ENERGY-LEVEL CROSSING OF A MATRIX SCHRÖDINGER OPERATOR AND SEMICLASSIAL RESONANCES

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Abstract. This short report is written for the mini-course in the conference *Spectral Analysis for Quantum Hamitonians* held on January 15-19, 2024 at CIRM, France, dedicated to the memory of **Georgi Raykov**. This report is a partial review of the joint works [5] and [6] with M. Assal (Santiago) and K. Higuchi (Ehime). These are continuation of the works [13], [14], [15] which the author talked about in the conference *Spectral Theory and Mathematical Physics* organized by Raikov et al., held in Santiago, Chile at the Pontifical Catholic University of Chile in December 2018.

We consider a model of 2×2 matrix Schrödinger operator $\mathscr{P}$ in 1D;

$$\mathscr{P} = \begin{pmatrix} P_1 & hW \\ hW^* & P_2 \end{pmatrix}$$

where $h > 0$ is a small parameter, $P_j = -h^2 \frac{d^2}{dx^2} + V_j(x)$, $j = 1, 2$ are scalar Schrödinger operators, and W, W^* are a first order differential operator and its adjoint. Assume that, near a fixed energy $E_0 \in \mathbb{R}$, the classical trajectory $\Gamma_j = p_j^{-1}(E_0)$ defined on the phase space by the underlying classical Hamiltonian $p_j(x, \xi) = \xi^2 + V_j(x)$ is periodic for p_1 and non-trapping for p_2. It is then expected that $\mathscr{P}$ has resonances close to the eigenvalues of P_1. The imaginary part of resonances (called width) represents the reciprocal of the life span of the system. It has been studied that, the width is exponentially small in h when the above two trajectories are disjoint, whereas it is of polynomial order if they cross each other.

The mini-course is concerned with the latter case with a finite number of crossing points $\Gamma_1 \cap \Gamma_2$. The order in h of the resonance width is governed by the microlocal structure of solutions at each crossing point, or in other words, the 'microlocal scattering matrix', which describes the 'probability amplitude' for particles to switch the trajectory from one to the other.

In the first part, we see how the global problem of resonances is reduced to a microlocal problem at crossing points, and in the second part, we focus on the study the asymptotic formula of the microlocal scattering matrix.

Contents

1. Introduction

Let P be the 1D scalar semiclassical Schrödinger operator

$$P := (hD_x)^2 + V(x), \quad D_x := -i\frac{d}{dx},$$

with a smooth potential $V(x) \in C^\infty(\mathbb{R})$. We assume that it has limits as $x \to \pm\infty$:

$$V(x) \to V_\pm \quad \text{as } x \to \pm\infty.$$

We denote by $p(x,\xi)$ the underlying classical Hamiltonian corresponding to the Schrödinger operator P:

$$p(x,\xi) = \xi^2 + V(x),$$

where the variable ξ stands for the momentum. The classical motion is described by the Hamitonian vector field

$$H_p = \frac{\partial p}{\partial \xi}\frac{\partial}{\partial x} - \frac{\partial p}{\partial x}\frac{\partial}{\partial \xi} = 2\xi\frac{\partial}{\partial x} - V'(x)\frac{\partial}{\partial \xi}.$$

The energy $p(x(t),\xi(t))$ is invariant in time t on the integral curve $(x(t),\xi(t)) = \exp tH_p(x_0,\xi_0)$ of H_p (we call it *classical trajectory*). In our 1D setting, the classical trajectory coincides with the characteristic set

$$\Gamma(E) := p^{-1}(E) = \{(x,\xi); p(x,\xi) = E\}$$

We fix a real energy $E_0 \in \mathbb{R}$ and consider the following two types of potentials.

Simple well potential. We say, in this mini-course, V is a *simple well potential at energy* E_0 if $V_+, V_- > E_0$ and there exist $a_0 < b_0$ such that

$$\frac{V(x) - E_0}{(x-a_0)(x-b_0)} > 0 \quad \forall x \in \mathbb{R}.$$

Remark that a_0 and b_0 are simple zeros of $V(x) - E_0$, called simple turning points. The classical trajectory for a simple-well potential is a simple periodic curve.

The spectrum of P near E_0 consists of (simple) eigenvalues. In the semiclassical limit $h \to +0$, they are approximated by the energies satisfying the so-called *Bohr-Sommerfeld quantization rule* (see [25], [36], [39]):

$$-e^{i\mathcal{A}(E)/h} = 1. \tag{1.1}$$

Here $\mathcal{A}(E)$ is the action integral

$$\mathcal{A}(E) := \int_{\Gamma(E)} \xi dx.$$

It is a smooth function of E with $\mathcal{A}'(E_0) > 0$. The condition (1.1) can be rewritten as

$$\mathcal{A}(E) = (2k+1)\pi h, \quad k \in \mathbb{Z},$$

and hence the eigenvalues near E_0 is a sequence with interval $\sim 2\pi h/\mathcal{A}'(E_0)$.

Non-trapping potential. We say (only in this mini-course) that V is a *non-trapping potential at energy* E_0 if $V_- < E_0 < V_+$ and there exists $c_0 \in \mathbb{R}$ such that

$$\frac{V(x) - E_0}{x - c_0} > 0 \quad \forall x \in \mathbb{R}.$$

Remark that c_0 is also a simple zero of $V(x) - E_0$. The classical trajectory for a non-trapping potential is an unbounded curve, whose real part comes from $-\infty$ and goes back to $-\infty$.

The spectrum of P near E_0 consists of continuous spectrum. It is also known that there is no resonance in a small complex neighborhood of E_0 for h sufficiently small with imaginary part of order 1 in the analytic potential case ([18]) and of order $h|\log h|$ in the C^∞ case ([34]).

Matrix-valued potential. Now we consider a matrix Schrödinger operator

$$\mathscr{P} = \begin{pmatrix} P_1 & hW \\ hW^* & P_2 \end{pmatrix},$$

with two scalar Schrödinger operators P_1 and P_2 on the diagonal. Such an operator appears as the Born-Oppenheimer approximation of the molecule predissociation (see [26], [27], [28]). We suppose here that P_1 has a simple-well potential and P_2 a non-trapping potential at a fixed energy E_0. In this mini-course, we suppose for simplicity that the interaction W is just a multiplication by a real-valued smooth function which is analytic in $\mathcal{S}$

$$W = W^* = r(x),$$

although it should be a first order differential operator in the application to the molecular predissociation.

We assume that $V_1(x)$, $V_2(x)$ and $r(x)$ are all analytic in an angular domain $\mathscr{S} := \{x \in \mathbb{C}; |\mathrm{Re}\,x| \geq R, |\mathrm{Im}\,x| < c|\mathrm{Re}\,x|\}$ for some $R, c > 0$. Moreover, $V_1(x)$, $V_2(x)$ have limits as $\mathrm{Re}\,x \to \pm\infty$ in this domain:

$$V_j(x) \to V_j^{\pm} \quad \text{as } \mathrm{Re}\,x \to \pm\infty \text{ in } \mathscr{S},$$

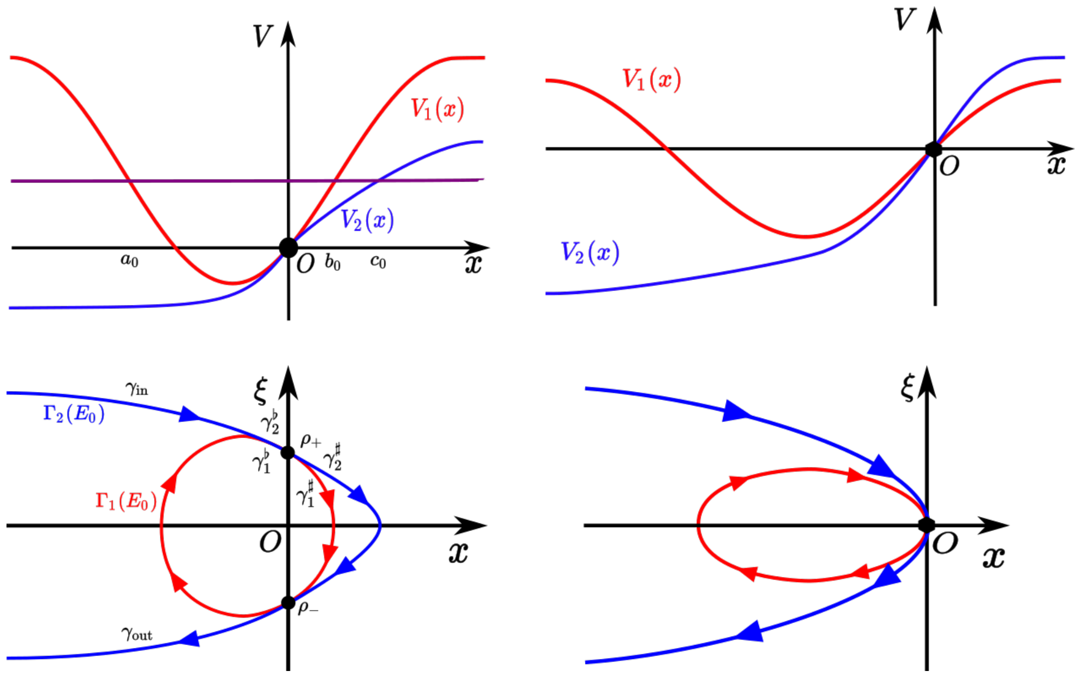
and $r(x)$ is bounded there.

The operator $\mathscr{P}$ is self-adjoint in $L^2(\mathbb{R};\mathbb{C}^2)$ with domain $H^2(\mathbb{R};\mathbb{C}^2)$. One can define the resonances of $\mathscr{P}$ in a sufficiently small complex neighborhood of E_0, as the values $E \in \mathbb{C}$ such that the equation $\mathscr{P}w = Ew$ has a non-trivial outgoing solution w. In our setting with small enough h, this means that $w(x)$ is decaying as $x \to +\infty$ and $w(te^{i\theta})$ for small $\theta > 0$ is decaying as $t \to -\infty$. In particular, the imaginary part of each resonance is negative. We denote $\mathrm{Res}(\mathscr{P})$ the set of resonances of $\mathscr{P}$.

In the trivial case where $W = r(x) \equiv 0$, the spectrum of $\mathscr{P}$ near E_0 is of course the union of the spectrum of P_1 and P_2, i.e. the continuous spectrum with embedded eigenvalues.

In the general case, it is expected that the eigenvalues shift in the lower complex plane as resonances (Fermi's golden rule). The imaginary part of these resonances (resonance width), interpreted as the reciprocal of the life time of the quantum particles, should be closely related with the interaction between the two classical dynamics of P_1 and P_2. Intuitively, particles trapped in the periodic trajectory Γ_1 may change the trajectory to Γ_2 thanks to the interaction and escape to the infinity. We naturally expect that the 'probability' of this change of trajectory from Γ_1 to Γ_2 should determine the resonance width, and higher the probability, wider the resonance width.

In the case where the two potentials have a gap $\inf_{x\in\mathbb{R}} |V_1(x) - V_2(x)| > 0$, then the two trajectories $\Gamma_1(E_0)$ and $\Gamma_2(E_0)$ are disjoint. Even in this case, an exponentially small in h interaction exists (called *phase space tunneling*), and it has been shown that the resonance width is exponentially small (e.g.[7], [16], [17], [32], [35]).



2. Our problem and results

We are interested in the case where

$$\Gamma_1(E_0) \cap \Gamma_2(E_0) \neq \emptyset. \tag{2.1}$$

This implies that the two potentials V_1 and V_2 cross at least at one point. We assume without loss of generality that this is the origin $x = 0$. We suppose moreover that this is a finite order contact: there exists $n \in \mathbb{N}$ such that

$$V_1^{(k)}(0) - V_2^{(k)}(0) = 0 \quad (0 \leq k \leq n-1), \qquad V_1^{(n)}(0) - V_2^{(n)}(0) \neq 0. \tag{2.2}$$

Remark that the crossing of the potentials (2.2) does not imply the intersection of the classical trajectories (2.1) depending on the energy E_0. In fact, if E_0 is below the crossing level 0, then $\Gamma_1(E_0)$ and $\Gamma_2(E_0)$ are disjoint. Ashida [3] gave the exponentially small asymptotics of the resonance width in such a case. The condition (2.1) implies that E_0 is above the crossing level ($E_0 > 0$), or at the crossing level ($E_0 = 0$).

Above the crossing level, $\Gamma_1(E_0)$ and $\Gamma_2(E_0)$ intersect at two points $(0, \sqrt{E_0})$ and $(0, -\sqrt{E_0})$ and the contact order m at these points are both n:

$$H_{p_1}^k p_2(0, \pm\sqrt{E_0}) = 0 \quad (0 \leq k \leq n-1), \quad H_{p_1}^n p_2(0, \pm\sqrt{E_0}) \neq 0. \tag{2.3}$$

On the contrary, at the crossing level, $\Gamma_1(E_0)$ and $\Gamma_2(E_0)$ intersect at one point $(0,0)$ and the contact order m at this point is $2n$:

$$H_{p_1}^k p_2(0,0) = 0 \quad (0 \leq k \leq 2n-1), \quad H_{p_1}^{2n} p_2(0,0) \neq 0. \tag{2.4}$$

The following results describe the asymptotic distribution of resonances in a rectangular complex neighborhood of E_0:

$$\mathcal{R} = \mathcal{R}(\delta_1, \delta_2) = \{E \in \mathbb{C}; \mathrm{Re}\, E \in [E_0 - \delta_1, E_0 + \delta_1], -\mathrm{Im}\, E \in [0, \delta_2]\}.$$

Let $\mathfrak{B}_h$ be the discrete set of approximated eigenvalues of P_1 in $[E_0-\delta_1, E_0+\delta_1]$:

$$\mathfrak{B}_h = \mathfrak{B}_h(\delta_1) := \{E \in [E_0-\delta_1, E_0+\delta_1];\ E \text{ satisfies } (1.1)\}. \tag{2.5}$$

In what follows, the scales δ_1, δ_2 are

$$\delta_1 = \begin{cases} \delta, & \text{when } E_0 > 0, \\ Lh^{\frac{2}{2n+1}}, & \text{when } E_0 = 0, \end{cases} \qquad \delta_2 = Lh$$

for a small enough h-independent $\delta > 0$ and an arbitrary large fixed constant L.

Suppose that V_1 and V_2 cross only at $x=0$ below the level E_0:

$$\{x \in \mathbb{R}; V_1(x) = V_2(x) \le E_0\} = \{0\},$$

and that the interaction does not vanish at $x=0$: $r(0) \neq 0$. Then we have the following theorem (see [31] for the case where $r(0)=0$).

Theorem 1. *There exists a bijective map for sufficiently small h*

$$z_h : \mathfrak{B}_h \to \mathrm{Res}(P) \cap \mathcal{R}$$

and a non-negative smooth function $\mathcal{D}(E) = \mathcal{D}(E;h)$ of E near E_0 uniformly bounded with respect to $h>0$ small enough (explicitly given in Remark 2.1) such that, for any $E \in \mathfrak{B}_h$,

$$|z_h(E) - E| = \mathcal{O}(h^{\frac{m+3}{m+1}}), \tag{2.6}$$

$$\operatorname{Im} z_h(E) = -\mathcal{D}(E) h^{\frac{m+3}{m+1}} + \mathcal{O}(h^{\frac{m+4-\epsilon}{m+1}}), \tag{2.7}$$

uniformly as $h \to 0^+$, for some $0 \le \epsilon < 1$.

Remark 2.1. *The coefficient $\mathcal{D}(E)$ is given by the following formulas. In the case $E_0 > 0$,*

$$\mathcal{D}(E) = 2\Gamma\left(\frac{m+2}{m+1}\right)^2 \left(\frac{2(m+1)!}{|v_n|}\right)^{\frac{2}{m+1}} \frac{E^{-\frac{m}{m+1}}}{\mathcal{A}'(E)} r(0)^2 R^2 \sin^2\left(\frac{\mathcal{S}(E)}{2h} + \theta\right), \tag{2.8}$$

where $v_n = V_2^{(n)}(0) - V_1^{(n)}(0)$, $R=1$, $\theta = \frac{\pi}{2(m+1)}\operatorname{sgn} v_n$ when m is odd, and $R = \cos\frac{\pi}{2(m+1)}$, $\theta = 0$ when m is even. $\mathcal{S}(E)$ is the action of the directed cycle γ:

$$\mathcal{S}(E) := \int_\gamma \xi dx = 2\int_{a(E)}^0 \sqrt{E - V_1(x)}\, dx + 2\int_0^{c(E)} \sqrt{E - V_2(x)}\, dx. \tag{2.9}$$

In the case $E_0 = 0$, one has with $m = 2n$,

$$\mathcal{D}(E) = \frac{4\pi^2}{v_0 q_n^{\frac{2}{2n+1}} \mathcal{A}'(E)} r(0)^2 A_n\left(-\frac{q_n^{\frac{2}{2n+1}}}{v_0}\eta\right)^2, \quad \eta = \frac{E}{h^{\frac{2}{2n+1}}}, \tag{2.10}$$

where A_n is a generalization of the Airy function ($A_1 = \mathrm{Ai}$)

$$A_n(\lambda) = \frac{1}{2\pi}\int_{\mathbb{R}} \exp\left(i\int_0^\eta (\lambda + \tau^2)^n d\tau\right) d\eta, \tag{2.11}$$

and $v_0 := \sqrt{V_1'(0)V_2'(0)} > 0$, $q_n = |v_n|/(n!\sqrt{v_0})$.

3. Microlocal scattering matrix

We now introduce a notion of microlocal solution. Let (x_0, ξ_0) be a point in the phase space. We say that $w \in L^2(\mathbb{R}; \mathbb{C}^2)$ is microlocally zero and write

$$w \equiv 0 \text{ at } (x_0, \xi_0),$$

if there exists cutoff functions $\chi_1(x) \in C_0^\infty(\mathbb{R})$ with $\chi_1(x_0) \neq 0$ and $\chi_2(\xi) \in C_0^\infty(\mathbb{R})$ with $\chi_2(\xi_0) \neq 0$ such that

$$\chi_2(\xi)\mathcal{F}_h(\chi_1(x)w) = \mathcal{O}(h^\infty).$$

Here $\mathcal{F}_h$ is the semiclassical Fourier transform

$$(\mathcal{F}_h w)(\xi) := \frac{1}{\sqrt{2\pi h}} \int_R e^{-ix\xi/h} w(x) dx.$$

We also say that w is a microlocal solution to the system $(\mathscr{P} - E)w = 0$ at (x_0, ξ_0) if $(\mathscr{P} - E)w \equiv 0$ at (x_0, ξ_0). It is well-known in the theory of semiclassical and microlocal analysis (see for example [12], [33], [40]) that any locally normalized solution w to $(\mathscr{P}-E)w = 0$ is microlocally supported on $\Gamma_1(E) \cup \Gamma_2(E)$, and that if w is microlocally zero at a point on a connected component γ of $\Gamma_1(E) \cup \Gamma_2(E) \setminus \Gamma_1(E) \cap \Gamma_2(E)$, then it is microlocally zero at all the points on γ (propagation of singularities). In our 1D setting, in particular, the space of microlocal solutions on γ is of dimension 1.

Let $\rho \in \mathbb{R}_x \times \mathbb{R}_\xi$ be a crossing point $\rho \in \Gamma_1(E_0) \cap \Gamma_2(E_0)$, and Ω a small neighborhood of ρ. In the setting of the previous section, $\rho = \rho_+ := (0, \sqrt{E_0})$ or $\rho_- := (0, -\sqrt{E_0})$ when $E_0 > 0$ and $\rho = \rho_0 := (0, 0)$ when $E_0 = 0$. But here we only assume the local crossing condition (2.2).

For each $j = 1, 2$, let $\gamma_j^\flat, \gamma_j^\sharp$ be respectively the incoming and outgoing component of $(\Gamma_j(E_0) \cap \Omega) \setminus \{\rho\}$. We can formally construct WKB solutions $f_1^\flat$, $f_1^\sharp$, $f_2^\flat$, $f_2^\sharp$ to the system $\mathscr{P}w = Ew$ in the form

$$f_j^\bullet = e^{i\phi_j^\bullet(x)/h} \sigma_j^\bullet(x, h). \tag{3.1}$$

The phase function $\phi_j^\bullet(x)$ is given by

$$\phi_j^\bullet(x) = \pm \int_0^x \sqrt{E - V_j(t)} dt,$$

where the sign $\pm$ for the phase function $\phi_j^\bullet$ is $+$ [resp. $-$] if $\gamma_j^\bullet$ is in the upper $(\xi > 0)$ [resp. lower $(\xi < 0)$] half plane of the phase space. The symbol $\sigma_j^\bullet(x, h)$ is of the form

$$\sigma_1^\bullet(x, h) = \begin{pmatrix} \sigma_{1,1}^\bullet(x, h) \\ h\sigma_{1,2}^\bullet(x, h) \end{pmatrix}, \quad \sigma_2^\bullet(x, h) = \begin{pmatrix} h\sigma_{2,1}^\bullet(x, h) \\ \sigma_{2,2}^\bullet(x, h) \end{pmatrix},$$

and each coefficient has a formal power series expansion in h

$$\sigma_{j,k}^\bullet(x, h) = \sum_{l=0}^\infty \sigma_{j,k,l}^\bullet(x) h^l. \tag{3.2}$$

In particular, the leading terms are, with the notation $\hat{j} := 3 - j$,

$$\sigma_{j,j,0}^\bullet(x) = \frac{1}{(E - V_j(x))^{\frac{1}{4}}}, \quad \sigma_{j,\hat{j},0}^\bullet(x) = \frac{r(x)}{(V_{\hat{j}}(x) - V_j(x))(E - V_j(x))^{\frac{1}{4}}}.$$

Let us take a resummation of the infinite series (3.2). Then the functions $f_j^\bullet$ defined by (3.1) are regarded as microlocal solutions on $\gamma_j^\bullet$:

$$(\mathscr{P}-E)f_j^\bullet \equiv 0 \quad \text{on } \gamma_j^\bullet, \quad j=1,2, \quad \bullet=\flat,\sharp, \tag{3.3}$$

Moreover, $f_j^\bullet$ generates the one-dimensional vector space of the microlocal solutions on $\gamma_j^\bullet$.

Theorem 2. *For $E\in\mathcal{R}$ and h small, there exists a 2×2 matrix $T=T(E,h)$ such that if a locally normalized vector-valued function w satisfies*

$$(\mathscr{P}-E)w \equiv 0 \qquad \text{in } \Omega \tag{3.4}$$

$$w \equiv \alpha_j^\bullet f_j^\bullet \quad \text{on } \gamma_j^\bullet \quad (j=1,2,\ \bullet=\flat,\sharp), \tag{3.5}$$

then we have

$$\begin{pmatrix}\alpha_1^\sharp\\ \alpha_2^\sharp\end{pmatrix} = T(E,h)\begin{pmatrix}\alpha_1^\flat\\ \alpha_2^\flat\end{pmatrix}. \tag{3.6}$$

Moreover, T satisfies the asymptotic formulas

$$T = \mathrm{Id} - ih^{\frac{1}{m+1}}\begin{pmatrix}0 & \overline{\omega}\\ \omega & 0\end{pmatrix} + \mathcal{O}(h^{\frac{2-\epsilon}{m+1}}), \quad \text{when } E_0>0 \text{ and } \rho=\rho_+, \tag{3.7}$$

$$T = \mathrm{Id} - ih^{\frac{1}{m+1}}\begin{pmatrix}0 & \omega\\ \overline{\omega} & 0\end{pmatrix} + \mathcal{O}(h^{\frac{2-\epsilon}{m+1}}), \quad \text{when } E_0>0 \text{ and } \rho=\rho_-, \tag{3.8}$$

$$iT = \mathrm{Id} - ih^{\frac{1}{m+1}}\begin{pmatrix}0 & \kappa\\ \kappa & 0\end{pmatrix} + \mathcal{O}(h^{\frac{2-\epsilon}{m+1}}), \quad \text{when } E_0=0 \text{ and } \rho=\rho_0, \tag{3.9}$$

where $\omega\in\mathbb{C}$ is given by

$$\omega = \mu_m\left(\frac{2(m+1)!}{|v_1-v_2|}\right)^{\frac{1}{m+1}} E^{-\frac{m}{2(m+1)}}\Gamma\left(\frac{m+2}{m+1}\right) r(0), \tag{3.10}$$

$$\mu_m = \begin{cases} e^{\frac{i\pi}{2(m+1)}\operatorname{sgn}(v_2-v_1)} & \text{when } m \text{ is odd,}\\ \cos\left(\dfrac{\pi}{2(m+1)}\right) & \text{when } m \text{ is even.}\end{cases} \tag{3.11}$$

and $\kappa\in\mathbb{R}$ is given by

$$\kappa(\eta) = \frac{2\pi r(0)}{\sqrt{v_0}} q_n^{-\frac{1}{2n+1}} A_n\left(-\frac{q_n^{\frac{2}{2n+1}}}{v_0}\eta\right), \quad \eta = \frac{E}{h^{\frac{2}{2n+1}}}. \tag{3.12}$$

Remark 3.1. *In terms of ω and κ, the coefficient $\mathcal{D}(E)$ of the resonance width in Theorem 1 are expressed as*

$$\mathcal{D}(E) = \frac{2|\omega|^2}{\mathcal{A}'(E)}\sin^2\left(\frac{\mathcal{S}(E)}{2h}+\arg\omega\right) \tag{3.13}$$

in the case $E_0>0$ and

$$\mathcal{D}(E) = \frac{\kappa^2}{\mathcal{A}'(E)} \tag{3.14}$$

in the case $E_0=0$.

4. Local exact solutions and microlocal scattering matrix

Let ρ be a crossing point in the phase space. Recall that $\rho=(0,\sqrt{E_0})$ or $(0,-\sqrt{E_0})$ when $E_0>0$ and $\rho=(0,0)$ when $E_0=0$. In order to compute the microlocal scattering matrix T at ρ, we construct local exact solutions to the system $(\mathscr{P}-E)w=0$ in a neighborhood I of $x=0$.

We study this in this course only in the case $E_0>0$. It is known that there exist exact solutions $u_j^\pm$ to the scalar Schrödinger equations $(P_j-E)u=0$ for each $j=1,2$ characterized by the semiclassical behavior in I, which is in the classically allowed region in the case $E_0>0$:

$$u_j^\pm \sim (E-V_j(x))^{-\frac{1}{4}}\exp\left(\pm\frac{i}{h}\int_0^x\sqrt{E-V_j(t)}dt\right),$$

and

$$\mathcal{W}(u_j^+,u_j^-)=u_j^+(u_j^-)'-u_j^-(u_j^+)'=-\frac{2i}{h}.$$

Let K_j^L and K_j^R be the operators acting on functions f in $C_0^\infty(\mathbb{R})$ defined by

$$K_j^Lf:=\frac{1}{2ih}\int_{-\infty}^x\Big(u_j^+(x)u_j^-(y)-u_j^-(x)u_j^+(y)\Big)f(y)dy, \tag{4.1}$$

$$K_j^Rf:=\frac{1}{2ih}\int_{\infty}^x\Big(u_j^+(x)u_j^-(y)-u_j^-(x)u_j^+(y)\Big)f(y)dy. \tag{4.2}$$

As is well known, they satisfy

$$(P_j-E)K_j^Lf=f,\quad (P_j-E)K_j^Rf=f.$$

Using these operators, we construct exact solutions to the system by successive approximation. Let $\widetilde{K}_j^L$ and $\widetilde{K}_j^R$ be the operators

$$\widetilde{K}_j^L=-hK_j^L\circ r(x),\quad \widetilde{K}_j^R=-hK_j^R\circ r(x).$$

We suppose here that the smooth function $r(x)$ has a compact support in I. Then $\widetilde{K}_j^L$ and $\widetilde{K}_j^R$ are well defined for any smooth function without compact support.

We use the following lemma without proof.

Lemma 4.1. *For $S=L,R$, we have*

$$\left\|\widetilde{K}_1^S\widetilde{K}_2^S\right\|_{\mathcal{B}(C(I))}=\mathcal{O}(h^{\frac{1}{m+1}}),\quad \left\|\widetilde{K}_2^S\widetilde{K}_1^S\right\|_{\mathcal{B}(C(I))}=\mathcal{O}(h^{\frac{1}{m+1}}). \tag{4.3}$$

This lemma implies that the infinite sums

$$J_{12}^S:=\sum_{k=0}^\infty(\widetilde{K}_1^S\widetilde{K}_2^S)^k,\quad J_{21}^S:=\sum_{k=0}^\infty(\widetilde{K}_2^S\widetilde{K}_1^S)^k$$

converge for sufficiently small h in the space $\mathcal{B}(C(I))$ of bounded operators on $C(I)$.

We then define 8 exact solutions: for $S=L,R$,

$$w_{1,\pm}^S:=\begin{pmatrix}J_{12}^Sw_1^\pm\\ \widetilde{K}_2^SJ_{12}^Sw_1^\pm\end{pmatrix}=\begin{pmatrix}w_1^\pm\\ \widetilde{K}_2^Sw_1^\pm\end{pmatrix}+\mathcal{O}(h^{\frac{1}{m+1}}), \tag{4.4}$$

$$w^S_{2,\pm} := \begin{pmatrix} \widetilde{K}^S_1 J^S_{21} w^\pm_2 \\ J^S_{21} w^\pm_2 \end{pmatrix} = \begin{pmatrix} \widetilde{K}^S_1 w^\pm_2 \\ w^\pm_2 \end{pmatrix} + \mathcal{O}(h^{\frac{1}{m+1}}). \tag{4.5}$$

We easily see the asymptotic behavior of $w^L_{j,\pm}$ on the left of the origin $I \cap \{x < 0\}$ and that of $w^R_{j,\pm}$ on the right of the origin $I \cap \{x > 0\}$. Let $\gamma^S_{j,\pm}$ be the eight portions of $\Gamma_j(E_0)$ in $I \times \mathbb{R}_\xi$ divided by crossing points:

$$\gamma^L_{j,\pm} = \Gamma_j(E_0) \cap \{(x,\xi) \in I \times \mathbb{R}; x < 0, \pm\xi > 0\},$$
$$\gamma^R_{j,\pm} = \Gamma_j(E_0) \cap \{(x,\xi) \in I \times \mathbb{R}; x > 0, \pm\xi > 0\}.$$

Lemma 4.2. *Modulo $\mathcal{O}(h)$, $w^L_{1,+}$ microlocally behaves like*

$$w^L_{1,+} \equiv \begin{cases} f^L_{1,+} & \text{on } \gamma^L_{1,+}, \\ 0 & \text{on } \gamma^L_{1,-} \cup \gamma^L_{2,+} \cup \gamma^L_{2,-}. \end{cases} \tag{4.6}$$

Similarly the other solutions $w^L_{j,\pm}$ and $w^R_{j,\pm}$ behave microlocally like

$$w^L_{j,\pm} \equiv \begin{cases} f^L_{j,\pm} & \text{on } \gamma^L_{j,\pm}, \\ 0 & \text{on } \gamma^L_{j,\mp} \cup \gamma^L_{\hat{j},\pm} \cup \gamma^L_{\hat{j},\mp}, \end{cases} \qquad w^R_{j,\pm} \equiv \begin{cases} f^R_{j,\pm} & \text{on } \gamma^R_{j,\pm}, \\ 0 & \text{on } \gamma^R_{j,\mp} \cup \gamma^R_{\hat{j},\pm} \cup \gamma^R_{\hat{j},\mp}. \end{cases} \tag{4.7}$$

Proof. We only prove (4.6). Recall that, modulo $\mathcal{O}(h^{\frac{2}{m+1}})$, we have

$$w^L_{1,+} = {}^t(u^+_1, \widetilde{K}^L_2 u^+_1),$$

and

$$\begin{aligned} \widetilde{K}^L_2 u^+_1(x) &= \frac{i}{2} \int_{-\infty}^x \left(u^+_2(x) u^-_2(y) - u^-_2(x) u^+_2(y)\right) r(y) u^+_1(y) dy \\ &= \frac{i}{2} u^+_2(x) \int_{-\infty}^x u^-_2(y) u^+_1(y) r(y) dy - \frac{i}{2} u^-_2(x) \int_{-\infty}^x u^+_2(y) u^+_1(y) r(y) dy. \end{aligned}$$

The last two integrals are oscillatory integral with integrand

$$u^-_2(y) u^+_1(y) r(y) = \frac{r(y)}{(E - V_1(y))^{\frac14} (E - V_2(y))^{\frac14}} \exp\left(\frac{i}{h} \int_0^y (E - V_1(t))^{\frac12} - (E - V_2(t))^{\frac12} dt\right),$$

$$u^+_2(y) u^+_1(y) r(y) = \frac{r(y)}{(E - V_1(y))^{\frac14} (E - V_2(y))^{\frac14}} \exp\left(\frac{i}{h} \int_0^y (E - V_1(t))^{\frac12} + (E - V_2(t))^{\frac12} dt\right).$$

The first integrand has a critical point at the crossing point $y = 0$ which is outside the integration range, and the second one no critical point. Therefore, the integrals are both $\mathcal{O}(h^\infty)$. This ends the proof. □

The four solutions $(w^L_{1,+}, w^L_{2,+}, w^L_{1,-}, w^L_{2,-})$ as well as the four solutions$(w^R_{1,+}, w^R_{2,+}, w^R_{1,-}, w^R_{2,-})$ make two bases of exact solutions to the system $\mathscr{P} w = 0$ in I. Therefore there exists a constant 4×4 matrix A such that

$$(w^L_{1,+}, w^L_{2,+}, w^L_{1,-}, w^L_{2,-}) = (w^R_{1,+}, w^R_{2,+}, w^R_{1,-}, w^R_{2,-}) A. \tag{4.8}$$

According to the previous lemma, we find the following lemma:

Lemma 4.3. *The microlocal scattering matrix T at the crossing point $(0,\sqrt{E_0})$ is the 2×2 block matrix of A_{11} of*

$$A=\begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}$$

defined by (4.8)

We are now going to compute A at the principal level modulo $\mathcal{O}(h^{\frac{2}{m+1}})$ in the semiclassical limit. We rewrite the definition (4.8) of A in the form

$$\begin{pmatrix} w^L_{1,+} & w^L_{2,+} & w^L_{1,-} & w^L_{2,-}\\ (w^L_{1,+})' & (w^L_{2,+})' & (w^L_{1,-})' & (w^L_{2,-})'\end{pmatrix} = \begin{pmatrix} w^R_{1,+} & w^R_{2,+} & w^R_{1,-} & w^R_{2,-}\\ (w^R_{1,+})' & (w^R_{2,+})' & (w^R_{1,-})' & (w^R_{2,-})'\end{pmatrix} A, \tag{4.9}$$

and look at this identity at $x=0$. Here $'$ stands for the derivative with respect to x. Notice that the both sides are 4×4 matrices. The first column vector on the left hand side is

$${}^t(u_1^+(0), (\widetilde{K}_2^L u_1^+)(0), (u_1^+)'(0), (\widetilde{K}_2^L u_1^+)'(0)).$$

The term $\widetilde{K}_2^L u_1^+(0)$ is expressed in a linear combination of $u_2^+(0)$ and $u_2^-(0)$.

$$\begin{aligned}\widetilde{K}_2^L u_1^+(0) &= \frac{i}{2}\int_{-\infty}^{x}\left(u_2^+(x)u_2^-(y)-u_2^-(x)u_2^+(y)\right)r(y)u_1^+(y)dy|_{x=0}\\ &= \frac{i}{2}\int_{-\infty}^{0}\left(u_2^+(0)u_2^-(y)-u_2^-(0)u_2^+(y)\right)r(y)u_1^+(y)dy\\ &= c^L_{1+}u_2^+(0)+c^L_{1-}u_2^-(0),\end{aligned}$$

with

$$c^L_{1+}=\frac{i}{2}\int_{-\infty}^{0}u_2^-(y)u_1^+(y)r(y)dy,\quad c^L_{1-}=-\frac{i}{2}\int_{-\infty}^{0}u_2^+(y)u_1^+(y)r(y)dy.$$

Similarly, $(\widetilde{K}_2^L u_1^+)'(0)$ is written in linear combination of $(u_2^+)'(0)$ and $(u_2^-)'(0)$:

$$(\widetilde{K}_2^L u_1^+)'(0)=c^L_{1+}(u_2^+)'(0)+c^L_{1-}(u_2^-)'(0).$$

We define a matrix B by

$$B=\begin{pmatrix} u_1^+(0) & 0 & u_1^-(0) & 0\\ 0 & u_2^+(0) & 0 & u_2^-(0)\\ (u_1^+)'(0) & 0 & (u_1^-)'(0) & 0\\ 0 & (u_2^+)'(0) & 0 & (u_2^-)'(0)\end{pmatrix}.$$

Then

$$\begin{pmatrix} w^L_{1,+} & w^L_{2,+} & w^L_{1,-} & w^L_{2,-}\\ (w^L_{1,+})' & (w^L_{2,+})' & (w^L_{1,-})' & (w^L_{2,-})'\end{pmatrix}|_{x=0}=B(I+C^L),$$

$$\begin{pmatrix} w^R_{1,+} & w^R_{2,+} & w^R_{1,-} & w^R_{2,-}\\ (w^R_{1,+})' & (w^R_{2,+})' & (w^R_{1,-})' & (w^R_{2,-})'\end{pmatrix}|_{x=0}=B(I+C^R),$$

where C^S, $S=L,R$ is given by

$$C_S=\begin{pmatrix} 0 & c^S_{2+} & 0 & c^S_{4+}\\ c^S_{1+} & 0 & c^S_{3+} & 0\\ 0 & c^S_{2-} & 0 & c^S_{4-}\\ c^S_{1-} & 0 & c^S_{3-} & 0\end{pmatrix},$$

where

$$c^L_{2+} = \frac{i}{2}\int_{-\infty}^{0} u_1^-(y)u_2^+(y)r(y)dy, \qquad c^L_{2-} = -\frac{i}{2}\int_{-\infty}^{0} u_1^+(y)u_2^+(y)r(y)dy,$$

$$c^L_{3+} = \frac{i}{2}\int_{-\infty}^{0} u_2^-(y)u_1^-(y)r(y)dy, \qquad c^L_{3-} = -\frac{i}{2}\int_{-\infty}^{0} u_2^+(y)u_1^-(y)r(y)dy,$$

$$c^L_{4+} = \frac{i}{2}\int_{-\infty}^{0} u_1^-(y)u_2^-(y)r(y)dy, \qquad c^L_{4-} = -\frac{i}{2}\int_{-\infty}^{0} u_1^+(y)u_2^-(y)r(y)dy,$$

and $c^R_{j\pm}$ is $c^L_{j\pm}$ with the lower endpoint of the integral replaced by $+\infty$.

Remark here that C^L, C^R are of $\mathcal{O}(h^{\frac{1}{m+1}})$. In fact, as in the proof of Lemma 4.2, the entries $c^S_{j\pm}$ are oscillatory integrals possibly with a critical point at the origin. The order of this critical point is the contact order $n = m$ of the two potentials, and hence by the degenerate stationary phase method, we see that it is of order $h^{\frac{1}{m+1}}$.

Then we obtain, for h small enough,

$$\begin{aligned} A &= (I + C^R)^{-1}(I + C^L) \\ &= I + C^L - C^R + \mathcal{O}(h^{\frac{2}{m+1}}) \\ &= I + \begin{pmatrix} 0 & c_{2+} & 0 & c_{4+} \\ c_{1+} & 0 & c_{3+} & 0 \\ 0 & c_{2-} & 0 & c_{4-} \\ c_{1-} & 0 & c_{3-} & 0 \end{pmatrix} + \mathcal{O}(h^{\frac{2}{m+1}}), \end{aligned}$$

where

$$c_{1+} = \frac{i}{2}\int_{-\infty}^{\infty} u_1^+(y)u_2^-(y)r(y)dy, \qquad c_{1-} = -\frac{i}{2}\int_{-\infty}^{\infty} u_1^+(y)u_2^+(y)r(y)dy,$$

$$c_{2+} = \frac{i}{2}\int_{-\infty}^{\infty} u_1^-(y)u_2^+(y)r(y)dy, \qquad c_{2-} = -\frac{i}{2}\int_{-\infty}^{\infty} u_1^+(y)u_2^+(y)r(y)dy,$$

$$c_{3+} = \frac{i}{2}\int_{-\infty}^{\infty} u_1^-(y)u_2^-(y)r(y)dy, \qquad c_{3-} = -\frac{i}{2}\int_{-\infty}^{\infty} u_1^-(y)u_2^+(y)r(y)dy,$$

$$c_{4+} = \frac{i}{2}\int_{-\infty}^{\infty} u_1^-(y)u_2^-(y)r(y)dy, \qquad c_{4-} = -\frac{i}{2}\int_{-\infty}^{\infty} u_1^+(y)u_2^-(y)r(y)dy.$$

The semiclassical asymptotics of these entries is obtained by the stationary phase method again as in the proof of of Lemma 4.2. The aniti-diagonal entries c_{1+}, c_{2+} of the block A_{11}, which is the scattering matrix at the crossing point $(0, \sqrt{E_0})$, and c_{3-}, c_{4-} of the block A_{22} have a degenerate stationary point at the origin, whereas those in the blocks A_{12}, A_{21} have no critical point.

More precisely, we have the following asymptotic formula, which gives Theorem 2.

Lemma 4.4.

(4.10) $$c_{1+} = -i\omega h^{\frac{1}{m+1}} + \mathcal{O}(h^{\frac{2}{m+1}}), \quad c_{2+} = -i\overline{\omega} h^{\frac{1}{m+1}} + \mathcal{O}(h^{\frac{2}{m+1}}),$$

where ω is given in Theorem 2 by (3.10).

5. From microlocal scattering matrix to resonances

Suppose E is a resonance in $\mathcal{R}$ and w a corresponding resonant state:

$$\mathscr{P}w = Ew.$$

Recall that w is exponentially decaying as $x \to \infty$ in our setting, and hence $w \in L^2([x_0, +\infty))$ for any $x_0 \in \mathbb{R}$. We take x_0 such that $x_0 < a_0$. Then $\{(x,\xi); x = x_0\}$ intersects with $\Gamma_2(E_0)$ at two points $(x_0, \pm\xi_0)$ with $\xi_0 = \sqrt{E_0 - V_2(x_0)} > 0$, and has no intersection with $\Gamma_1(E_0)$. The point (x_0, ξ_0) is on the incoming classical trajectory $\gamma_{\rm in}$ from infinity to a crossing point, and $(x_0, -\xi_0)$ is on the outgoing classical trajectory $\gamma_{\rm out}$ from a crossing point to infinity.

We have

$$\begin{aligned}0 &= \langle (\mathscr{P} - E)w, w\rangle_{L^2([x_0,+\infty))}\\ &= \|hD_x w\|^2_{L^2([x_0,+\infty))} - E\|w\|^2_{L^2([x_0,+\infty))} - h^2\langle w'(x_0), w(x_0)\rangle.\end{aligned}$$

Taking the imaginary part, one has

$$\operatorname{Im} E \cdot \|w\|^2_{L^2([x_0,+\infty))} = -h^2(w_1'(x_0)\overline{w_1(x_0)} + w_2'(x_0)\overline{w_2(x_0)}).$$

Near $x = x_0$, we are far from the crossing points and the turning points, and the WKB expression is valid. Modulo $\mathcal{O}(h)$, the first component w_1 of w is 0, and the second component w_2 is of outgoing WKB form with its derivative:

$$\begin{aligned}w_2 &\sim \frac{\alpha_{\rm out}}{(E - V_2(x))^{\frac14}} \exp\left(-\frac{i}{h}\int_0^x \sqrt{E - V_2(t)}dy\right),\\ \overline{w_2'} &\sim \frac{i}{h}\overline{\alpha_{\rm out}}(E - V_2(x))^{\frac14} \exp\left(\frac{i}{h}\int_0^x \sqrt{E - V_2(t)}dy\right),\end{aligned}$$

for some constant $\alpha_{\rm out}$, which yields

$$w_2'(x_0)\overline{w_2(x_0)} \sim \frac{i}{h}|\alpha_{\rm out}|^2.$$

The microlocal asymptotic behavior of w for x near x_0 is expressed as

$$w \equiv \alpha_{\rm out} f_{\rm out} \quad \text{on } \gamma_{\rm out}, \tag{5.1}$$

$$w \equiv 0 \quad \text{on } \gamma_{\rm in}. \tag{5.2}$$

On the other hand, if w is normalized on any portion γ_1 of Γ_1 free from crossing point and turning point as

$$w \equiv f_{\gamma_1} \quad \text{on } \gamma_1, \tag{5.3}$$

then it turns out (see [15] and [5]) that

$$\|w\|^2_{L^2([x_0,+\infty))} = 2\mathcal{A}'(E) + \mathcal{O}(h^{\frac13} + h^{\frac{1}{m+1}}).$$

Thus we obtained a formula of the resonance width:

Lemma 5.1. *Let $\alpha_{\rm out}$ be the constant determined by (5.1) for the resonant state w normalized by (5.3). Then the resonance width is asymptotically given by*

$$\operatorname{Im} E = -\frac{h|\alpha_{\rm out}|^2}{2\mathcal{A}'(E)} + \mathcal{O}(h^{\frac43} + h^{\frac{m+2}{m+1}}). \tag{5.4}$$

Thanks to Lemma 5.1, the study of resonance width is reduced to the computation of the microlocal data $\alpha_{\rm out}$ on the outgoing tail $\gamma_{\rm out}$ of a resonant state w which is normalized microlocally at a point on $\Gamma_1(E_0)$.

This is just an elementary computation using the microlocal scattering matrices. Let us do it in the case $E_0 > 0$. $(\Gamma_1(E_0)\cup\Gamma_2(E_0))\setminus(\Gamma_1(E_0)\cap\Gamma_2(E_0))$ consists of 5 connected components: two unbounded components $\gamma_{\rm in}$, $\gamma_{\rm out}\in\Gamma_2(E_0)$ and three bounded components $\gamma_{1,\uparrow}$, $\gamma_{1,\downarrow}\subset\Gamma_1(E_0)$ and $\gamma_2\subset\Gamma_2(E_0)$. Here $\gamma_{1,\uparrow}$ starts from ρ_-, and $\gamma_{1,\downarrow}$ starts from ρ_+.

We normalize w on $\gamma_{1,\uparrow}$ near ρ_+

$$w\equiv f_{1,\uparrow}\quad \text{on } \gamma_{1,\uparrow},$$

where $f_{1,\uparrow}$ is the microlocal WKB solution whose phase is normalized at ρ_+, i.e. defined by $\int_{\rho_+}^{\rho}\xi dx$ for $\rho\in\gamma_{1,\uparrow}$ near ρ_+.

Since w is a resonant state, it should be microlocally zero on $\gamma_{\rm in}$. Then the two microlocal data on the incoming trajectories are given near ρ_+, and hence the microlocal scattering matrix there gives the two microlocal data on the outgoing trajectories $\gamma_{1,\downarrow}$ and γ_2:

$$w\equiv t_{11}^+f_{1,\downarrow}^+\quad\text{on } \gamma_{1,\downarrow},$$
$$w\equiv t_{21}^+f_{2}^+\quad\text{on } \gamma_{2},$$

where t_{jk}^+ are the (j,k) entry of the microlocal scattering matrix T at ρ_+. Remark that the microlocal WKB solutions $f_{1,\downarrow}$ and f_2 are normalized at ρ_+.

Now we continue the microlocal WKB solutions $f_{1,\downarrow}^+$ and f_2^+ along $\gamma_{1,\downarrow}$ and γ_2 respectively. The continuation is unique by the propagation of singularities and explicitly given again in the WKB form near ρ_-:

$$w\equiv t_{11}^+\exp\left(\frac{i}{h}\int_{\gamma_{1,\downarrow}}\xi dx-\frac{\pi}{2}i\right)f_{1,\downarrow}^-\quad\text{on } \gamma_{1,\downarrow},$$
$$w\equiv t_{21}^+\exp\left(\frac{i}{h}\int_{\gamma_{2}}\xi dx-\frac{\pi}{2}i\right)f_{2}^-\quad\text{on } \gamma_{2}.$$

Here $\frac{\pi}{2}i$ is the Maslov index counted at the turning points $\gamma_{1,\downarrow}\cap\{\xi=0\}$ and $\gamma_2\cap\{\xi=0\}$.

Finally we go through the crossing point ρ_- using the microlocal scattering matrix at this point. We then obtain

$$w\equiv -i\left(t_{11}^+t_{21}^-\exp\left(\frac{i}{h}\int_{\gamma_{1,\downarrow}}\xi dx\right)+t_{21}^+t_{22}^-\exp\left(\frac{i}{h}\int_{\gamma_{2}}\xi dx\right)\right)f_{\rm out}\quad\text{on } \gamma_{\rm out},$$

that is,

$$\alpha_{\rm out}=-it_{11}^+t_{21}^-\exp\left(\frac{i}{h}\int_{\gamma_{1,\downarrow}}\xi dx\right)-it_{21}^+t_{22}^-\exp\left(\frac{i}{h}\int_{\gamma_{2}}\xi dx\right).$$

Recall that $t_{11}^+=t_{22}^-=1$, $t_{21}^+=-i\omega h^{\frac{1}{m+1}}$, $t_{21}^-=-i\overline{\omega}h^{\frac{1}{m+1}}$ modulo $\mathcal{O}(h^{\frac{1}{m+1}})$, and that

$$\int_{\gamma_{1,\downarrow}}\xi dx-\int_{\gamma_2}\xi dx=\mathcal{A}(E)-\mathcal{S}(E).$$

This, together with the fact that E satisfies the Bohr-Sommerfeld quantization rule (1.1), give the formula (3.13) of the resonance width in Remark 3.1.

The formula (3.14) in the case $E_0 = 0$ is obtained similarly, and we omit the proof.

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