Combining fixpoint and differentiation theory

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Zeinab Galal, Jean-Simon Pacaud Lemay

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In math and computer science

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- Various notions of differentiation for categories (differential categories, Cartesian (closed) differential categories, tangent categories, coherent categories, ...)
- Various notions of recursion (fixpoint operators, trace, ...)

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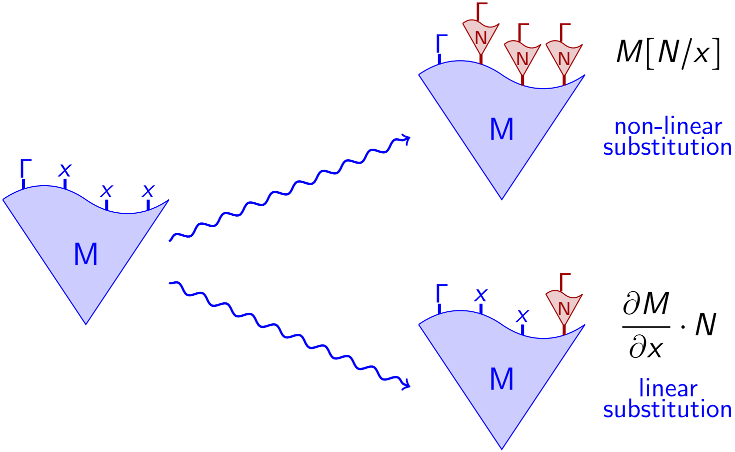
- Various notions of recursion (fixpoint operators, trace, ...)

Goal: general account of the interactions between fixpoints, trace and differentiation

Differential λ-calculus

$$\frac{\Gamma, x : A \vdash M : B \qquad\qquad \Gamma \vdash N : A}{\Gamma, x : A \vdash \dfrac{\partial M}{\partial x} \cdot N : B}\ \textsc{diff}$$



Cartesian (closed) differential categories

Blute, R. F. and Cockett, J. R. B. and Seely, R. A. G. *Cartesian Differential Categories*, 2009.

Bucciarelli, A., Ehrhard, T., Manzonetto, G. *Categorical Models for Simply Typed Resource Calculi*, 2010.

A Cartesian differential category is a Cartesian left additive category $\mathbb{C}$ equipped with an operator **D**

$$\begin{array}{rcl} \mathbf{D} : \mathbb{C}(A, B) & \longrightarrow & \mathbb{C}(A \times A, B) \\ f : A \to B & \longmapsto & \mathbf{D}[f] : A \times A \to B \end{array}$$

satisfying seven axioms.

Cartesian differential categories are Tangent categories

$$\frac{f : A \to B}{\mathbf{D}(f) : A \times A \to B} \quad \rightsquigarrow \quad \frac{f : A \to B}{\mathbf{T}(f) : A \times A \to B \times B}$$

Directional derivative

$$(a, b) \longmapsto f'(a) \cdot b$$

Tangent bundle

$$(a, b) \longmapsto (f(a), f'(a) \cdot b)$$

$$\mathbf{T} : \mathbb{C} \longrightarrow \mathbb{C}$$

$$\begin{array}{ccc} A & \longmapsto & A \times A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \langle f\pi_1, \mathbf{D}[f] \rangle} \\ B & \longmapsto & B \times B \end{array}$$

$$\begin{array}{c} (a, b) \\ \big\downarrow \\ \left(f(a), \frac{\mathrm{d}f(x)}{\mathrm{d}x}(a) \cdot b\right) \end{array}$$

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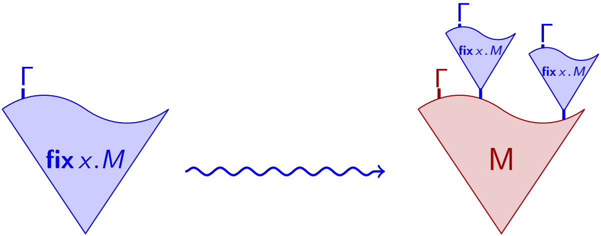
$\mathbf{T}$ preserves composition $\Leftrightarrow$ the chain rule holds

Fixpoint operator for terms

$$\frac{\Gamma, x : A \vdash M : A}{\Gamma \vdash \mathbf{fix}\, x.M : A}\ \textsc{fix}$$



$$\mathbf{fix}\, x.M \to M[\mathbf{fix}\, x.M/x]$$

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For a Cartesian category $\mathbb{C}$, a parametrized fixpoint operator on $\mathbb{C}$ is family of functions

$$\begin{aligned} \mathbf{fix}_X^{\Gamma} : \mathbb{C}(\Gamma \times X, X) &\longrightarrow \mathbb{C}(\Gamma, X) \\ f &\longmapsto \mathbf{fix}_X^{\Gamma}(f) \end{aligned}$$

indexed by pairs of objects in $\mathbb{C}$ verifying:

- fixpoint axiom: for all $f : \Gamma \times X \to X$,

$$\mathbf{fix}(f) = f \circ \langle \mathrm{id}_{\Gamma}, \mathbf{fix}(f) \rangle$$

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- naturality axiom: for all morphisms $g : \Gamma \to \Delta$ and $f : \Delta \times X \to X$,

$$\mathbf{fix}(f) \circ g = \mathbf{fix}(f \circ (g \times \mathrm{id}_X)).$$

Examples

- for domains Γ, X and a Scott-continuous map $f : \Gamma \times X \to X$,

$$f(a, \bigvee_{n\in\omega} f^n(a, \bot)) = \bigvee_{n\in\omega} f^n(a, \bot)$$

  is a least parametrized fixpoint for f.

- co-Kleisli category (weighted) relations with the finite multisets comonad

- special case: power series over continuous semi-rings

- any cpo-enriched cartesian closed category

- complete metric spaces and contractive maps

Additional axioms

A fixpoint operator **fix** is a Conway operator if it satisfies

- Dinaturality axiom:

$$\mathbf{fix}\, x.g\,(f(a,x)) \quad = \quad g\,(\mathbf{fix}\, y.f(a,g(y)))$$

- Bekić axiom: for systems of equations

$$\begin{array}{r} \mathbf{fix}(x,y).(f(x,y), \\ g(x,y)) \end{array} = \begin{array}{l} (\mathbf{fix}\, x.f(x, \mathbf{fix}\, y.g(x,y)), \\ \;\mathbf{fix}\, y.g(\mathbf{fix}\, x.f(x,y), y)) \end{array}$$

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Conway fixpoint operator $\Leftrightarrow$ Trace operator with $\times$ as tensor

Particular cases

Cartesian closed

$$\mathbf{Y} : X \Rightarrow X \to X$$

$$\mathbf{Y} = \mathrm{eval}\langle \mathrm{id}, \mathbf{Y} \rangle$$

general trees

Cartesian

$$\frac{f : \Gamma \times X \to X}{\mathbf{fix}(f) : \Gamma \to X}$$

$$\mathbf{fix}\, f = f\langle \mathrm{id}, \mathbf{fix}\, f \rangle$$

linear trees

Finite biproducts

$$\frac{f : X \to X}{f^* : X \to X}$$

$$f^* = \mathrm{id} \oplus f^* f$$

Repetition operator

For a category $\mathbb{C}$ with finite biproducts, a repetition operator is a family of functions

$$\begin{aligned}(-)^* : \mathbb{C}(X,X) &\longrightarrow \mathbb{C}(X,X)\\ f &\longmapsto f^*\end{aligned}$$

indexed by objects in $\mathbb{C}$ verifying:

- fixpoint axiom: $f^* = \mathrm{id}_X \oplus f \circ f^*$
- addition axiom: $(f \oplus g)^* = (f^* \circ g)^* \circ f^*$
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For a category with finite biproducts:

Conway fixpoint $\Leftrightarrow$ Repetition operator

How should derivatives and fixpoints interact?

The function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ has a parametrized fixpoint $g : \mathbb{R} \to \mathbb{R}$

$(x, y) \mapsto 1 - x^2 y$ $\qquad$ $x \mapsto \frac{1}{1+x^2}$

$$g(x) = f(x, g(x)) \tag{1}$$

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By the chain rule,

$$g'(x) = \frac{\partial f}{\partial x}(x, g(x)) + \frac{\partial f}{\partial y}(x, g(x)) \cdot g'(x) = \frac{-2x}{(1+x^2)^2} \tag{2}$$

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Another way of computing $g'(x)$:

- Compute the tangent of f, $\mathbf{T}(f) : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$:

$$\mathbf{T}(f)(x, y, a, b) = \left(f(x, y), \frac{\partial f}{\partial x}(x, y) \cdot a + \frac{\partial f}{\partial y}(x, y) \cdot b \right)$$

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- (1) and (2) imply $\mathbf{T}(g)$ is a fixpoint for $\mathbf{T}(f)$

$$\mathbf{T}(g)(x, a) = (g(x), g'(x) \cdot a) = \mathbf{T}(f)(x, g(x), a, g'(x) \cdot a)$$

Cartesian differential fixpoint categories

Computing the derivative of the fixpoint is equivalent to computing the fixpoint of the tangent.

- **Option 1** differentiating the fixpoint

$$\dfrac{\dfrac{\Gamma \times X \xrightarrow{f} X}{\Gamma \xrightarrow{\mathbf{fix}(f)} X}}{\Gamma \times \Gamma \xrightarrow{\mathbf{D}[\mathbf{fix}(f)]} X}$$

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- **Option 2** fixpoint of the tangent

$$\dfrac{\dfrac{\dfrac{\Gamma \times X \xrightarrow{f} X}{\Gamma \times X \times \Gamma \times X \xrightarrow{\mathbf{T}(f)} X \times X}}{\Gamma \times \Gamma \times X \times X \xrightarrow{c} \Gamma \times X \times \Gamma \times X \xrightarrow{\mathbf{T}(f)} X \times X}}{\Gamma \times \Gamma \xrightarrow{\mathbf{fix}(\mathbf{T}(f)\circ c)} X \times X \xrightarrow{\pi_2} X}$$

where $c := \mathrm{id}_\Gamma \times \langle \pi_2, \pi_1 \rangle \times \mathrm{id}_X : \Gamma \times \Gamma \times X \times X \xrightarrow{\cong} \Gamma \times X \times \Gamma \times X$

Cartesian differential fixpoint categories

For a Cartesian differential category $\mathbb{C}$, a parametrized fixpoint operator **fix** satisfies the

- **differential-fixpoint axiom** if for every $f : \Gamma \times X \to X$

$$\mathbf{D}[\mathbf{fix}(f)] = \pi_2 \, \mathbf{fix}(\mathbf{T}(f \circ c))$$

- **tangent-fixpoint axiom** if for every $f : \Gamma \times X \to X$

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Lemma

If the fixpoint operator **fix** *is Conway, the following are equivalent:*

- **fix** *satisfies the differential-fixpoint axiom;*
- **fix** *satisfies the tangent-fixpoint axiom.*

Examples

- Any category with finite biproducts where $\mathbf{D}[f] = f \circ \pi_2$ provided that the fixpoint is uniform wrt projection maps
- Weighted relations
- Special case: formal power series over continuous semi-rings
- Quantale enriched profunctors with the free exponential
- More generally: any Cartesian differential category where the fixpoint operator is obtained from free bialgebras (bifree algebras) and is uniform wrt linear maps

Cartesian closed differential fixpoint categories

Ehrhard, T. *A coherent differential PCF*, 2023.

For a Cartesian closed differential category $\mathbb{C}$, a fixpoint combinator $\mathbf{Y} : X \Rightarrow X \to X$ satisfies the

- **differential-fixpoint axiom** if

$$\mathbf{D}[\mathbf{Y}] = \pi_2 \mathbf{Y} \lambda(\mathbf{T}(\mathrm{eval}) \circ c)$$

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Examples cpo-enriched cartesian closed categories, categories with fixpoint objects

Fixpoints and linearity

In a Cartesian differential category $\mathbb{C}$, a morphism $f : X \to Y$ is linear if

$$\mathbf{D}[f] = X \times X \xrightarrow{\pi_2} X \xrightarrow{f} Y$$

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Lemma

If $\mathbb{C}$ is a Cartesian differential fixpoint category, then the fixpoint of a linear morphism is linear.

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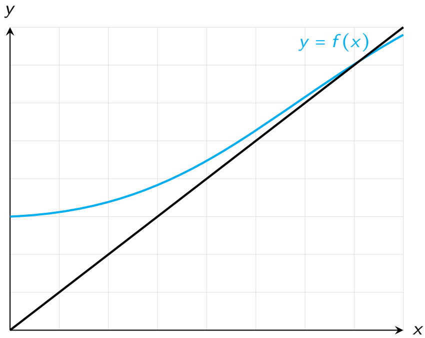
Lemma

If $\mathbb{C}$ is a Cartesian differential fixpoint category, then the fixpoint of a linear morphism is linear.

- Differential fixpoint axiom $\Rightarrow$ $\mathbf{Lin}(\mathbb{C})$ has a fixpoint operator.
- Differential fixpoint axiom + Conway fixpoint $\Rightarrow$ $\mathbf{Lin}(\mathbb{C})$ has a repetition operator.

Newton-Raphson iteration



- We want to solve $x = f(x)$

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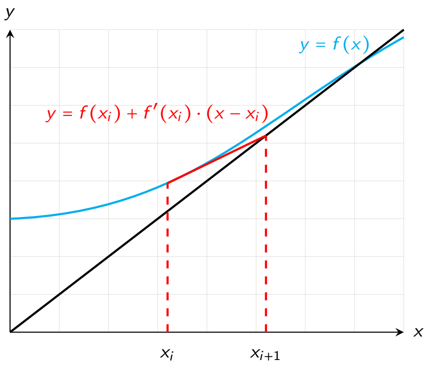


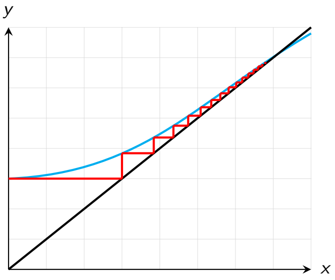
- We want to solve $x = f(x)$
- At each iteration, we approximate f by its Taylor expansion of order 1:

$$x_{i+1} = f(x_i) + f'(x_i) \cdot (x_{i+1} - x_i)$$

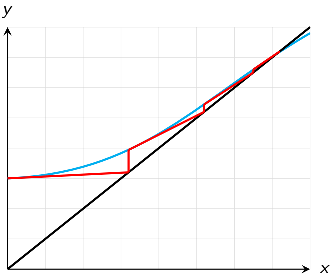
$$\iff \quad x_{i+1} = \frac{1}{1 - f'(x_i)} (f(x_i) - f'(x_i) \cdot x_i)$$

Newton-Raphson iteration

Fast computation of fixpoints using derivatives



Kleene/Scott iteration



Newton iteration

$$S_0 = x_0$$
$$S_{i+1} = f(S_i)$$

$$N_0 = x_0$$
$$N_{i+1} = \frac{1}{1 - f'(N_i)}\left(f(N_i) - f'(N_i) \cdot N_i\right)$$

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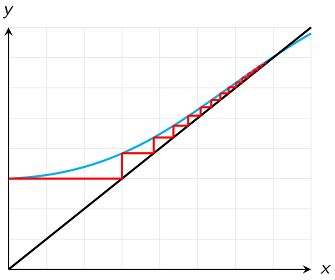

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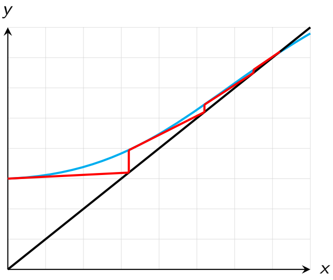

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The sequence $\{N_i\}_i$ does not always converge but under certain conditions, it converges to a fixpoint of f and we have quadratic convergence:

$$\forall i \geq 0, \quad |\mathbf{fix}(f) - N_{i+1}| \;\leq\; \frac{1}{2}\,|\mathbf{fix}(f) - N_i|^2$$

Generalizing Newton's method

How can we generalize Newton's method if we are not working with real numbers?

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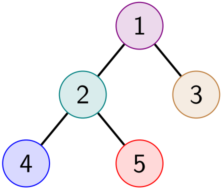
- Subtraction: set difference for combinatorial species, prove the existence of an element y such that $y + f'(N_i) \cdot N_i = f(N_i)$ in the semi-ring case.

Combinatorial species

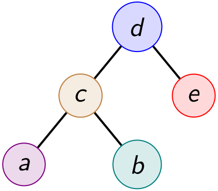
- **Joyal:** species of structures

set of structures $+$ relabelling action

$$\{1, 2, 3, 4, 5\} \overset{\sim}{\underset{\sigma}{\longrightarrow}} \{a, b, c, d, e\}$$



$$F(\sigma)\ \overset{\sim}{\longrightarrow}$$



generating series

$$\textbf{BinTree} : x \mapsto 1 + x + 2x^2 + 5x^3 + 14x^4 + \dots$$

Newton's method for combinatorial species

- Labelle, Lacoste, Pivoteau, Salvy, Soria, ...

species defined implicitly $\rightsquigarrow$ (fast computation using) species of sequences + species derivative

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species defined implicitly $\xrightarrow[\text{using}]{\text{fast computation}}$ species of sequences + species derivative

- Effective computations for constructible species (Pivoteau, Salvy, Soria)
- Applications to Boltzmann sampling: efficient random generators for implicit combinatorial structure (many people)

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Monotony and positivity imply that Newton's method always converges.

Newton's method for power series over a continuous semi-ring

- Dataflow analysis, formal languages: Etessami, Yannakakis, Esparza, Kiefer, Luttenberger, Schlund, ...

language generated by a context-free grammar ⇝ (fast computation using) Kleene $(-)^*$ operator + Brzozowski derivative

Newton's method for power series over a continuous semi-ring

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Monotony and positivity imply that Newton's method always converges.

Scott/Kleene iteration

$$\mathbf{BinTree}(x) = 1 + x^2 \cdot \mathbf{BinTree}(x)$$

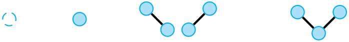
Scott/Kleene iteration

$$\mathbf{BinTree}(x) = 1 + x + 2 \cdot x^2 + 5 \cdot x^3 + 14 \cdot x^4 + 42 \cdot x^3 + \ldots$$

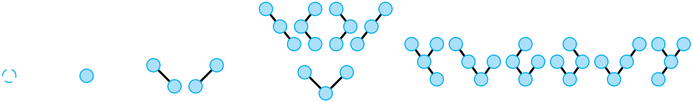
$$\mathbf{S}_0 = 0$$

$$\mathbf{S}_1 = 1 \cdot x^0$$

$$\mathbf{S}_2 = 1 \cdot x^0 + 1 \cdot x^1$$



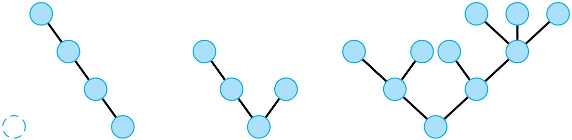
$$\mathbf{S}_3 = 1 \cdot x^0 + 1 \cdot x^1 + 2 \cdot x^2 + 1 \cdot x^3$$



$$\mathbf{S}_4 = 1 \cdot x^0 + 1 \cdot x^1 + 2 \cdot x^2 + 5 \cdot x^3 + 6 \cdot x^4 + \ldots$$

Strahler-Horton number of a tree

Strahler number of tree = height of the largest perfect binary tree that can be embedded into it



Strahler number 0 1 2 3

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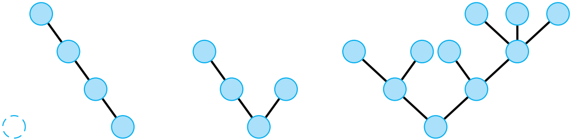

Strahler number 0 1 2 3

- Strahler, Horton: measure of the branching complexity of rivers
- Ershov: minimum number of registers needed to evaluate an arithmetic expression

Viennot, X., *The Strahler analysis of binary trees in computer science and in other sciences*, 2013.

Esparza, J., Luttenberger, M., Schlund, M.,*A Brief History of Strahler Numbers*, 2014.

Newton iteration

$$\mathbf{BinTree}(x) = 1 + x + 2 \cdot x^2 + 5 \cdot x^3 + 14 \cdot x^4 + 42 \cdot x^5 + 132 \cdot x^6 + \ldots$$

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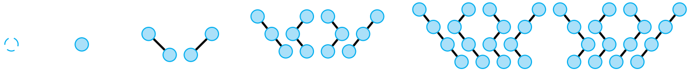
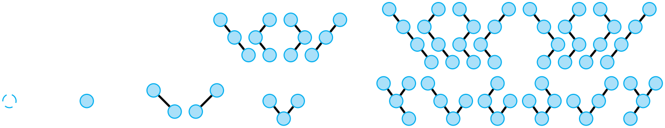


$\mathbf{N}_2 = 1 \cdot x^0 + 1 \cdot x^1 + 2 \cdot x^2 + 4 \cdot x^3 + 8 \cdot x^4 + \ldots$



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Newton iteration for Cartesian differential categories

non-linear fixpoint operator $\overset{\text{fast computation}}{\underset{\text{using}}{\rightsquigarrow}}$ linear repetition operator + differentiation operator

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We assume that:

- the fixpoint-differentiation axiom holds
- it is a Taylor category

  $\rightsquigarrow$ for $f : X \to Y$, $f(a+b) = f(a) + \frac{df(x)}{dx}(a) \cdot b + \ldots$

Newton iteration for Cartesian differential categories

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- we have truncated subtractions: the commutative monoid operation on hom-sets has a right adjoint $\ominus$ for the natural order

  $\rightsquigarrow$ for $f, g, h : X \to Y$, $f \leq h \ominus g \quad \Leftrightarrow \quad f + g \leq h \quad \Leftrightarrow \quad g \leq h \ominus f$

Newton method for Cartesian closed differential categories

Key ingredient: derivatives are linear in the second argument.

For $f : A \to A$, and for any point $\top \xrightarrow{a} A$, $\mathbf{D}[f] : A \times A \to A$ verifies

$$\mathbf{D}_a[f] := \mathbf{D}[f] \circ (a \times \mathrm{id}_A) : A \to A$$

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Newton iterator in the Cartesian closed setting

$$\begin{array}{ccc} A \Rightarrow A & \xrightarrow{\mathbf{N}} & A \Rightarrow A \\ f & \longmapsto & \lambda a.\,((\mathbf{D}_a[f])^* \cdot (f(a) \ominus \mathbf{D}_a[f] \cdot a)) \end{array}$$

Newton method for Cartesian closed differential categories

- If we have a least fixpoint $\mathbf{Y} : A \Rightarrow A \to A$ obtained from ω-cpo enrichment as $\mathbf{Y} = \bigvee_{i \in \omega} \mathbf{Y}_i$ with

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- We obtain an ω-chain converging to the least fixpoint from below:

$$\forall i,\ \mathbf{Y}_i \leq \mathbf{N}_i \leq \mathbf{N}_{i+1} \leq \mathbf{Y}$$

Convergence rate

How do we measure distance?

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We use a metric based on *n*-th Taylor monomials:

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$\rightsquigarrow$ metric on each homset $d : \mathbb{C}(A, B) \times \mathbb{C}(A, B) \longrightarrow [0, 1]$

$$(f, g) \longmapsto \begin{cases} 2^{-i} & \text{where } i := \inf\{n \in \mathbb{N} \mid \mathcal{M}_n(f) \neq \mathcal{M}_n(g)\} \\ 0 & \text{if for all } i \in \mathbb{N},\ \mathcal{M}_i(f) = \mathcal{M}_i(g) \end{cases}$$

Informal idea in a Taylor category:

$$d(f, g) = 2^{-i} \quad \Rightarrow \quad f(x) - g(x) = o(x^i)$$

Convergence rate

Properties of the Taylor metric:

- In general, d is only a pseudo metric, we do not have:

$$d(f,g) = 0 \quad \Rightarrow \quad f = g$$

- The stronger triangle inequality holds:

$$d(f,h) \leq \max\{d(f,g), d(g,h)\}$$

- If $\mathbb{C}$ is a Taylor category, then it is an ultrametric.

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- If $\mathbb{C}$ is a Taylor category, then it is an ultrametric.

The convergence rate is quadratic, for all i:

$$d(\mathbf{N}_{i+1}, \mathbf{Y}) = \frac{1}{2}(d(\mathbf{N}_i, \mathbf{Y}))^2$$

Informally, we have:

$$\mathbf{N}_i(f) - \mathbf{Y}(f) = o(x^m) \quad \Rightarrow \quad \mathbf{N}_{i+1}(f) - \mathbf{Y}(f) = o(x^{2m+1})$$

Future work

- Interaction between differentiation and trace:

$$\frac{f : A \to B}{\mathbf{D}[f] : !A \otimes A \to B}$$

- Syntactic counterpart for differential λ-calculus (or in general for a signature with a linear and non-linear substitution)
- More general iterative schemes than Newton-Raphson
- Approximation of solutions of differential equations
- Reverse differential categories and fixpoints
- Local fixpoint operators to capture local implicit function theorems

Thank you