)$, and one can take $\nu < H$.
- It allows to have a pathwise (almost sure) meaning for the SDE.

Path-by-path type uniqueness : multiplicative case

Theorem (Lyons, Davie, Gubinelli, Friz-Victoir...)

In the scope of the previous definition, when $b \in Lip$ and bounded and $\sigma \in C_b^3$, There is a unique solution to the previous RDE. Furthermore, it defines a flow which is continuous with respect to the initial condition and to the driving signal $\mathbf{W} = (W, \mathbb{W})$.

- The process $\mathbb{W}$ is a data of the problem.
- When $W = B$ is a standard Brownian motion, one can take $\nu < \frac{1}{2}$ and $\mathbb{B}_{s,t} = \int_s^t (B_r - B_s) \otimes \circ \mathrm{d}B_r$, and we retrieve standard (Stratonovitch) solutions for SDEs.
- One can also take $W = B^H$ a standard fractional Brownian motion of Hurst parameter $H \in (\frac{1}{3}, \frac{1}{2})$ (this is a centered continuous Gaussian process of covariance $s, t \to \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}) I_d)$, and one can take $\nu < H$.
- It allows to have a pathwise (almost sure) meaning for the SDE.
- The whole theory would work for more general Gaussian rough paths ($\nu > \frac{1}{4}$).

MULTIPLICATIVE ROUGH CASE : RESULTS

- Davie [Dav11].
  - w geometric Brownian rough path,
  - RDE in the sense of Davie/Friz-Victoir
  - $\sigma \in C_b^3$ and $b \in L^\infty$
  - Tools : Girsanov transform and $T(1)$ Theorem for Kolmogorov equations.
- Athreya Bhar Shekhar [ABS17]
  - w geometric fractional Brownian rough path, $1/2 \geq H > \frac{1}{3}$.
  - RDE in the sense of Davie/Friz-Victoir.
  - $b \in \mathcal{C}^0$ and bounded (or $\mathcal{C}^\varepsilon$ for semiflow)
  - $\sigma \in C_b^3$, is strictly elliptic and σ^{-1} is conservative. Namely there exists $F : \mathbb{R}^d \mapsto \mathbb{R}^d$ such that $$\nabla F = \sigma^{-1}.$$
  - Tools : Results of [CG16] and rough Lamperti transform.
- Dareiotis and Gerenscer [DG22] (simultaneously as our work).
  - w geometric fractional Brownian rough path for $H > \frac{1}{3}$,
  - RDE in the sense of Gubinelli
  - $\sigma \in C_b^3$ and $\sigma\sigma^T$ stricly elliptic
  - $b \in \mathcal{C}^{\varepsilon \vee \left(1 - \frac{1}{2H} + \varepsilon\right)}$
  - Continuous semi-flow
  - Same techniques for Young and smooth cases
  - Tools : stochastic sewing lemma and additive translation of the solution.

MAIN RESULT

Theorem (C., Duboscq)

Let $\frac{1}{4} < H \leq \frac{1}{2}$, $(B^H, \mathbb{B}^H)$ *be the rough path associated to the fractional Brownian motion. Let* $\sigma \in C^\infty(\mathbb{R}^d; \mathbb{R}^{d\times d})$ *being strictly elliptic, namely a constant* $c > 0$ *exists such that for all* $y, z \in \mathbb{R}^d$,

$$|\sigma(y)z|^2 \geq c|z|^2.$$

Let $b \in \mathcal{C}^{\varepsilon\vee(\frac{3}{2}-\frac{1}{2H}+\varepsilon)}(\mathbb{R}^d; \mathbb{R}^d)$*. Then path-by-path existence and uniqueness holds for the RDE (interpreted in the sense of Davie/Friz-Victoir)*

$$\mathrm{d}x_t = b(x_t)\,\mathrm{d}t + \sigma(x_t)\,\mathrm{d}B^H_t.$$

Furthermore, the solution semi-flow is locally Lispchitz continuous with respect to the initial condition. Finally, if $b^n \to b$*, then so does the flow (almost surely).*

MAIN RESULT

Theorem (C., Duboscq)

Let $\frac{1}{4} < H \leq \frac{1}{2}$, $(B^H, \mathbb{B}^H)$ *be the rough path associated to the fractional Brownian motion. Let* $\sigma \in C^\infty(\mathbb{R}^d; \mathbb{R}^{d\times d})$ *being strictly elliptic, namely a constant* $c > 0$ *exists such that for all* $y, z \in \mathbb{R}^d$,

$$|\sigma(y)z|^2 \geq c|z|^2.$$

Let $b \in \mathcal{C}^{\varepsilon \vee (\frac{3}{2} - \frac{1}{2H} + \varepsilon)}(\mathbb{R}^d; \mathbb{R}^d)$*. Then path-by-path existence and uniqueness holds for the RDE (interpreted in the sense of Davie/Friz-Victoir)*

$$\mathrm{d}x_t = b(x_t)\,\mathrm{d}t + \sigma(x_t)\,\mathrm{d}B_t^H.$$

Furthermore, the solution semi-flow is locally Lispchitz continuous with respect to the initial condition. Finally, if $b^n \to b$*, then so does the flow (almost surely).*

The result still holds for some more genereal Gaussian rough path under a local non-determinism condition. Three main ideas for the proof:

Main result

Theorem (C., Duboscq)

Let $\frac{1}{4} < H \leq \frac{1}{2}$, $(B^H, \mathbb{B}^H)$ be the rough path associated to the fractional Brownian motion. Let $\sigma \in C^\infty(\mathbb{R}^d; \mathbb{R}^{d \times d})$ being strictly elliptic, namely a constant $c > 0$ exists such that for all $y, z \in \mathbb{R}^d$,

$$|\sigma(y)z|^2 \geq c|z|^2.$$

Let $b \in \mathcal{C}^{\varepsilon \vee (\frac{3}{2} - \frac{1}{2H} + \varepsilon)}(\mathbb{R}^d; \mathbb{R}^d)$. Then path-by-path existence and uniqueness holds for the RDE (interpreted in the sense of Davie/Friz-Victoir)

$$\mathrm{d}x_t = b(x_t)\,\mathrm{d}t + \sigma(x_t)\,\mathrm{d}B^H_t.$$

Furthermore, the solution semi-flow is locally Lispchitz continuous with respect to the initial condition. Finally, if $b^n \to b$, then so does the flow (almost surely).

The result still holds for some more genereal Gaussian rough path under a local non-determinism condition. Three main ideas for the proof:

- a flow transformation,
- Malliavin calculus,
- Besov spaces and a martingale decomposition.

FLOW TRANSFORMATION

Ideas from Riedel and Scheutzow [RS17]. Let $(\varphi_t(x))_{t\in[0,T]}$ be the flow of the RDE

$$d\varphi_t(x) = \sigma(\varphi_t(x))\, d\mathbf{W}_t, \quad \varphi_0(x) = x.$$

where $\mathbf{W}$ is a geometric rough path for $\nu \in (\frac{1}{4}, \frac{1}{2}]$.

Theorem (C., Duboscq)

Let b be continuous and bounded. Let $\sigma \in C_b^{\lfloor \frac{1}{\nu} \rfloor + 2}$. A path $(x_t)_{t\in[0,T]}$ is a solution (in the sense of Davie) of the RDE

$$dx_t = b(x_t)\, dt + \sigma(x_t)\, d\mathbf{W}_t$$

if and only if $(x_t)_{t\in[0,T]} = (\varphi_t(\theta_t))_{t\in[0,T]}$, where θ is a solution of the ODE

$$\theta_t = \theta_0 + \int_0^t (\nabla\varphi_r(\theta_r))^{-1}\, b(\varphi_r(\theta_r))\, dr.$$

- Restriction : $\sigma \in C^{\lfloor \frac{1}{\nu} \rfloor + 2}$
- Strength : Averaging operator (along the flow), focus on "standard" *ODE*.

Regularization properties of stochastic rough flow

Let

$$(Tb)_{s,t}(x) = \int_s^t (\nabla \varphi_r(x))^{-1} \, b(\varphi_r(x)) \, \mathrm{d}r.$$

How can we obtain a regularization effect with φ? Malliavin calculus!

Regularization properties of stochastic rough flow

Let

$$(Tb)_{s,t}(x) = \int_s^t (\nabla \varphi_r(x))^{-1} b(\varphi_r(x)) \, \mathrm{d}r.$$

How can we obtain a regularization effect with φ? Malliavin calculus!

Isonormal Gaussian processes

An Isonormal Gaussian process is a set of

1. a real and separable Hilbert space $\mathcal{H}$ (whose scalar product is denoted $\langle \cdot, \cdot \rangle_{\mathcal{H}}$),
2. a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$,
3. a real-valued Gaussian process $W : h \in \mathcal{H} \mapsto W(h)$, *i.e.* $(W(h))_{h \in \mathcal{H}}$ is a family of centered Gaussian random variables such that $\mathbb{E}[W(h)W(g)] = \langle h, g \rangle_{\mathcal{H}}$, for any $h, g \in \mathcal{H}$.

REGULARIZATION PROPERTIES OF STOCHASTIC ROUGH FLOW

Let

$$(Tb)_{s,t}(x) = \int_s^t (\nabla\varphi_r(x))^{-1}\, b(\varphi_r(x))\, \mathrm{d}r.$$

How can we obtain a regularization effect with φ? Malliavin calculus!

Isonormal Gaussian processes

An Isonormal Gaussian process is a set of

1. a real and separable Hilbert space $\mathcal{H}$ (whose scalar product is denoted $\langle \cdot, \cdot \rangle_{\mathcal{H}}$),
2. a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$,
3. a real-valued Gaussian process $W : h \in \mathcal{H} \mapsto W(h)$, *i.e.* $(W(h))_{h\in\mathcal{H}}$ is a family of centered Gaussian random variables such that $\mathbb{E}[W(h)W(g)] = \langle h, g\rangle_{\mathcal{H}}$, for any $h, g \in \mathcal{H}$.

An example of such process is given by the Wiener process and defined by setting $\mathcal{H} = L^2(\mathbb{R}^+; \mathbb{R})$ and defining, for any $h \in \mathcal{H}$,

$$W(h) = \int_0^{+\infty} h(s) dB_s.$$

We now assume that $\mathcal{H} = L^2([0,1]; \mathbb{R})$ and denote $\mathcal{H}(s,t) = \mathcal{H}\mathbf{1}_{[s,t]}$, for any $[s,t] \subset [0,1]$.

REGULARIZATION PROPERTIES OF STOCHASTIC ROUGH FLOW

Let S be the set of smooth cylindrical fields given by

$$S = \left\{ F = f(W(h_1), W(h_2), \dots, W(h_n)) : \; n \in \mathbb{N}^*, \, f \in \mathcal{C}_p^\infty(\mathbb{R}^n), \; (h_k)_{1 \le k \le n} \in \mathcal{H}^n \right\}$$

Malliavin derivative/Divergence operator

Let $[s,t] \subset [0,1]$. For any $F \in S$, we define the operator $D_{[s,t]} : S \mapsto \mathcal{H}(s,t)$, the Malliavin derivative restricted to $[s,t]$, as

$$D_{[s,t]} F = \sum_{k=1}^{n} \partial_k f(W(h_1), W(h_2), \dots, W(h_n)) h_k \mathbf{1}_{[s,t]}.$$

It is linear and closable from S to $L^p(\Omega; \mathcal{H}(s,t))$, with $p \ge 1$.

REGULARIZATION PROPERTIES OF STOCHASTIC ROUGH FLOW

Let S be the set of smooth cylindrical fields given by

$$S = \left\{ F = f(W(h_1), W(h_2), \dots, W(h_n)) :\ n \in \mathbb{N}^*,\ f \in \mathcal{C}_p^\infty(\mathbb{R}^n),\ (h_k)_{1 \le k \le n} \in \mathcal{H}^n \right\}$$

Malliavin derivative/Divergence operator

Let $[s, t] \subset [0, 1]$. For any $F \in S$, we define the operator $D_{[s,t]} : S \mapsto \mathcal{H}(s, t)$, the Malliavin derivative restricted to $[s, t]$, as

$$D_{[s,t]}F = \sum_{k=1}^{n} \partial_k f(W(h_1), W(h_2), \dots, W(h_n)) h_k \mathbf{1}_{[s,t]}.$$

It is linear and closable from S to $L^p(\Omega; \mathcal{H}(s, t))$, with $p \geq 1$. It admits an adjoint which is called the divergence operator, denoted $\delta_{[s,t]}$, which satisfies, for any $u \in L^2(\Omega, \mathcal{H}(s, t))$, the integration by parts formula

$$\mathbb{E}[\langle D_{[s,t]}F, u\rangle_{\mathcal{H}(s,t)} | \mathcal{F}_s] = \mathbb{E}[F\delta_{[s,t]}(u) | \mathcal{F}_s],$$

where $\mathcal{F}_s = \sigma(W(h) :\ h \in \mathcal{H}(0, s))$.

Regularization properties of stochastic rough flow

We remark that, for a vector-valued $F \in S$, we have

$$D_{[s,t]}(f(F)) = \sum_{k=1}^{d} \partial_k f(F) D_{[s,t]} F_k,$$

REGULARIZATION PROPERTIES OF STOCHASTIC ROUGH FLOW

We remark that, for a vector-valued $F \in S$, we have

$$D_{[s,t]}(f(F)) = \sum_{k=1}^{d} \partial_k f(F) D_{[s,t]} F_k,$$

so that

$$\begin{aligned}\langle D_{[s,t]}(f(F)), D_{[s,t]} F_\ell \rangle_{\mathcal{H}([s,t])} &= \sum_{k=1}^{d} \partial_k f(F) \langle D_{[s,t]} F_k, D_{[s,t]} F_\ell \rangle_{\mathcal{H}([s,t])} \\ &= \sum_{k=1}^{d} \partial_k f(F) (\gamma_{F,[s,t]})_{k,\ell} = (\gamma_{F,[s,t]} \nabla f(F))_\ell,\end{aligned}$$

where

$$\gamma_{F,[s,t]} = \big(\langle D_{[s,t]} F_k, D_{[s,t]} F_\ell \rangle_{\mathcal{H}([s,t])} \big)_{1 \le k,\ell \le d},$$

is the covariance matrix associated to F on $[s,t]$ (which is symmetric).

Regularization properties of stochastic rough flow

In particular, this yields the relation

$$\partial_k f(F) = \langle D_{[s,t]}(f(F)), R_{[s,t],k} \rangle_{\mathcal{H}([s,t])},$$

where we denote

$$R_{[s,t],k} = \left((\gamma_{F,[s,t]})^{-1} D_{[s,t]} F \right)_k ,$$

the k-th row of $(\gamma_{F,[s,t]})^{-1} D_{[s,t]} F$. The integration by parts formula yields, for any $G_r \in L^p(\Omega)$ that is $\mathcal{F}_r$-measurable with $r \in [s,t]$,

$$\mathbb{E}\left[\partial_k f(F) G_r | \mathcal{F}_s\right] = \mathbb{E}\left[f(F) \delta_{[s,t]}\left(R_{[s,t],k} G_r\right) \middle| \mathcal{F}_s \right].$$

REGULARIZATION PROPERTIES OF STOCHASTIC ROUGH FLOW

With $f = b$, $F = \varphi_r(x)$ and $G_r = (\nabla\varphi_r(x))^{-1}$, we obtain

Regularization by the flow (C., Duboscq)

There exists a positive adapted stochastic process $(Z_s)_{s\in[0,1]}$, such that for all $q \geq 2$,

$$\sup_{s\in[0,1]} \mathbb{E}[Z_s^q] < +\infty$$

and such that for all $\beta \in \mathbb{N}^d$, $f \in \mathcal{S}$, $0 \leq s \leq r \leq 1$

$$\mathbb{E}\left[(\nabla\varphi_r(x))^{-1}\partial^\beta f(\varphi_r(x)) \middle| \mathcal{F}_s \right] \leq (r-s)^{-|\beta|H} Z_s \|f\|_{L^\infty},$$

with $H = 1/2$ in the Brownian case.

Regularization properties of stochastic rough flow

With $f = b$, $F = \varphi_r(x)$ and $G_r = (\nabla\varphi_r(x))^{-1}$, we obtain

Regularization by the flow (C., Duboscq)

There exists a positive adapted stochastic process $(Z_s)_{s\in[0,1]}$, such that for all $q \geq 2$,

$$\sup_{s\in[0,1]} \mathbb{E}[Z_s^q] < +\infty$$

and such that for all $\beta \in \mathbb{N}^d, f \in \mathcal{S}, 0 \leq s \leq r \leq 1$

$$\mathbb{E}\left[(\nabla\varphi_r(x))^{-1}\partial^\beta f(\varphi_r(x)) \middle| \mathcal{F}_s \right] \leq (r-s)^{-|\beta|H} Z_s \|f\|_{L^\infty},$$

with $H = 1/2$ in the Brownian case.

- Inversion of the Malliavin covariance matrix: strict ellipticity of σ.
- Estimate on the multiplicative term [GOT20]: local non-determinism assumption on the gaussian rough path

$$\inf_{0\leq s<t\leq 1} (t-s)^H \mathrm{Var}\left(B_t^H - B_s^H \middle| \mathcal{F}_{[0,s]} \vee \mathcal{F}_{[t,1]} \right) \geq c_W > 0.$$

Besov spaces and Martingale decomposition

We rely on the Paley-Littlewood blocks $(\Delta_j)_{j\geq -1}$ which are such that, in some sense,

$$b = \sum_{j=-1}^{\infty} \Delta_i b$$

and, for $\beta \in \mathbb{N}^d$, $p \in [1, \infty]$

$$\|\partial^\beta \Delta_j b\|_{L^p} \approx 2^{j|\beta|} \|\Delta_j b\|_{L^p}. \tag{3}$$

We use the Besov spaces $B^s_{\infty,\infty}$ (Hölder-Zygmund space: Hölder for $s \in \mathbb{R}^+ \backslash \mathbb{N}$ and Zygmund otherwise) which are the $f \in S'$ such that

$$\|f\|_{B^s_{\infty,\infty}} = \sup_{j\geq -1} 2^{js} \|\Delta_j f\|_{L^\infty} < \infty.$$

Besov spaces and Martingale decomposition

We rely on the Paley-Littlewood blocks $(\Delta_j)_{j\geq -1}$ which are such that, in some sense,

$$b = \sum_{j=-1}^{\infty} \Delta_i b$$

and, for $\beta \in \mathbb{N}^d$, $p \in [1,\infty]$

$$\|\partial^\beta \Delta_j b\|_{L^p} \approx 2^{j|\beta|} \|\Delta_j b\|_{L^p}. \tag{3}$$

We use the Besov spaces $B^s_{\infty,\infty}$ (Hölder-Zygmund space: Hölder for $s \in \mathbb{R}^+ \setminus \mathbb{N}$ and Zygmund otherwise) which are the $f \in S'$ such that

$$\|f\|_{B^s_{\infty,\infty}} = \sup_{j\geq -1} 2^{js} \|\Delta_j f\|_{L^\infty} < \infty.$$

A lemma

Relation (3), the regularization property and some interpolation enables to deduce that

$$\begin{aligned}\left\|\mathbb{E}\left[(\nabla\varphi_r(\cdot))^{-1}\Delta_j f(\varphi_r(\cdot))\middle|\mathcal{F}_s\right]\right\|_{L^\infty} &\approx \left\|\mathbb{E}\left[(\nabla\varphi_r(\cdot))^{-1}(\partial^\beta)^{-1}\partial^\beta\Delta_j f(\varphi_r(\cdot))\middle|\mathcal{F}_s\right]\right\|_{L^\infty} \\ &\lesssim (r-s)^{-(1-\eta)} 2^{-j\frac{1-\eta}{H}} Z_s \|\Delta_j f\|_{L^\infty},\end{aligned}$$

for $\eta \in [0,1]$.

Besov spaces and Martingale decomposition

Recall that

$$T(\Delta_j b)_{s,t}(x) = \int_s^t (\nabla\varphi_r(x))^{-1}\,\Delta_j b(\varphi_r(x))\,\mathrm{d}r.$$

We remark that, for any $-1 \le j \le \ell := \min\{j \in \mathbb{N} :\ 2^{-j/H} \le (t-s)\}$, we have, for any $\eta \in [0,1]$,

$$\|T(\Delta_j b)_{s,t}\|_{L^\infty} \lesssim (t-s)\|\Delta_j b\|_{L^\infty} \overset{?}{\le} (t-s)^{\frac{1+\eta}{2}}\,2^{-\frac{1-\eta}{2H}j}\|\Delta_j b\|_{L^\infty}\,.$$

The previous lemma yields, for any $j \ge -1$,

$$\left\|\mathbb{E}\left[T(\Delta_j b)_{s,t}\middle|\mathcal{F}_s\right]\right\|_{L^\infty} \lesssim (t-s)^{\frac{1+\eta}{2}}\,2^{-j\frac{1-\eta}{2H}}\,Z_s\|\Delta_j b\|_{L^\infty}\,.$$

Besov spaces and Martingale decomposition

For $N \geq 0$ and $t_k = k\frac{t-s}{N} + s$ one can decompose

$$T(\Delta_j b)_{s,t}(x) - \mathbb{E}[T(\Delta_j b)_{s,t}(x)|\mathcal{F}_s] = \sum_{k=0}^{N-1} \underbrace{\mathbb{E}[T(\Delta_j b)_{s,t}(x)|\mathcal{F}_{t_{k+1}}] - \mathbb{E}[T(\Delta_j b)_{s,t}(x)|\mathcal{F}_{t_k}]}_{\text{martingale increment}}$$

$$= \sum_{k=0}^{N-1} T(\Delta_j b)_{t_k,t_{k+1}}(x) - \mathbb{E}[T(\Delta_j b)_{t_k,t}(x)|\mathcal{F}_{t_k}] + \mathbb{E}[T(\Delta_j b)_{t_{k+1},t}(x)|\mathcal{F}_{t_{k+1}}]$$

BDG inequality, interpolation in Besov spaces, smart choice of the sequence (t_k), the regularity lemma and Kolmogorov continuity theorem give the following result of regularity for the averaged field.

REGULARIZATION PROPERTIES OF STOCHASTIC ROUGH FLOW

Theorem

For any $q \in [2, \infty)$, $\varepsilon_3 > \varepsilon_2 > \varepsilon_1 > 0$ and $\zeta > d/q$, we have, for any $b \in B_{\infty,\infty}^{-\frac{1}{2H}-\varepsilon_2}$,

$$\mathbb{E}\left[\sup_{0\leq s<t\leq 1}\left(\frac{\|(Tb)_{s,t}\|_{\mathcal{C}^{\kappa}_{\chi}}}{|t-s|^{\frac{1+\varepsilon_1}{2}-\frac{1}{q}}}\right)^q\right]^{\frac{1}{q}} \lesssim \|b\|_{B_{\infty,\infty}^{\kappa-\frac{1}{2H}-\varepsilon_3}},$$

with $\chi(x) = (1 + |x|)^{\zeta}$.

We then have the existence/uniqueness of a solution θ to

$$\theta_t = \theta_0 + \int_0^t (Tb)_{\mathrm{d}r}(\theta_r).$$

Thank you very much for your attention!

References I

[[ABS17]]Siva Athreya, Suprio Bhar, and Atul Shekhar.
Smoothness of Flow and Path-by-Path Uniqueness in Stochastic Differential Equations.
arXiv:1709.02115 [math], September 2017.

[[Anz22]]Lukas Anzeletti.
Comparison of classical and path-by-path solutions to SDEs.
(arXiv:2204.07866), 2022.

[[CG16]]Rémi Catellier and Massimiliano Gubinelli.
Averaging along irregular curves and regularisation of ODEs.
Stochastic Processes and their Applications, 126(8):2323–2366, August 2016.

[[Dav07]]Alexander M. Davie.
Uniqueness of Solutions of Stochastic Differential Equations.
International Mathematics Research Notices, 2007, January 2007.

[[Dav11]]Alexander M. Davie.
Individual path uniqueness of solutions of stochastic differential equations.
In *Stochastic Analysis 2010*, pages 213–225. Springer, 2011.

[[DG22]]Konstantinos Dareiotis and Máté Gerencsér.
Path-by-path regularisation through multiplicative noise in rough, Young, and ordinary differential equations, July 2022.

References II

[[FdLP06]]Denis Feyel and Arnaud de La Pradelle.
Curvilinear integrals along enriched paths.
Electronic Journal of Probability, 11, 2006.

[[Gal23]]Lucio Galeati.
Nonlinear young differential equations: a review.
Journal of Dynamics and Differential Equations, 35(2):985–1046, 2023.

[[GG21]]Lucio Galeati and Massimiliano Gubinelli.
Noiseless regularisation by noise.
Revista Matemática Iberoamericana, July 2021.

[[GOT20]]Benjamin Gess, Cheng Ouyang, and Samy Tindel.
Density bounds for solutions to differential equations driven by Gaussian rough paths.
Journal of Theoretical Probability, 33(2):611–648, 2020.

[[HL17]]Yaozhong Hu and Khoa Lê.
Nonlinear young integrals and differential systems in hölder media.
Transactions of the American Mathematical Society, 369(3):1935–2002, 2017.

[[Pri20]]Enrico Priola.
On Davie's uniqueness for some degenerate SDEs.
Theory of Probability and Mathematical Statistics, 103:41–58, December 2020.

References III

[[RS17]]Sebastian Riedel and Michael Scheutzow.
Rough differential equations with unbounded drift term.
Journal of Differential Equations, 262(1):283–312, January 2017.

[[SW22]]Alexander Shaposhnikov and Lukas Wresch.
Pathwise vs. path-by-path uniqueness.
Annales de l'Institut Henri Poincaré, Probabilités et Statistiques, 58(3):1640–1649, August 2022.