

The attractive log gas: stability, uniqueness, and propagation of chaos

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Outline of the Talk

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Introduction

The microscopic attractive log gas I

The microscopic *attractive log gas* for N particles at temperature $1/\beta \geq 0$ on the flat torus \mathbb{T}^d , $d \geq 1$, is described by the free energy

$$\frac{1}{\beta} \int_{(\mathbb{T}^d)^N} \log(f_N) df_N - \frac{1}{2N} \sum_{1 \leq i \neq j \leq N} \int_{(\mathbb{T}^d)^N} g(x_i - x_j) df_N,$$

where f_N is a probability density on $(\mathbb{T}^d)^N$ and g is the solution of

$$(1.1) \quad (-\Delta)^{d/2} g = c_d(\delta_0 - 1), \quad c_d := \frac{\Gamma(d/2)(4\pi)^{d/2}}{2}.$$

I.e. g is the zero-mean periodization of $-\log|x|$ on \mathbb{R}^d and behaves like $-\log|x|$ as $|x| \rightarrow 0$.

- ▶ If $d = 2$, then g is the *Coulomb* potential.
- ▶ Attractiveness is reflected in the $-$ in front of the interaction energy term.
- ▶ The logarithmic interaction is the most singular one for which the attraction can be balanced by the effectively logarithmic repulsion of the entropy. In contrast, gases with attractions that are stronger than logarithmic do not even have finite free energy.

The microscopic attractive log gas II

The *overdamped Langevin dynamics* of the free energy are given by the system of SDES

$$\begin{cases} dx_i^t = \frac{1}{N} \sum_{1 \leq j \leq N: j \neq i} \nabla g(x_i^t - x_j^t) + \sqrt{\frac{2}{\beta}} dW_i^t \\ x_i^t|_{t=0} = x_i^0 \in \mathbb{T}^d. \end{cases}$$

where W_1, \dots, W_N are independent standard d-dim. Brownian motions.

Making sense of the SDE system is delicate due to the positive probability of collisions (see [Fournier-Jourdain 2017](#), [Cattiaux-Pédèches 2016](#), [Fournier-Tardy 2021](#)). For us, more natural to work with *forward Kolmogorov equation*

$$\begin{cases} \partial_t f_N = - \sum_{i=1}^N \operatorname{div}_{x_i} \left(f_N \frac{1}{N} \sum_{1 \leq i \neq j \leq N} \nabla g(x_i - x_j) \right) + \frac{1}{\beta} \sum_{i=1}^N \Delta_{x_i} f_N \\ f_N|_{t=0} = f_N^0. \end{cases}$$

Questions of interest: mean-field limit I

What are the limiting dynamics of the empirical measure

$$\mu_N^t := \frac{1}{N} \sum_{i=1}^N \delta_{x_i^t} \in \mathcal{P}(\mathbb{T}^d)$$

as $N \rightarrow \infty$?

Formally expect that if $\mu_N^0 \xrightarrow{N \rightarrow \infty} \mu^0$, then $\mu_N^t \xrightarrow{N \rightarrow \infty} \mu^t$, where μ^t is a solution to the *mean-field equation*

$$\begin{cases} \partial_t \mu + \operatorname{div}(\mu \nabla g * \mu) = \frac{1}{\beta} \Delta \mu \\ \mu|_{t=0} = \mu^0. \end{cases}$$

When $\beta = \infty$, μ_N^t is a genuine weak solution. This equation is the gradient flow (with respect to W_2 on $\mathcal{P}(\mathbb{T}^d)$) of the macroscopic free energy

$$\mathcal{E}_\beta := \frac{1}{\beta} \int_{\mathbb{T}^d} \log(\mu) d\mu - \frac{1}{2} \int_{(\mathbb{T}^d)^2} g(x-y) d\mu^{\otimes 2}(x,y),$$

obtained by considering product states $f_N = \mu^{\otimes N}$ and evaluating the average as $N \rightarrow \infty$.

Establishing the mean-field limit refers to proving this convergence.

Questions of interest: propagation of chaos I

Suppose the initial positions $X_N^0 = (x_1^0, \dots, x_N^0)$ are independently and identically distributed with some law μ^0 :

$$f_N^0 = (\mu^0)^{\otimes N}.$$

What is the limiting behavior as $N \rightarrow \infty$ of the law $f_N^t(x_1, \dots, x_N)$ of the positions $X_N^t = (x_1^t, \dots, x_N^t)$ of the particles at time t ?

If μ^t is the solution of the mean-field equation with initial datum μ^0 , does it hold that

$$f_N^t \approx (\mu^t)^{\otimes N} \quad \text{as } N \rightarrow \infty ?$$

Propagation of chaos refers to the asymptotic factorization of k -point marginals $f_{N;k}^t \rightarrow (\mu^t)^{\otimes k}$.¹

Known that mean-field convergence and propagation of chaos are closely related; qualitatively, they are equivalent [Hauray-Mischler 2014](#).

¹Recall that the k -point marginal $f_{N;k} := \int_{(\mathbb{T}^d)^{N-k}} f_N(\cdot, x_{k+1}, \dots, x_N) dx_{k+1} \cdots dx_N$.

Questions of interest: generation of chaos I

Related notion of **generation of chaos**.² Even when the initial law f_N^0 is not asymptotically chaotic, one still has chaos as $t \rightarrow \infty$ and $N \rightarrow \infty$.

Interpreted in an *entropic sense*, we have generation of chaos if

$$H_N(f_N^t | (\mu^t)^{\otimes N}) = o_t(1) + o_N(1) \quad \text{as } N \rightarrow \infty \text{ and } t \rightarrow \infty,$$

where H_N is the $1/N$ normalized relative entropy and

- ▶ $o_t(1)$ depends on N only through $H_N(f_N^0 | (\mu^0)^{\otimes N})$ and is uniform in N assuming $H_N(f_N^0 | (\mu^0)^{\otimes N}) = O(1)$,
- ▶ $o_N(1)$ is uniform in t .

²This term was recently coined by Jani Lukkarinen.

Motivation and relevance I

g falls into the class of *Riesz interactions*

$$g(x) = \frac{1}{c_{s,d}} \begin{cases} -\log |x|, & s = 0 \\ |x|^{-s}, & s \neq 0 \end{cases}$$

where $s = d - 2$ is Coulomb.

Numerous applications...Refer to surveys [Jabin 2014](#), [Jabin-Wang 2017](#), [Chaintron-Diez 2022](#), [Golse 2022](#) and book [Borodachov-Hardin-Saff 2019](#)

The $d = 2$ attractive log case is a version of the Patlak-Keller-Segel model for the aggregation of cells by chemotaxis [Patlak 1953](#), [Keller-Segel 1970](#), [Nanjundiah 1973](#). “Critical” temperature $\beta_c = 2d$, below which diffusion dominates leading to global existence and above which aggregation dominates, leading to finite time blow-up

- ▶ β_c is precisely the value at which the macroscopic free energy ceases to be bounded from below.
- ▶ β_c is also the exponent for which $e^{-\frac{\beta}{2}|x|}$ is not locally integrable, corresponding divergence of the partition function.

For $\beta = \infty$, studied as a model for vortex densities in superconductors [Chapman-Rubinstein-Schatzman 1996](#) and superfluids [E 1994](#), as well as a model for adhesion dynamics [Nieto-Poupaud-Soler 2001](#), [Poupaud 2002](#).

Also mention that for $d = 1$, the repulsive case is widely studied in random matrix theory [Forrester 2010](#).

Motivation and relevance II

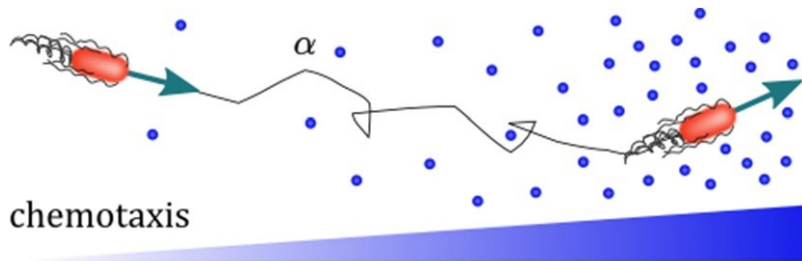


Figure: Courtesy of Jakuszeit-LindseyJones-Peaudecerf-Croze 2021

Previous results on mean-field limits/propagation of chaos I

Many previous works on general mean-field limits for gradient and Hamiltonian dynamics!

- ▶ Coupling method [Sznitman 1991](#), [Hauray-Jabin 2015](#), [Boers-Pickl 2016](#), [Lazarovici-Pickl 2017](#), [Graß 2021](#), [Guillin-Le Bris-Monmarché 2021](#),...
- ▶ Wasserstein stability [Braun-Hepp 1977](#), [Dobrushin 1979](#), [Neunzert-Wick 1974](#), [Hauray 2009](#), [Carrillo-Choi-Hauray 2014](#),...
- ▶ Relative entropy [Jabin-Wang 2016, 2018](#), [Guillin-Le Bris-Monmarché 2021](#)
- ▶ Control of microscopic dynamics and compactness for well-chosen point configurations [Goodman-Hou-Lowengrub 1990](#), [Schochet 1996](#),...
- ▶ Displacement convexity for Wasserstein gradient flow [Carrillo-Ferreira-Precioso 2012](#), [Berman-Onnheim 2015](#),...
- ▶ Compactness via diffusion [Osada 1985-1987](#), [Rogers-Shi 1993](#), [Cépa-Lepingle 1997](#), [Fournier-Hauray-Mischler 2014](#), [Wang-Zhao-Zhu 2022](#),...
- ▶ Stability for BBGKY hierarchy [Lacker 2021](#), [Han 2022](#), [Jabin-Poyato-Soler 2021](#), [Bresch-Jabin-Soler 2022](#), [Lacker-Le Flem 2023](#)
- ▶ Modulated energy/free energy method [Duerinckx 2016](#), [Serfaty 2020](#), [R. 2020-2021](#), [Nguyen-R.-Serfaty 2021](#), [R.-Serfaty 2021](#) / [Bresch-Jabin-Wang 2019-2020](#), [Chodron de Courcel-R.-Serfaty 2023](#)

Previous results on mean-field limits/propagation of chaos II

Much of the literature concerns *repulsive* interactions, while we are interested in *attractive*.

- ▶ Weak forms of convergence — Fournier-Jourdain 2017, Cattiaux-Pédèches 2016
 - ▶ Shows tightness of sequence of empirical measures and convergence along a subsequence to a weak solution of limiting PDE. No uniqueness for class of weak solutions prevents deducing convergence along the full sequence.
 - ▶ Recently extended up to the critical temperature $\beta_c = 2d$ Tardy 2022
- ▶ Models with regularization of singularity — Haskovec-Schmeiser 2011, Godinho-Quiñao 2015, Garcia-Pickl 2017, Olivera-Richard-Tomasevic 2020,...
- ▶ Quantitative propagation of chaos (in entropic sense) — Bresch-Jabin-Wang 2019-2020
 - ▶ Modulated free energy method – functional combining relative entropy and modulated energy (obeys a dissipation identity for the dynamics)
 - ▶ Modulated energy (attractive) competes with relative entropy (repulsive); key difficulty is to show that the latter controls the former, implying coercivity; requires $\beta < \beta_c$
 - ▶ Local-in-time

Goal

One of the questions posed by [Bresch-Jabin-Wang 2019-2020](#) is the possibility of a *uniform-in-time* rate for propagation of chaos, as their rate deteriorates exponentially in time.

We affirmatively answer this question if β is sufficiently small. Doing so requires us to

1. Establish a functional inequality (*modulated logarithmic Hardy-Littlewood-Sobolev (mLHLS) inequality*) expressing the control of modulated energy by relative entropy.
2. Understand thoroughly the asymptotic behavior of solutions to the limiting equation on \mathbb{T}^d .

We also show that a uniform in time estimate *cannot hold* if β is too large (namely, $\beta > \beta_s$, the threshold for stability of the uniform distribution).

Along the way, show that uniform-in-time propagation of chaos is inherently linked with uniqueness and stability of equilibria for the limiting equation.

One of the main consequences of our analysis will be that $\beta_c = 2d$ is not always the right notion to speak of the “criticality” of the system.

The limiting equation

Comparison with \mathbb{R}^d I

Recall the mean-field equation

$$\partial_t \mu + \operatorname{div}(\mu \nabla g * \mu) = \frac{1}{\beta} \Delta \mu.$$

On \mathbb{R}^d , straightforward computation that all classical solutions with finite second moment blow up in finite time if $\beta > 2d$.

- ▶ Computation doesn't work on \mathbb{T}^d , since $g \neq -\log|x|$ (the second moment is not meaningful to measure spatial localization on \mathbb{T}^d anyway).
- ▶ Need additional condition that density sufficiently concentrated around a point [Nagai 2001](#), [Senba-Suzuki 2002](#).

In fact, it's not true that all classical solutions blow up in finite time if $\beta > 2d$ (always have the uniform measure which is stationary).

Comparison with \mathbb{R}^d II

Recall that the limiting equation is the gradient flow with respect to W_2 of the (macroscopic) free energy

$$\mathcal{E}_\beta(\mu) := \frac{1}{\beta} \int_{\mathbb{T}^d} \log(\mu) d\mu - \frac{1}{2} \int_{(\mathbb{T}^d)^2} g(x-y) d\mu^{\otimes 2}(x,y),$$

hence $\mathcal{E}_\beta(\mu^t)$ is decreasing.

- ▶ *A priori* unclear if \mathcal{E}_β is coercive (i.e. controls some distance) or even bounded from below.
- ▶ Competition between the entropy (repulsive) and interaction energy (attractive)
- ▶ The sharp *logarithmic Hardy-Littlewood-Sobolev inequality* [Carlen-Loss 1992](#), [Beckner 1993](#) (also [Carlen-Carrillo-Loss 2010](#)) implies that \mathcal{E}_β is bounded from below iff $\beta \leq \beta_c$ and coercive (i.e. controls relative entropy) if $\beta < \beta_c$.

Comparison with \mathbb{R}^d III

On \mathbb{R}^2 (Patlak-Keller-Segel), extensive literature...

- ▶ Local/global well-posedness — Biler 1995, Senba-Suzuki 2002, Blanchet 2013, Bedrossian-Masmoudi 2014, Fernández-Mischler 2016, Wei 2018,...
- ▶ Stationary solutions — Naito 2001, Biler-Karch-Laurencot-Nadzieja 2006,...
- ▶ Finite-time blow-up ($\beta > \beta_c$) — Jäger-Luckhaus 1992, Herrero-Velázquez 1996-1997, Velázquez 2002-2004, Dolbeault-Schmeiser 2009, Raphaël-Schweyer 2014, Collot-Ghoul-Masmoudi-Nguyen 2021-2022, Buseghin-Dávila-del Pino-Musso 2023,...
- ▶ Asymptotic behavior ($\beta < \beta_c$) — Blanchet-Dolbeault-Perthame 2006, Campos-Dolbeault 2014,...
- ▶ Asymptotic behavior ($\beta = \beta_c$) — Blanchet-Carrillo-Masmoudi 2008, Blanchet-Carlen-Carrillo 2012, Carlen-Figalli 2013, Ghoul-Masmoudi 2018, Dávila-del Pino-Musso-Wei 2023, Carlen 2023,...

Log dimensional generalization on \mathbb{R}^d — Calvez-Carrillo 2012, Calvez-Perthame-Sharifi Tabar 2007

Coulomb dimensional generalization on \mathbb{R}^d — Corrias-Perthame-Zaag 2004, Blanchet-Carrillo-Laurencot 2009, Calvez-Corrias-Ebde 2012, Ogawa-Wakui 2016, Biler-Karch-Zienkiewicz 2018, Souplet-Winkler 2019, Naito 2021,...

Big gap in the literature when it comes \mathbb{T}^d !

Global well-posedness I

Local well-posedness follows by fixed point argument. Given initial data $\mu^0 \in \mathcal{P}_{ac}(\mathbb{T}^d)$, there exists a unique mild solution $\mu \in C_w([0, T], \mathcal{P}_{ac}(\mathbb{T}^d))$ such that

$$\sup_{0 < t \leq T} (\beta t)^{\frac{1}{4}} \|\mu^t\|_{L^{\frac{2d}{2d-1}}} < \infty.$$

This built-in hypercontractivity + iteration argument imply that μ is instantaneously L^∞ . Exploiting the smoothing of the heat kernel, one can further show that μ is instantaneously smooth. Valid for all $\beta < \infty$.

To show the solution is global, one just needs to show that $\|\mu^t\|_{L^{\frac{2d}{2d-1}}}$ cannot blow up in finite time. For this, $\beta < \beta_c$ is crucial because the free energy controls the entropy...

Global well-posedness II

From the sharp log HLS, follows that

$$\left(\frac{1}{\beta} - \frac{1}{\beta_c}\right) \int_{\mathbb{T}^d} \log_+(f) df \leq C_1 \left(\frac{1}{\beta} - \frac{1}{\beta_c}\right) + (\mathcal{E}_\beta(f) + C_2),$$

for $C_1, C_2 > 0$ independent of β . Defining the *dissipation functional*

$$\mathcal{D}_\beta(f) := \int_{\mathbb{T}^d} \left| \frac{1}{\beta} \nabla \log f - \nabla g * f \right|^2 df,$$

the free energy satisfies

$$\forall t \geq 0, \quad \mathcal{E}_\beta(\mu^t) + \beta \int_0^t \mathcal{D}_\beta(\mu^\tau) d\tau = \mathcal{E}_\beta(\mu^0).$$

On the other hand, if $\beta < \beta_c$, the dissipation functional + entropy control the *Fisher information*

$$I(f) := \int_{\mathbb{T}^d} \frac{|\nabla f|^2}{f} dx = \int_{\mathbb{T}^d} |\nabla \log f|^2 df.$$

Through a Grönwall argument, this yields a linear-in- t bound for $\|\mu^t\|_{L^2}^2$. Since $\frac{2d}{2d-1} \leq 2$, this yields global existence.

Trend to equilibrium I

Using the decrease of the free energy + compactness argument, one can show that $H(\mu^t | \mu^\beta) \rightarrow 0$ (hence, also in L^1 by Pinsker), as $t \rightarrow \infty$, where μ_β solves the *Kirkwood-Monroe/Lane-Emden equation*

$$\mu_\beta = \frac{e^{\beta g * \mu_\beta}}{Z_\beta}, \quad Z_\beta := \int_{\mathbb{T}^d} e^{\beta g * \mu_\beta} dx,$$

In particular, μ_β is a stationary solution of the mean-field PDE.

Define the *uniqueness threshold*

$$\beta_u := \sup\{\beta_0 \geq 0 : \forall \beta \in [0, \beta_0], \mu_{\text{unif}} \text{ is the unique solution to (KM)}\}.$$

Through a contraction argument, we show that $\beta_u > 0$. In particular, this implies that for $\beta < \beta_u$, μ_{unif} is the unique minimizer of \mathcal{E}_β .

- ▶ For $d = 2$, previously known that uniqueness holds for $\beta \leq 4$ [Lin-Lucia 2006](#), [Gui-Moradifam 2019](#), [Gu-Gui-Hu-Li 2021](#) and fails for $\beta > 4$ [Ricciardi-Tarantello 1998](#), [Struwe-Tarantello 1998](#). I.e. $\beta_u = \beta_c = 4$.
- ▶ For $d = 1$, we also show that $\beta_u = \beta_c = 2$. Only other case where value of β_u is known.

Trend to equilibrium II

Suppose $\beta < \beta_s = \frac{(2\pi)^d}{c_d} = \frac{2\pi^{d/2}}{\Gamma(d/2)}$. Then there exists $T_0 \geq 0$ such that if

1. $\|\mu^t - \mu_{\text{unif}}\|_{L^1} \ll 1$ for all $t \geq T_0$, then $\|\mu^t - \mu_{\text{unif}}\|_{L^2} = O(e^{-ct})$ as $t \rightarrow \infty$;
2. $\|\mu^t - \mu_{\text{unif}}\|_{L^2} \ll 1$ for $t = T_0$, then this is true for all $t \geq T_0$ and $\|\mu^t - \mu_{\text{unif}}\|_{L^2} = O(e^{-ct})$ as $t \rightarrow \infty$.

Our previous qualitative L^1 convergence implies scenario 1 is satisfied for $\beta < \beta_u$. Scenario 2 is obviously satisfied if we choose initial data μ° close to μ_{unif} in L^2 .

Combining this exponential-in-time convergence in L^2 with the local smoothing effect, we also obtain exponential-in-time decay of derivatives to all order: for any $k \geq 1$,

$$\|\nabla^{\otimes k} \mu^t\|_{L^\infty} = O(e^{-ct}) \quad \text{as } t \rightarrow \infty.$$

N.B. $\beta_c \neq \beta_s$ in general. E.g. for $d = 2$, $\beta_c = 4 < 2\pi = \beta_s$. In particular, this shows the global existence and trend to equilibrium for classical solutions on \mathbb{T}^2 corresponding to the supercritical regime of Patlak-Keller-Segel.

Nonlinear (in)stability of uniform distribution

The *stability threshold* β_s is such that for $\beta < \beta_s$, the linearization around μ_{unif} ,

$$L_\beta := \left(\frac{1}{\beta} \Delta - c_d(-\Delta)^{\frac{2-d}{2}} \right),$$

has no positive eigenvalues, while for $\beta > \beta_s$, L_β has positive eigenvalues, implying β_s is sharp for linear stability of μ_{unif} .

Recall that a stationary state μ_* of a PDE is

- ▶ *nonlinearly stable* (with respect to the norms $\|\cdot\|_1, \|\cdot\|_2$) if for every $\epsilon > 0$, there is a $\delta > 0$ such that if one takes initial data μ^0 such that $\|\mu^0 - \mu_*\|_1 < \delta$, then $\sup_{t \geq 0} \|\mu^t - \mu_*\|_2 < \epsilon$.
- ▶ *nonlinearly unstable* if it is not nonlinearly stable.

Result of previous slide shows that μ_{unif} is nonlinearly stable (in L^2) if $\beta < \beta_s$.

If $\beta > \beta_s$, μ_{unif} is nonlinearly unstable: given $\epsilon > 0$, there exists an $O(\epsilon)$ perturbation μ_ϵ^0 of μ_{unif} such that in time $O(\log \frac{1}{\epsilon})$, the associated solution μ_ϵ is $\frac{1}{2}$ distance from μ_{unif} (in L^1). Prove this using the corrector method of Grenier 2000 introduced for fluid equations.

Instability, minimization, and uniqueness

Instability tied to the fact that μ_{unif} does not minimize \mathcal{E}_β when $\beta > \min(\beta_s, \beta_c)$.

- ▶ If $\beta > \beta_c$, then this implied by unboundedness from below of the free energy.
- ▶ If $\beta > \beta_s$, then can perturb around μ_{unif} to show upper bound

$$\inf_{\mu \in \mathcal{P}_{ac}(\mathbb{T}^d)} \mathcal{E}_\beta(\mu) \leq -\frac{\beta}{432} \left| \frac{1}{\beta_s} - \frac{1}{\beta} \right|^3 < 0 = \mathcal{E}_\beta(\mu_{\text{unif}}).$$

Value of the minimal free energy is tied to uniqueness for KM equation.

- ▶ If $\beta_s < \beta_c$, then since minimizers satisfy the KM equation, it follows that for $\beta \in (\beta_s, \beta_c)$, solutions of the KM equation are nonunique.
- ▶ If $\beta_c < \beta_s$, then for $\beta \in (\beta_c, \beta_s)$, μ_{unif} is a local minimizer, but can still use the mountain pass geometry of \mathcal{E}_β to show nonuniqueness.
- ▶ Conclude that $\beta_u \leq \min(\beta_s, \beta_c)$.

The one-dimensional case: exactly solvable

Remarkably, the $d = 1$ case is exactly solvable. Seen through looking at system of ODEs satisfied by Fourier coefficients

$$\begin{cases} \partial_t \hat{\mu}(k) = -2\pi k \left(\frac{2\pi}{\beta} k + \pi \right) \hat{\mu}(k) - \pi^2 k \sum_{j=1}^{k-1} \hat{\mu}(j) \hat{\mu}(k-j) \\ \hat{\mu}(k)|_{t=0} = \hat{\mu}^\circ(k). \end{cases}$$

This simplified structure (special consequence of $\hat{g}(k) = -i\pi \operatorname{sgn}(k)$) was first observed in passing [Cépa-Lepingle 1997](#) (for the repulsive version) to prove uniqueness.

If $\beta \notin -2\mathbb{N}_+$, unique stationary solution is μ_{unif} . If $\beta = -2n$, for $n \geq 1$, then (modulo translations) there is family of smooth probability density solutions $(\mu_{\beta,c})_{c \in (0,1)}$ interpolating between μ_{unif} and $\frac{1}{n} \sum_{i=0}^{n-1} \delta_{i/n}$.

Can explicitly describe necessary and sufficient conditions for nonlinear stability, sharp rates of relaxation, blow-up times, etc.

Modulated free energy and logarithmic HLS inequality

Modulated free energy I

Proof of [Bresch-Jabin-Wang 2019-2020](#) through *modulated free energy*,

$$E_N(f_N, \mu) := \frac{1}{\beta} H_N(f_N | \mu^{\otimes N}) - \mathbb{E}_{f_N} [F_N(X_N, \mu)],$$

which is well-suited to studying overdamped Langevin dynamics at nonzero temperature.

- (normalized) relative entropy [Jabin-Wang 2016-2018](#)

$$H_N(f_N | \mu^{\otimes N}) := \frac{1}{N} \int_{(\mathbb{R}^d)^N} \log \left(\frac{f_N}{\mu^{\otimes N}} \right) df_N$$

- (average) modulated energy $\mathbb{E}_{f_N} [F_N(X_N, \mu)]$

$$F_N(X_N, \mu) := \frac{1}{2} \int_{(\mathbb{R}^d)^2 \setminus \Delta} g(x-y) d \left(\frac{1}{N} \sum_{i=1}^N \delta_{x_i} - \mu \right)^{\otimes 2} (x, y).$$

Total interaction of system of N discrete charges at x_i against neutralizing background of charge μ , with (infinite) self-interaction of points.

Modulated free energy II

The points $X_N = (x_1, \dots, x_N) \in (\mathbb{R}^d)^N$ are viewed as randomly distributed according to the law f_N , and we take the expectation of the modulated energy $F_N(X_N, \mu)$.

Modulated energy first appeared in stat mech of Coulomb/Riesz gases [Sandier-Serfaty 2015](#), [Rougerie-Serfaty 2016](#), [Petrache-Serfaty 2017](#), [Leblé-Serfaty 2017-2018](#) as a *next-order energy*. Later used to study dynamics [Duerinckx 2016](#), [Serfaty 2020](#), [Nguyen-R.-Serfaty 2021](#),...

Enlightening to re-express the modulated free energy in a different form [R.-Serfaty 2023](#)...

Modulated free energy III

Given a probability density μ , we can define the *modulated Gibbs measure* and *modulated partition function*

$$\mathbb{Q}_{N,\beta}(\mu) := \frac{1}{K_{N,\beta}(\mu)} e^{\beta N F_N(X_N, \mu)} d\mu^{\otimes N}(X_N),$$
$$K_{N,\beta}(\mu) := \int_{(\mathbb{R}^d)^N} e^{\beta N F_N(X_N, \mu)} d\mu^{\otimes N}(X_N).$$

Using the explicit form of $\mathbb{Q}_{N,\beta}(\mu)$, we may rewrite

$$E_N(f_N, \mu) = \frac{1}{\beta} \left(H_N(f_N | \mathbb{Q}_{N,\beta}(\mu)) + \frac{\log K_{N,\beta}(\mu)}{N} \right).$$

With this rewriting, a crucial condition appearing in all that follows, called *smallness of the free energy*, is

$$|\log K_{N,\beta}(\mu)| = o(N).$$

Up to an error related to the smallness of free energy condition, the modulated free energy is another relative entropy!

Coercivity of the modulated free energy? I

MFE satisfies the dissipation identity

$$\begin{aligned} \frac{d}{dt} E_N(f_N^t, \mu^t) &\leq -\frac{1}{\beta^2} I_N(f_N^t | \mathbb{Q}_{N,\beta}(\mu^t)) \\ &+ \frac{1}{2} \mathbb{E}_{f_N^t} \left[\int_{(\mathbb{R}^d)^2 \setminus \Delta} (u^t(x) - u^t(y)) \cdot \nabla g(x - y) d \left(\frac{1}{N} \sum_{i=1}^N \delta_{x_i} - \mu^t \right)^{\otimes 2} (x, y) \right]. \end{aligned}$$

where the velocity field associated to the mean-field dynamics is

$$u^t := \frac{1}{\beta} \nabla \log \mu^t - \nabla g * \mu^t.$$

- ▶ The first term on RHS may be ignored, while the second term may be estimated in terms of the relative entropy $H_N(f_N^t | (\mu^t)^{\otimes N})$.
- ▶ To close the Grönwall loop, one needs to estimate $H_N(f_N^t | (\mu^t)^{\otimes N})$ in terms of $E_N(f_N^t, \mu^t)$.
- ▶ *A priori* unclear that MFE is coercive (i.e. it controls the relative entropy) due to competition between the relative entropy and modulated energy.

Coercivity of the modulated free energy? II

In fact, Bresch et al. were unable to prove this coercivity. Instead, they truncate g into a “short-range” and “long-range” parts:

$$g = g\chi_r + g(1 - \chi_r) =: g_{sr} + g_{lr}.$$

Provided $0 < r \ll 1$, depending on $d, \beta < \beta_c$, they prove that there is a $C = C(d, \beta, \|\mu\|) > 0$ and $\gamma = \gamma(d) > 0$ such that for all $N \gg 1$, $f_N \in \mathcal{P}_{ac}((\mathbb{T}^d)^N)$, and “nice” $\mu \in \mathcal{P}_{ac}(\mathbb{T}^d)$,

$$\beta \mathbb{E} [F_{N, sr}(X_N, \mu)] \leq H_N(f_N | \mu^{\otimes N}) + CN^{-\gamma},$$

where $F_{N, sr}$ is the modulated energy with g replaced by g_{sr} .

Coercivity of the modulated free energy? III

Their proof of this functional inequality proceeds through large deviation arguments (cf. [Bodineau-Guionnet 1999](#)). At one point, one has to show that the LD rate functional

$$\mathcal{I}(\mu) := \sup_{\rho \in \mathcal{P}(\mathbb{T}^d)} \left(\beta \int_{(\mathbb{T}^d)^2} \tilde{g}_{sr}(x-y) d(\rho - \mu)^{\otimes 2}(x, y) - \int_{\mathbb{T}^d} \log \left(\frac{d\rho}{d\mu} \right) d\rho \right)$$

vanishes, where \tilde{g}_{sr} is a regularization of g_{sr} .

- ▶ Done by showing uniqueness for the EL equation through a contraction argument.
- ▶ Requires that $\|g_{sr}\|_{L^1}$ is sufficiently small, which can, of course, be guaranteed by taking r sufficiently small.

The disadvantage of this truncation is that one still has to account for the long-range contribution of g_{lr} . Since g_{lr} is C^∞ , one can directly perform a Grönwall argument on

$$\mathbb{E}_{\mu^t} \left[F_{N,lr}(X_N, \mu^t) \right],$$

where $F_{N,lr}$ is the modulated energy with g replaced by g_{lr} , and bound by the relative entropy.

This spoils the possibility of a uniform-in-time estimate, as one encounters terms that have no clear means of decay in time!

Modulated logarithmic Hardy-Littlewood-Sobolev inequality I

- ▶ We observe that it is possible to prove this inequality *without truncation*, provided β is *sufficiently small*.
- ▶ Such an inequality *cannot hold* if β is *too large*, namely $\beta > \min(\beta_s, \beta_c)$, as a consequence of the strict negativity of the minimal macroscopic free energy.

Modulated logarithmic Hardy-Littlewood-Sobolev inequality II

Theorem (mLHLS)

Let $d \geq 1$. There exists a maximal $\beta_i > 0$, such that for every $0 \leq \beta < \beta_i$, there exists a $\delta_\beta > 0$, tending to ∞ as $\beta \rightarrow 0$, such that for all $\mu \in \mathcal{P}_{ac}(\mathbb{T}^d)$ with $\|\log \mu\|_{L^\infty} \leq \delta_\beta$ and $f_N \in \mathcal{P}_{ac}((\mathbb{T}^d)^N)$, it holds that for all $N = N(d, \beta)$ sufficiently large,

$$\mathbb{E}_{f_N} [F_N(X_N, \mu)] \leq \frac{1}{\beta} H_N(f_N | \mu^{\otimes N}) + \mathbf{C}(\beta, \|\log \mu\|_{W^{1,\infty}}) N^{-\gamma},$$

where $\gamma > 0$ depends only on d and $\mathbf{C} : [0, \infty)^2 \rightarrow [0, \infty)$ is a continuous, increasing function of its arguments, vanishing if any of them are zero.

Conversely, if $\beta > \min(\beta_c, \beta_s)$, then there exists a $\eta_\beta > 0$ such that the following holds: for any N , there is an $f_N \in \mathcal{P}_{ac}((\mathbb{T}^d)^N) \cap C^\infty((\mathbb{T}^d)^N)$ such that

$$\frac{1}{\beta} H_N(f_N | \mu_{unif}^{\otimes N}) \leq \mathbb{E}_{f_N} [F_N(X_N, \mu_{unif})] - \eta_\beta.$$

Link with Kazhdan-Warner equations

Proof follows a similar strategy to [Bresch-Jabin-Wang 2020](#). One of the most important steps (and the source of the restriction on β) is to show for a reference measure μ , the functional

$$\nu \mapsto \frac{1}{\beta} \int_{\mathbb{T}^d} \log \left(\frac{\nu}{\mu} \right) d\nu - \frac{1}{2} \int_{\mathbb{T}^d} g(x-y) d(\mu - \nu)^{\otimes 2}(x, y).$$

is minimized at $\nu = \mu$. One way to do this by showing uniqueness of solutions for the EL equation

$$\nu = \mu \frac{e^{\beta g * (\nu - \mu)}}{\int_{\mathbb{T}^d} e^{\beta g * (\nu - \mu)} d\mu} \iff \frac{1}{c_d} (-\Delta)^{d/2} h = \mu \left[\frac{e^{\beta h}}{\int_{\mathbb{T}^d} e^{\beta h} d\mu} - 1 \right],$$

where $h = g * (\nu - \mu)$. By perturbative arguments, can show that $\|\nu - \mu\|_{L^1} < \|\nu - \mu\|_{L^1}$, if $\beta, \|\log \mu\|_{L^\infty}$ sufficiently small.

As we learned, this equation appears (in equivalent form) in geometry in the context of the *prescribed curvature problem*, sometimes called a *Kazhdan-Warner* type equation (for $d = 2$). Uniqueness for nonuniform μ seems challenging. Even for $d > 2$ and $\mu = \mu_{\text{unif}}$, the sharp threshold for uniqueness (i.e. β_u from before) is unknown.

Counterexample to mLHLS

If $\beta > \min(\beta_s, \beta_c)$, then μ_{unif} does not minimize \mathcal{E}_β : there exists $\eta_\beta > 0$ and a smooth probability density μ_{min} such that

$$\frac{1}{\beta} \int_{\mathbb{T}^d} \log(\mu_{\text{min}}) d\mu_{\text{min}} - \frac{1}{2} \int_{(\mathbb{T}^d)^2} g(x-y) d\mu_{\text{min}}^{\otimes 2}(x,y) \leq -\eta_\beta.$$

Setting $f_N := \mu_{\text{min}}^{\otimes N}$, exchangeability and $\int_{\mathbb{T}^d} g = 0$ imply

$$\mathbb{E}_{f_N}(F_N(X_N, \mu_{\text{unif}})) = \frac{(N-1)}{2N} \int_{(\mathbb{T}^d)^2} g(x-y) d\mu_{\text{min}}^{\otimes 2}(x,y).$$

Since

$$H_N(f_N | \mu_{\text{unif}}^{\otimes N}) = \frac{1}{N} \int_{(\mathbb{T}^d)^N} \log(\mu_{\text{min}}^{\otimes N}) d\mu_{\text{min}}^{\otimes N} = \int_{\mathbb{T}^d} \log(\mu_{\text{min}}) d\mu_{\text{min}},$$

now follows that

$$\frac{1}{\beta} H_N(f_N | \mu_{\text{unif}}^{\otimes N}) = \mathbb{E}_{f_N}(F_N(X_N, \mu_{\text{unif}})) + \frac{\|\mu_{\text{min}}\|_{H^{-\frac{d}{2}}}^2}{2N} - \eta_\beta.$$

Choosing $N = N(\beta)$ sufficiently large so that $\frac{\|\mu_{\text{min}}\|_{H^{-d/2}}^2}{2N} \leq \frac{\eta_\beta}{2}$, conclude

$$\frac{1}{\beta} H_N(f_N | \mu_{\text{unif}}^{\otimes N}) \leq \mathbb{E}_{f_N}(F_N(X_N, \mu_{\text{unif}})) - \frac{\eta_\beta}{2}.$$

Uniform-in-time propagation of chaos

Motivation I

Beyond aesthetics, problem of uniform-in-time propagation of chaos motivates proving the mLHLS. Remember that the truncation in [Bresch-Jabin-Wang 2020](#) otherwise be an issue.

A number of works on uniform-in-time mean-field convergence/propagation of chaos [Malrieu 2003](#), [Cattiaux-Guillin-Malrieu 2008](#), [Salem 2018](#), [Durmus-Eberle-Guillin-Zimmer 2020](#), [Arnaudon-Del Moral 2020](#), [Delarue-Tse 2021](#), [Guillin-Le Bris-Monmarche 2021](#), [R.-Serfaty 2021](#), [Guillin-Le Bris-Monmarche 2022](#), [R.-Serfaty 2021](#), [Delgadino-Gvalani-Pavliotis-Smith 2023](#), [Chodron de Courcel-R.-Serfaty 2023](#)...

But these all concern nonsingular interactions or singular, repulsive interactions. Not aware of any work on singular, attractive interactions.

Main result in a nutshell I

Once we have the mLHLS and our relaxation estimates, a uniform-in-time bound for MFE is straightforward.

- ▶ Follow strategy of [Chodron de Courcel-R.-Serfaty 2023](#) (perspective advanced first in [R.-Serfaty 2021](#)).
- ▶ By coercivity of MFE, this implies a uniform bound for the relative entropy $H_N(f_N^t | (\mu^t)^{\otimes N})$.

On the other hand, our quantitative instability result for the uniform distribution implies that if $\beta > \beta_s$, not possible to have a uniform relative entropy estimate

$$\forall t \geq 0, \quad H_N(f_N^t | (\mu^t)^{\otimes N}) \leq C \left(o_N(1) + H_N(f_N^0 | (\mu^0)^{\otimes N}) \right).$$

Remark: If $d = 2$ and $\mu^0 = \mu_{\text{unif}}$, then have a uniform-in-time bound for all $\beta < \beta_c = 4$. Unsatisfactory, though, because this corresponds to the system being prepared in the macroscopic equilibrium state (which we expect is close to the microscopic equilibrium).

Dissipation identity

Recall the MFE dissipation identity

$$\begin{aligned} \frac{d}{dt} E_N(f_N^t, \mu^t) &\leq -\frac{1}{\beta^2} I_N(f_N^t | \mathbb{Q}_{N,\beta}(\mu^t)) \\ &+ \frac{1}{2} \mathbb{E}_{f_N^t} \left[\int_{(\mathbb{R}^d)^2 \setminus \Delta} (u^t(x) - u^t(y)) \cdot \nabla g(x - y) d \left(\frac{1}{N} \sum_{i=1}^N \delta_{x_i} - \mu^t \right)^{\otimes 2} (x, y) \right]. \end{aligned}$$

where $u^t := \frac{1}{\beta} \nabla \log \mu^t - \nabla g * \mu^t$.

1. The first term on the RHS is ≤ 0 and may be discarded.
2. Expressions of the form of the second term appear naturally when computing variations of the ME along a transport field.
 - ▶ Called “commutator terms” because they have the structure of a quadratic form associated to a commutator.
 - ▶ Functional inequalities developed to show they are bounded by the modulated energy for general log/Riesz potentials [Leblé-Serfaty 2018](#), [Serfaty 2020](#), [Bresch-Jabin-Wang 2019](#), [R. 2020-2023](#), [Nguyen-R.-Serfaty 2021](#), [R.-Serfaty 2022](#)

Commutator-type estimate

For log-type potentials, even easier because MVT implies

$$(u^t(x) - u^t(y)) \cdot \nabla g(x - y) \in L_{x,y}^\infty((\mathbb{T}^d)^2),$$

if u^t is Lipschitz. Can use a cruder argument combining Donsker-Varadhan + LD type arguments originating in [Jabin-Wang 2018](#). Ultimately, show that commutator term is bounded by

$$C^t \left(H_N(f_N^t | (\mu^t)^{\otimes N}) + \frac{C^t}{N} \right),$$

for some constant $C^t > 0$ depending on $\|\log \mu^\tau\|_{W^{2,\infty}}$.

To close the Grönwall loop, one applies mLHLS to bound relative entropy in terms of the MFE E_N . Since $\|\log \mu^\tau\|_{W^{2,\infty}} = O(e^{-ct})$, all time-dependent factors in the Grönwall relation are uniformly bounded.

Remark: The dependence on $\|\log \mu^\tau\|_{W^{2,\infty}}$ motivates the restriction to \mathbb{T}^d on a technical level. Can satisfy by assuming $\mu^\circ \geq c > 0$ (not possible on \mathbb{R}^d).

Counterexample

Our instability result for μ_{unif} when $\beta > \beta_s$ means that given $\epsilon > 0$ small, there is an $O(\epsilon)$ perturbation μ_ϵ° of μ_{unif} such that for $t_\epsilon = O(\log(1/\epsilon))$, we have $\|\mu_\epsilon^{t_\epsilon} - \mu_{\text{unif}}\|_{L^1} \geq \frac{1}{2}$.

Taking f_N^t to be an entropy solution to forward Kolmogorov with initial data $(\mu_\epsilon^\circ)^{\otimes N}$, triangle inequality implies that

$$\|f_{N;1}^{t_\epsilon} - \mu_\epsilon^{t_\epsilon}\|_{L^1} + \|f_{N;1}^{t_\epsilon} - \mu_{\text{unif}}\|_{L^1} \geq \frac{1}{2}.$$

Since $H_N(f_N^\circ | (\mu_\epsilon^\circ)^{\otimes N}) = 0$ and $H_N(f_N^\circ | \mu_{\text{unif}}^{\otimes N}) = O(\epsilon^2)$, conclude from subadditivity and Pinsker's inequality that preceding LHS cannot be bounded solely in terms of N , $H_N(f_N^\circ | \mu_{\text{unif}}^{\otimes N})$.

Main result: uniform-in-time propagation of chaos

Theorem (Uniform PoC)

Let $d \geq 1$ and $\beta < \beta_i$. There exists $\delta_\beta > 0$, depending on d, β and tending to ∞ as $\beta \rightarrow 0$, and functions $\mathbf{C}_1, \mathbf{C}_2 : [0, \infty)^2 \rightarrow [0, \infty)$, continuous, increasing, and vanishing if any of them are zero, and constant $\gamma > 0$, depending on d , such that the following holds. Let $\mu \in L^\infty([0, \infty), W^{2, \infty})$ be a mean-field solution with $\|\log \mu^0\|_{L^\infty} \leq \delta_\beta$. Let $f_N \in L^\infty([0, \infty), \mathcal{P}_{ac})$ be an entropy solution to the forward Kolmogorov equation. Then $\forall t \geq 0$,

$$H_N(f_N^t | (\mu^t)^{\otimes N}) \leq \mathbf{C}_1(\beta, \|\log \mu^0\|_{W^{2, \infty}}) \left(E_N(f_N^0, \mu^0) + \mathbf{C}_2(\beta, \|\log \mu^0\|_{W^{2, \infty}}) N^{-\gamma} \right).$$

If $\beta > \beta_s$, then for all $\epsilon > 0$ sufficiently small depending on d, β , there exists a C^∞ solution μ_ϵ to the mean-field equation with $\|\mu_\epsilon^0 - \mu_{unif}\|_{W^{n, \infty}} = O(\epsilon)$, for any n , such that for time $t_\epsilon = O(\log \frac{1}{\epsilon})$,

$$\|f_{N;1}^{t_\epsilon} - \mu_{unif}\|_{L^1} + \|f_{N;1}^{t_\epsilon} - \mu_\epsilon^{t_\epsilon}\|_{L^1} \geq \frac{1}{2},$$

where f_N is an entropy solution of the forward Kolmogorov equation with initial datum $(\mu_\epsilon^0)^{\otimes N}$.

Summary and outlook

Comparison of thresholds I

Let us explain the ordering between the various $\beta_{(\cdot)}$ parameters introduced in this talk.

- ▶ $\beta_c = 2d$ is the critical inverse temperature corresponding to boundedness from below of the macroscopic free energy \mathcal{E}_β .
- ▶ $\beta_s = \frac{(2\pi)^d}{c_d} = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ is the threshold for stability of the uniform distribution μ_{unif} .
- ▶ β_u is the threshold for uniqueness of solutions of the KM equation.
- ▶ β_i is the maximal value such that for $\beta < \beta_i$, the mLHLS inequality holds for sufficiently small $\log \mu$ and sufficiently large N .

For $d = 1$, we have the ordering

$$0 < \beta_i ? \beta_u \leq \beta_c = \beta_s < \infty.$$

For dimensions $2 \leq d \leq 10$,

$$0 < \beta_i ? \beta_u \leq \beta_c < \beta_s < \infty.$$

While for dimensions $d > 10$,

$$0 < \beta_i ? \beta_u \leq \beta_s < \beta_c < \infty.$$

For the special case $d = 2$, we can be explicit and sharp with respect to all the thresholds, except β_i .

β_i	β_u	β_c	β_s
?	4	4	2π

Comparison of thresholds II

- ▶ For $\beta < \beta_c$, we have global existence of all solutions and convergence to a steady state.
- ▶ If $\beta < \beta_u$, then this steady state is uniform and convergence at an exponential rate holds to arbitrary regularity.
- ▶ If $\beta < \beta_s$ and the initial data is sufficiently close to μ_{unif} , we have global existence and exponential trend to μ_{unif} . In particular, for $2 \leq d \leq 10$, this implies global existence of classical solutions for $\beta \in (\beta_c, \beta_s)$, in contrast to the behavior on \mathbb{R}^d , where all classical solutions with finite second moment blow up in finite time!
- ▶ If $\beta < \beta_i$, then the mLHLS holds for all reference measures μ such that $\|\log \mu\|_{L^\infty}$ is not too large and we have uniform-in-time propagation of chaos.

Remaining questions

- ▶ *Going beyond $\beta < \beta_i$?* While β_i is not explicit, we know that it is not sharp in general. Improving the range would require us to better understand uniqueness for the Kazhdan-Warner equation.
- ▶ *Going beyond β_c ?* Entropy methods will likely not work in this regime and would at best only expect a local-in-time result. Some hope for compactness-based methods, at least for $d = 1$, since we have uniqueness of weak solutions.
- ▶ *Nonlinear stability of nonuniform stationary states?*
- ▶ *On \mathbb{R}^d ?* One has difficulty of decay at infinity (estimates for $\log \mu$), but we now know how to handle this (at least in repulsive case). On the other hand, one gains rotational symmetry, which simplifies questions like uniqueness of stationary states.
- ▶ *Sharp rate of convergence?* Our bound for E_N in terms of N is almost certainly not sharp. Would expect $1/N$ to be sharp. Though, this would still imply a suboptimal bound for propagation of chaos (cf. [Lacker 2023](#), [Lacker-Le Flem 2023](#), [Bresch-Jabin-Soler 2022](#), [HessChilds-Rowan 2023](#)).
- ▶ *Generation of chaos?* Showing this would require exploiting the modulated Fisher information (which we discarded) through a LSI for the modulated Gibbs measure $\mathbb{Q}_{N,\beta}(\mu^t)$. This is extremely hard! Only known for the 1D repulsive Riesz case [R.-Serfaty 2023](#), as a consequence of convexity.
- ▶ *Fluctuations, cumulant bounds?* We have a general method for using ME/MFE bounds to deduce Gaussian CLTs [J. Huang-R.-Serfaty](#). But would need sharper MFE bounds...

The End

Thank you for your attention!