

DiλLL2024 at the CIRM

An introduction to Differential Linear Logic

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Thank you to the organizers !



Objective and Methodology

Objective : Make DiLL a comfortable place to think

1. A reminder or a quick review of Linear Logic,
2. An introduction of DiLL' rules,
3. A detailed explanation of its cut-elimination procedure,
4. An introduction to Differential λ -calculus,
5. A few examples of its essential models,

Methodology: All of that done with a denotational flavor

- ▶ I am much more a semantician than a syntactician.
- ▶ The semantics will however stay very much informal and in the background

Linear Logic

A decomposition of the implication

$$A \Rightarrow B \simeq !A \multimap B$$

Linear Logic

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- Usual **non-linear** implication

Linear Logic

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- ▶ Usual **non-linear** implication
- ▶ **Linear** implication

Linear Logic

A decomposition of the implication

$$A \Rightarrow B \simeq !A \multimap B$$

- ▶ Usual **non-linear** implication
- ▶ **Linear** implication
- ▶ **Exponential**: Usually, the duplicable copies of A .

Linear Logic

A decomposition of the implication

- ▶ Usual **non-linear** implication
- ▶ **Linear** implication

A linear proof is in particular non-linear.

A proof of $A \vdash B$ is linear.

A proof of $!A \vdash B$ is non-linear.

$$\frac{A \vdash \Gamma}{!A \vdash \Gamma} \text{dereliction}$$

Slogan: ! in the hypotheses, speaking of resources.

- ▶ A linear proof will make use only once of its hypothesis A
- ▶ A non-linear proof will make use only once of its hypothesis A

Classical Linear Logic

I'll be considering only classical linear logic, where sequents are monolateral by default but can be made bilateral for better intuitions

$$\Gamma \vdash A \quad \vdash \Gamma^\perp, A$$

Formulas for (Differential) Linear Logic

$$A, B := a \mid a^\perp \mid 0 \mid 1 \mid \top \mid \perp \mid A \otimes B \mid A \wp B \mid A \oplus B \mid A \& B \mid !A \mid ?A$$

where $\otimes$ (resp $\&$) denotes the multiplicative (resp. additive) conjunction and $\wp$ (resp $\oplus$) denotes the multiplicative (resp. additive) disjunction.

An involutive linear negation

$$\begin{aligned}(A \& B)^\perp &= A^\perp \oplus B^\perp & (A \oplus B)^\perp &= A^\perp \& B^\perp \\(A \wp B)^\perp &= A^\perp \otimes B^\perp & (A \otimes B)^\perp &= A^\perp \wp B^\perp \\!A^\perp &= ?A^\perp & ?A^\perp &= !A^\perp\end{aligned}$$

Proofs of Linear Logic

Multiplicative Additive Linear Logic

$$\frac{}{\vdash A, A^\perp} \text{ axiom}$$

$$\frac{\vdash \Gamma, A \quad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta} \text{ cut}$$

$$\frac{}{\vdash 1} (1)$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, \perp} \perp$$

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \wp B} \wp$$

$$\frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B} \otimes$$

$$\frac{}{\vdash \Gamma, \top} \top$$

$$\frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \& B} \&$$

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B} \oplus_L$$

$$\frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B} \oplus_R$$

$$\Gamma \vdash \Delta \quad A_1 \otimes \cdots \otimes A_n \vdash B_1 \wp \cdots \wp B_n$$

Linear and non-linear implication

$$A \multimap B := A^\perp \wp B \quad A^\perp \equiv A \multimap \perp$$

$$A \multimap B := !A \multimap B$$

Denotational Intuitions for Linear Logic (DILL)

$$\Gamma, A \rightsquigarrow \llbracket \Gamma \rrbracket \llbracket A \rrbracket$$

- ▶ Some *additive* structure
- ▶ For example: multi-sets with unions,
- ▶ Or general *vector spaces*: spaces of sequences, topological vector spaces...

$$A \vdash B, \vdash A^\perp, B \rightsquigarrow \ell : \llbracket A \rrbracket \multimap \llbracket B \rrbracket$$

Where ℓ stands for a linear map

$$!A \vdash B \rightsquigarrow f : \llbracket A \rrbracket \Rightarrow \llbracket B \rrbracket$$

where f stands for a non-linear, maybe continuous, maybe differentiable map

Denotational Intuitions for "Why Not" "?"

In classical Linear Logic

$$\llbracket ?A \rrbracket \simeq \llbracket A^\perp \Rightarrow \perp \rrbracket$$

Let's assume, as in most models of Differential Linear Logic:

$$\llbracket \perp \rrbracket = \mathbb{R}$$

$$\llbracket A \Rightarrow B \rrbracket = \mathcal{C}^\infty(\llbracket A \rrbracket, \llbracket B \rrbracket)$$

Then

$$\llbracket ?A \rrbracket \subseteq \{f \mid f \in \mathcal{C}^\infty(\llbracket A^\perp \rrbracket, \mathbb{R})\}$$

$$!A \vdash \perp \rightsquigarrow f \in \mathcal{C}^\infty(\llbracket A \rrbracket, \mathbb{R})$$

$$\vdash ?A^\perp \rightsquigarrow \vdash f : ?A^\perp$$

Linear Logic, structural rules

In a bilateral presentation

$$\frac{\vdash \Gamma}{!A \vdash \Gamma} \text{ w}$$

$$\frac{!A, !A \vdash \Gamma}{!A \vdash \Gamma} \text{ c}$$

$$\frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B} \text{ d}$$

$$\frac{! \Gamma \vdash A}{! \Gamma \vdash !A} \text{ p}$$

In a monolateral presentation:

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A} \text{ w}$$

$$\frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A} \text{ c}$$

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, ?A} \text{ d}$$

$$\frac{\vdash ? \Gamma, A}{\vdash ? \Gamma, !A} \text{ p}$$

Denotational Intuitions for monolateral exponential rules

Weakening and contraction

$$\frac{\vdash \Gamma}{\vdash \Gamma, (\text{cst}_1 : a' \mapsto 1) : ?A} w$$

The constant function is non-linear

$$\frac{\vdash \Gamma, f : ?A, f : ?A}{\vdash \Gamma, f \cdot g : ?A} c$$

The multiplication of scalar functions

Dereliction and promotion

$$\frac{\vdash \Gamma, v : A}{\vdash \Gamma, ((x : A^\perp) \mapsto x(v)) : ?A} d$$

linear $\rightsquigarrow$ non-linear

$$\frac{\vdash ?\Gamma, v : A}{\vdash ?\Gamma, \delta_v : !A} p$$

Diracs

Denotational Intuitions for bilateral exponential rules

Weakening and contraction

$$\frac{\vdash \gamma : \Gamma}{(\text{cst}_1 : a \mapsto \gamma) : (!A \vdash \Gamma)} w$$

The constant function is non-linear

$$\frac{h : (!A, !A \vdash \Gamma)}{(x \mapsto h(x, x)) : (!A \vdash \Gamma)} c$$

Identifying arguments

Dereliction and promotion

$$\frac{\ell : (A \vdash \Gamma)}{\ell : (!A \vdash \Gamma)} d$$

linear $\rightsquigarrow$ non-linear

$$\frac{f : (!\Gamma \vdash A)}{\delta_f : (!\Gamma \vdash !A)} p$$

Diracs

Denotational Intuitions about "Bang" !

In classical Linear Logic

$$\llbracket !A \rrbracket \simeq \llbracket (A \Rightarrow \perp)^\perp \rrbracket$$

Let's assume, as in most models of Differential Linear Logic:

$$\llbracket \perp \rrbracket = \mathbb{R} \quad \llbracket A \Rightarrow B \rrbracket = \mathcal{C}^\infty(\llbracket A \rrbracket, \llbracket B \rrbracket)$$

$$\llbracket A^\perp \rrbracket = \llbracket A \rrbracket' = \mathcal{L}(A, \mathbb{R}) \quad \text{Linear Negation is interpreted by Dual}$$

Then the ! is interpreted as a space of distributions with compact support:

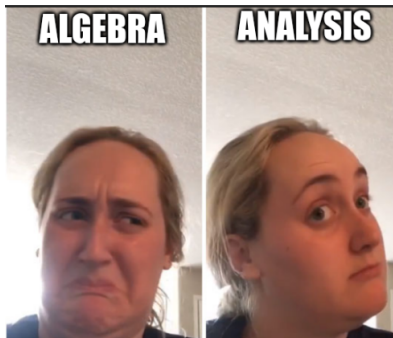
$$\llbracket !A \rrbracket \subseteq \mathcal{C}^\infty(\llbracket A \rrbracket, \mathbb{R})'$$

- Distributions are linear scalar maps acting on (some subspace of) smooth functions.
- e.g. $\delta_x : f \mapsto f(x)$

Convolution, the monoidal operation on distributions:

$$\phi * \psi := f \mapsto \phi(x \mapsto \psi(y \mapsto f(x + y))) \quad \delta_x * \delta_y = \delta_{x+y}$$

Differential Linear Logic



Dereliction and co-dereliction:



$$\frac{\ell : A \vdash B}{\ell : !A \vdash B} \text{d}$$

linear $\hookrightarrow$ *non-linear*.

$$\frac{\vdash \Delta, v : A}{\vdash \Delta, (f \mapsto D_0(f)(v)) : !A} \bar{\text{d}}$$

non-linear $\hookrightarrow$ *linear*

Dereliction and co-dereliction:



$$\frac{\ell : A \vdash B}{\ell : !A \vdash B} \text{d}$$

linear $\hookrightarrow$ *non-linear*.

$$\frac{\vdash \Delta, v : A}{\vdash \Delta, (f \mapsto D_0(f)(v)) : !A} \bar{\text{d}}$$

non-linear $\hookrightarrow$ *linear*

Cut-elimination:

$$\frac{\frac{\vdash \Gamma, v : A}{\vdash \Gamma, D_0(-)(v) : !A} \bar{\text{d}} \quad \frac{\ell : A \vdash B}{\ell : !A \vdash B} \text{d, dereliction}}{\vdash \Gamma, \Delta} \text{cut}$$

$\rightsquigarrow$

$$\frac{\vdash \Gamma, v : A \quad \ell : \vdash \Delta, A^\perp}{\vdash \Gamma, \Delta, D_0(\ell)(v) = \ell(v)} \text{cut}$$

Exponential Rules of DiLL

To the previous structural rules, DiLL adds the following co-structural rules

Monolateral co-structural rules

$$\frac{}{\vdash !A} \bar{w}$$

$$\frac{\vdash \Gamma, !A \quad \vdash \Delta, !A}{\vdash \Gamma, \Delta, !A} \bar{c}$$

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, !A} \bar{d}$$

Bilateral co-structural rules derivable from the above:

$$\frac{!A \vdash B}{\vdash B} \bar{w}$$

$$\frac{!A \vdash B}{!A \vdash !B} \bar{c}$$

$$\frac{!A \vdash B}{A \vdash B} \bar{d}$$

Denotational intuitions for co-structural rules of DiLL

Co-weakening

Computing the value at 0 of a smooth map f

$$\frac{}{\vdash \delta_0 : ?A} \bar{w}$$

$$\frac{f : !A \vdash B}{\vdash f(0) : B} \bar{w}$$

Co-contraction

Summing in the domain of smooth maps

$$\frac{\vdash \Gamma, \phi : !A \quad \vdash \Delta, \psi : !A}{\vdash \Gamma, \Delta, \psi * \phi : !A} \bar{c}$$

$$\frac{f : !A, \vdash B}{(x, y) \mapsto f(x + y) : !A, !A \vdash !B} \bar{c}$$

Denotational intuitions for co-structural rules of DiLL

Co-dereplication

Differentiating

$$\frac{\vdash \Gamma, x : A}{\vdash \Gamma, D_0(-)(x) : !A} \bar{d}$$

$$\frac{\vdash \Delta, v : A}{\vdash \Delta, (f \mapsto D_0(f)(v)) : !A} \bar{d}$$

Given a vector v , $f \mapsto D_0(f)(v)$ is also a distribution

Cut-Elimination



Contraction and co-weakening

$$\frac{\frac{\vdash \Gamma, f : ?(A^\perp), g : ?(A^\perp)}{\vdash \Gamma, f \cdot g : ?(A^\perp)} c \quad \frac{\vdash}{\vdash \delta_0 : !A} \bar{w}}{\vdash \Gamma, f(0).g(0) : \mathbb{R}} \text{cut}$$

$\rightsquigarrow$

Contraction and co-weakening

$$\frac{\frac{\vdash \Gamma, f : ?(A^\perp), g : ?(A^\perp)}{\vdash \Gamma, f \cdot g : ?(A^\perp)} c \quad \frac{\vdash}{\vdash \delta_0 : !A} \bar{w}}{\vdash \Gamma, f(0).g(0) : \mathbb{R}} \text{cut}$$

$\rightsquigarrow$

$$\frac{\frac{\vdash \Gamma, f : ?(A^\perp), g : ?(A^\perp)}{\vdash \Gamma, f(0) : \mathbb{R}, g : ?(A^\perp)} \quad \frac{\vdash}{\vdash \delta_0 : !A} \bar{w}}{\vdash \Gamma, f(0) : \mathbb{R}, g : ?(A^\perp)} \text{cut} \quad \frac{\vdash}{\vdash \delta_0 : !A} \bar{w}}{(\vdash \Gamma, f(0) : \mathbb{R}, g(0) : \mathbb{R}) \equiv (\vdash \Gamma, f(0).g(0) : \mathbb{R})} \text{cut}$$

Tensor for scalar is multiplication, $\mathfrak{A} = \otimes = \cdot$ in $\mathbb{R}$

$$(f \cdot g)(0) = f(0)g(0) \quad h(0_A, 0_A) = h(0_{A \times A})$$

Weakening and co-contraction

$$\frac{\frac{\frac{\vdash \Gamma, \phi : !A \quad \vdash \Gamma', \psi : !A}{\vdash \Gamma, \Gamma', \phi * \psi : !A} \bar{c}}{\vdash \Delta, \Gamma, \Gamma', \phi * \psi(cst_1) : \mathbb{R}} \text{cut} \quad \frac{\vdash \Delta}{\vdash \Delta, cst_1 : A \Rightarrow \mathbb{R}} w$$

Weakening and co-contraction

$$\frac{\frac{\vdash \Gamma, \phi : !A \quad \vdash \Gamma', \psi : !A}{\vdash \Gamma, \Gamma', \phi * \psi : !A} \bar{c} \quad \frac{\vdash \Delta}{\vdash \Delta, \text{cst}_1 : A \Rightarrow \mathbb{R}} w}{\vdash \Delta, \Gamma, \Gamma', \phi * \psi(\text{cst}_1) : \mathbb{R}} \text{cut}$$

$\rightsquigarrow$

$$\frac{\frac{\vdash \Gamma, \phi : !A \quad \frac{\vdash \Delta}{\vdash \Delta, \text{cst}_1 : A \Rightarrow \mathbb{R}} w}{\vdash \Gamma, \Delta, \phi(\text{cst}_1) : \mathbb{R}} \text{cut}}{\frac{\vdash \Gamma, \Delta, \phi(\text{cst}_1) : \mathbb{R}, \text{cst}_1 : A \Rightarrow \mathbb{R} \quad \vdash \Gamma', \psi : !A}{\vdash \Delta, \Gamma, \Gamma', \psi(\text{cst}_1) : \mathbb{R}, \phi(\text{cst}_1) : \mathbb{R}} w} \text{cut}$$

$$\begin{aligned} (\phi * \psi)(\text{cst}_1) &= \phi(\text{cst}_1)\psi(\text{cst}_1) \\ (x, y) \mapsto 1 &\equiv (x \mapsto 1)(y \mapsto 1) \end{aligned}$$

Weakening and co-weakening

$$\frac{\frac{\frac{\vdash \Gamma}{\vdash \Gamma, cst_1 : A \Rightarrow \mathbb{R}} w \quad \frac{\frac{\vdash}{\vdash \delta_0 : !A} \bar{w}}{\vdash \Gamma, 1 : \mathbb{R}} \text{cut}}{\vdash \Gamma, 1 : \mathbb{R}} \rightsquigarrow$$

Weakening and co-weakening

$$\frac{\frac{\frac{\vdash \Gamma}{\vdash \Gamma, cst_1 : A \Rightarrow \mathbb{R}} w \quad \frac{\vdash}{\vdash \delta_0 : !A} \bar{w}}{\vdash \Gamma, 1 : \mathbb{R}} \text{cut} \rightsquigarrow \vdash \Gamma$$

(Co)-weakening and (co)-dereliction

$$\frac{\frac{\vdash \Gamma}{\vdash \Gamma, \textcolor{teal}{cst}_1 : A \Rightarrow \mathbb{R}} w \quad \frac{\vdash \Delta, \textcolor{brown}{v} : A}{\vdash \Delta, \textcolor{brown}{D}_0(-)(\textcolor{brown}{v}) : !A} \bar{d}}{\vdash \Gamma, \Delta, D_0(\textcolor{teal}{cst}_1)(\textcolor{brown}{v}) : \mathbb{R}} \text{cut} \rightsquigarrow$$

(Co)-weakening and (co)-dereliction

$$\frac{\frac{\vdash \Gamma}{\vdash \Gamma, \text{cst}_1 : A \Rightarrow \mathbb{R}} w \quad \frac{\vdash \Delta, v : A}{\vdash \Delta, D_0(-)(v) : !A} \bar{d}}{\vdash \Gamma, \Delta, D_0(\text{cst}_1)(v) : \mathbb{R}} \text{cut} \rightsquigarrow \mathbf{o}$$

$D_0(\text{cst}_1)(v) = 0$: Differentiating a constant function leads to 0

(Co)-weakening and (co)-dereliction

$$\frac{\frac{\vdash \Gamma}{\vdash \Gamma, \text{cst}_1 : A \Rightarrow \mathbb{R}} w \quad \frac{\vdash \Delta, v : A}{\vdash \Delta, D_0(-)(v) : !A} \bar{d}}{\vdash \Gamma, \Delta, D_0(\text{cst}_1)(v) : \mathbb{R}} \text{cut} \rightsquigarrow \mathbf{o}$$

$D_0(\text{cst}_1)(v) = 0$: Differentiating a constant function leads to 0

$$\frac{\frac{\vdash}{\vdash \delta_0 : !A} \bar{w} \quad \frac{\vdash \Gamma, \ell : A \multimap \mathbb{R}}{\vdash \Gamma, \ell : A \Rightarrow \mathbb{R}} d}{\vdash \Gamma, \ell(0) : \mathbb{R}} \text{cut} \rightsquigarrow$$

(Co)-weakening and (co)-dereliction

$$\frac{\frac{\vdash \Gamma}{\vdash \Gamma, \text{cst}_1 : A \Rightarrow \mathbb{R}} w \quad \frac{\vdash \Delta, v : A}{\vdash \Delta, D_0(-)(v) : !A} \bar{d}}{\vdash \Gamma, \Delta, D_0(\text{cst}_1)(v) : \mathbb{R}} \text{cut} \rightsquigarrow \mathbf{o}$$

$D_0(\text{cst}_1)(v) = 0$: *Differentiating a constant function leads to 0*

$$\frac{\frac{\vdash}{\vdash \delta_0 : !A} \bar{w} \quad \frac{\vdash \Gamma, \ell : A \multimap \mathbb{R}}{\vdash \Gamma, \ell : A \Rightarrow \mathbb{R}} d}{\vdash \Gamma, \ell(0) : \mathbb{R}} \text{cut} \rightsquigarrow \mathbf{o}$$

$\ell(0) = 0$ if ℓ is linear

Proofs of DiLL

Formulas:

$$A, B := a \mid a^\perp \mid 0 \mid 1 \mid \top \mid \perp \mid A \otimes B \mid A \wp B \mid A \oplus B \mid A \times B \mid !A \mid ?A$$

Proofs

- ▶ **Sums of proofs** generated by multiplicative additive rules for $\otimes, \wp, \oplus, \&$ as well as structural and co-structural rules.
- ▶ Every proof admits a zero-proof !

Contraction and co-dereliction

$$\frac{\frac{\frac{\vdash \Gamma, f : A \Rightarrow \mathbb{R}, g : A \Rightarrow \mathbb{R}}{\vdash \Gamma, f \cdot g : A \Rightarrow \mathbb{R}} c \quad \frac{\vdash \Delta, v : A}{\vdash \Delta, D_0(-)(v) : !A} \bar{d}}{\vdash \Gamma, \Delta, D_0(f \cdot g)(v) : \mathbb{R}} \text{cut} \rightsquigarrow$$

Contraction and co-dereliction

$$\begin{array}{c}
 \frac{\frac{\vdash \Gamma, f : A \Rightarrow \mathbb{R}, g : A \Rightarrow \mathbb{R}}{\vdash \Gamma, f \cdot g : A \Rightarrow \mathbb{R}} c \quad \frac{\vdash \Delta, v : A}{\vdash \Delta, D_0(-)(v) : !A} \bar{d}}{\vdash \Gamma, \Delta, D_0(f \cdot g)(v) : \mathbb{R}} \text{cut} \rightsquigarrow \\
 \\
 \frac{\vdash \Gamma, f : A \Rightarrow \mathbb{R}, g : A \Rightarrow \mathbb{R} \quad \frac{\vdash \Delta, v : A}{\vdash \Delta, D_0(-)(v) : !A} \bar{d}}{\vdash \Gamma, \Delta, D_0(f) : (v) : \mathbb{R}, g : A \Rightarrow \mathbb{R}} \text{cut} \quad \frac{\vdash}{\vdash \delta_0 : !A} \bar{w}}{\vdash \Delta, \Gamma, D_0(f) : (v) : \mathbb{R}, g(0) : \mathbb{R}} \text{cut}
 \end{array}$$

Contraction and co-dereliction

$$\begin{array}{c}
 \frac{\frac{\vdash \Gamma, f : A \Rightarrow \mathbb{R}, g : A \Rightarrow \mathbb{R}}{\vdash \Gamma, f \cdot g : A \Rightarrow \mathbb{R}} \text{ } c \quad \frac{\vdash \Delta, v : A}{\vdash \Delta, D_0(-)(v) : !A} \bar{d}}{\vdash \Gamma, \Delta, D_0(f \cdot g)(v) : \mathbb{R}} \text{ } \bar{d} \text{ } \text{cut} \rightsquigarrow \\
 \\
 \frac{\vdash \Gamma, f : A \Rightarrow \mathbb{R}, g : A \Rightarrow \mathbb{R} \quad \frac{\vdash \Delta, v : A}{\vdash \Delta, D_0(-)(v) : !A} \bar{d}}{\vdash \Gamma, \Delta, D_0(f) : (v) : \mathbb{R}, g : A \Rightarrow \mathbb{R}} \text{ } \bar{d} \text{ } \text{cut} \quad \frac{\vdash}{\vdash \delta_0 : !A} \bar{w}}{\vdash \Delta, \Gamma, D_0(f) : (v) : \mathbb{R}, g(0) : \mathbb{R}} \text{ } \bar{w} \text{ } \text{cut} \\
 \\
 + \frac{\vdash \Gamma, f : A \Rightarrow \mathbb{R}, g : A \Rightarrow \mathbb{R} \quad \frac{\vdash \Delta, v : A}{\vdash \Delta, D_0(-)(v) : !A} \bar{d}}{\vdash \Gamma, \Delta, g : A \Rightarrow \mathbb{R}, D_0(g) : (v) : \mathbb{R}} \text{ } \bar{d} \text{ } \text{cut} \quad \frac{\vdash}{\vdash \delta_0 : !A} \bar{w}}{\vdash \Delta, \Gamma, f(0) : \mathbb{R}, D_0(g) : (v) : \mathbb{R}} \text{ } \bar{w} \text{ } \text{cut} \\
 \\
 D_0(f\dot{g}) = D_0(f)\dot{g}(0) + f(0)\dot{D}_0(g)
 \end{array}$$

Dereliction and co-contraction

$$\frac{\frac{\frac{\vdash \Gamma, \phi : !A \quad \vdash \Gamma', \psi : !A}{\vdash \Gamma, \Gamma', \phi * \psi : !A} \bar{c} \quad \frac{\vdash \Delta, \ell : A \multimap \mathbb{R}}{\vdash \Delta, \ell : A \Rightarrow \mathbb{R}} d}{\vdash \Gamma, \Gamma', \Delta, (\phi * \psi)(\ell)} \text{cut} \rightsquigarrow$$

Dereliction and co-contraction

$$\begin{array}{c}
 \frac{\frac{\frac{\vdash \Gamma, \phi : !A \quad \vdash \Gamma', \psi : !A}{\vdash \Gamma, \Gamma', \phi * \psi : !A} \bar{c} \quad \frac{\frac{\vdash \Delta, \ell : A \multimap \mathbb{R}}{\vdash \Delta, \ell : A \Rightarrow \mathbb{R}} d}{\vdash \Gamma, \Gamma', \Delta, (\phi * \psi)(\ell)} \text{cut} \quad \sim \\
 \\
 \frac{\frac{\frac{\vdash \Gamma, \phi : !A \quad \frac{\frac{\vdash \Delta, \ell : A \multimap B}{\vdash \Delta, \ell : A \Rightarrow B} d}{\vdash \Gamma, \Delta, \phi(\ell) : \mathbb{R} \wp B} \text{cut}}{\vdash \Gamma, \Delta, \phi(\ell) : \mathbb{R} \wp B, \text{cst}_1 A \Rightarrow \mathbb{R}} w \quad \vdash \Gamma', \psi : !A}{\vdash \Gamma', \Gamma, \Delta, \phi(\ell) : \mathbb{R} \wp B, \psi(\text{cst}_1) : \mathbb{R}} \text{cut}
 \end{array}$$

Dereliction and co-contraction

$$\begin{array}{c}
 \frac{\frac{\frac{\vdash \Gamma, \phi : !A \quad \vdash \Gamma', \psi : !A}{\vdash \Gamma, \Gamma', \phi * \psi : !A} \bar{c} \quad \frac{\frac{\vdash \Delta, \ell : A \multimap \mathbb{R}}{\vdash \Delta, \ell : A \Rightarrow \mathbb{R}} d}{\vdash \Gamma, \Gamma', \Delta, (\phi * \psi)(\ell)} \text{cut} \quad d}{\vdash \Gamma, \Gamma', \Delta, \phi(\ell) : \mathbb{R} \wp B} \text{cut} \quad \rightsquigarrow \\
 \\
 \frac{\frac{\frac{\vdash \Gamma, \phi : !A \quad \frac{\frac{\vdash \Delta, \ell : A \multimap B}{\vdash \Delta, \ell : A \Rightarrow B} d}{\vdash \Gamma, \Delta, \phi(\ell) : \mathbb{R} \wp B} \text{cut}}{\vdash \Gamma, \Delta, \phi(\ell) : \mathbb{R} \wp B, \text{cst}_1 A \Rightarrow \mathbb{R}} w \quad \vdash \Gamma', \psi : !A}{\vdash \Gamma', \Gamma, \Delta, \phi(\ell) : \mathbb{R} \wp B, \psi(\text{cst}_1) : \mathbb{R}} \text{cut} \\
 \\
 + \quad \frac{\frac{\frac{\vdash \Gamma', \psi : !A \quad \frac{\frac{\vdash \Delta, \ell : A \multimap B}{\vdash \Delta, \ell : A \Rightarrow B} d}{\vdash \Gamma', \Delta, \psi(\ell) : \mathbb{R} \wp B} \text{cut}}{\vdash \Gamma', \Delta, \psi(\ell) : \mathbb{R} \wp B, \text{cst}_1 A \Rightarrow \mathbb{R}} w \quad \vdash \Gamma, \phi : !A}{\vdash \Gamma, \Gamma', \Delta, \psi(\ell) : \mathbb{R} \wp B, \phi(\text{cst}_1) : \mathbb{R}} \text{cut} \\
 \\
 \ell(x + y) = \ell(x) + \ell(y)
 \end{array}$$

Cut-elimination in a nutshell

Cut-elimination in DiLL is symmetric

- ▶ Cut-elimination of co-structural rules between them are alike cut-eliminations of structural rules.
- ▶ Cut-elimination between structural and co-structural rules are alike.
- ▶ $\bar{d}; w = 0$ and $\bar{w}; d = 0$
- ▶ $\bar{w}; w = id$ and $\bar{d}; d = id$
- ▶ $\bar{c}; w = w \otimes w$ and $\bar{w}; c = \bar{w} \otimes \bar{w}$
- ▶ $c; \bar{d} = \bar{w} \otimes \bar{d} + \bar{d} \otimes w$ and $d; \bar{c} = w \otimes d + d \otimes w$
- ▶ $(!, w, c, \bar{w}, \bar{c})$ is a commutative bialgebra

This can be made more synthetic in Categorical Models of DiLL.

See Jean-Simon Pacaud Lemay's talk on Tuesday !

Finitary differential Linear Logic

The first version by Ehrhard and Regnier in 2006:

$$\frac{\vdash \Gamma}{\vdash \Gamma, \textcolor{teal}{cst}_1 : ?A} w$$

$$\frac{\vdash \Gamma, \textcolor{teal}{f} : ?A, \textcolor{teal}{g} : ?A}{\vdash \Gamma, \textcolor{teal}{f}.g : ?A} c$$

$$\frac{\vdash \Gamma, \textcolor{teal}{\ell} : A}{\vdash \Gamma, \textcolor{teal}{\ell} : ?A} d$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, \textcolor{brown}{\delta}_0 : !A} \bar{w}$$

$$\frac{\vdash \Gamma, \textcolor{brown}{\phi} : !A \quad \vdash \Delta, \textcolor{brown}{\psi} : !A}{\vdash \Gamma, \Delta, \textcolor{brown}{\psi} * \textcolor{brown}{\phi} : !A} \bar{c}$$

$$\frac{\vdash \Gamma, \textcolor{brown}{x} : A}{\vdash \Gamma, \textcolor{brown}{D}_0(-)(x) : !A} \bar{d}$$

The Prom Queen

Exponential rules of Linear Logic (Resources)

$$\frac{\vdash \Gamma}{\vdash \Gamma, \text{cst}_1 : ?A} w \quad \frac{\vdash \Gamma, f : ?A, g : ?A}{\vdash \Gamma, f.g : ?A} c \quad \frac{\vdash \Gamma, \ell : A}{\vdash \Gamma, \ell : ?A} d \quad \boxed{\frac{!\Gamma \vdash x : A}{!\Gamma \vdash \delta_x : !A} p}$$

Exponential rules added by Differential Linear Logic (Distributions)

$$\frac{\vdash \Gamma}{\vdash \Gamma, \delta_0 : !A} \bar{w} \quad \frac{\vdash \Gamma, \phi : !A \quad \vdash \Delta, \psi : !A}{\vdash \Gamma, \Delta, \psi * \phi : !A} \bar{c} \quad \frac{\vdash \Gamma, x : A}{\vdash \Gamma, D_0(-)(x) : !A} \bar{d}$$

The promotion rule $p : !A \rightarrow !!A$ $\delta_x \mapsto \delta_{\delta_x}$:

- Makes $(!, d, p)$ a co-monad : $p; d = \text{id}$.
- What about the cut-elimination between p and $\bar{d}$?

Unchained Melody

Differentiation is non-functorial, hence the chain rule

$$D_{\mathbf{0}}(g \circ f) = D_{f(\mathbf{0})}(g) \circ D_{\mathbf{0}}(f)$$

See Michele Pagani's talk on Thursday for more version of the chain rule !

Cut-elimination between promotion and co-dereliction

$$\frac{\frac{!A \vdash B}{!A \vdash !B} \text{ p} \quad \frac{\Gamma \vdash A}{\Gamma \vdash !A} \bar{\text{d}}}{A \vdash B} \text{ cut}$$

$\rightsquigarrow$

...

Cut-elimination between promotion and co-dereliction

$$\frac{\frac{f : (x : !A \vdash f(x)) : B}{\delta_f : x \mapsto \delta_{f(x)} !A \vdash !B} \text{p} \quad \frac{\Gamma \vdash v : A}{\Gamma \vdash D_0(-)(v) : !A} \bar{\text{d}}}{\Gamma \vdash D_0(\delta_f)(v) : !B} \text{cut}$$

$\rightsquigarrow$

$$\begin{aligned} D_0(\delta_f)(v) &= (g \mapsto D_0(g \circ f)(v)) \\ &= (g \mapsto D_{f(0)}(g)(D_0(f)(v))) \end{aligned}$$

How to interpret $D_{f(0)}(D_0(f)(v))$?

Back to distributions

Example of distributions:

- For $x : A$, $\delta_x = (f \mapsto f(x)) : !A$
- For $v : A$, $D_0(-)(v) = (f \mapsto D_0(f)(v)) : !A$

Convolutions of distributions

- $\phi * \psi := f \mapsto \phi(x \mapsto \psi(y \mapsto f(x + y)))$
- E.g. $\delta_x * \delta_y = \delta_{x+y}$
- E.g. $D_0(-)(v) * \delta_x = D_x(-)(v)$
- E.g. $D_0(-)(v) * D_0(-)(v) = D_0^2(-)(v)$

The we can compute the chain rule with the basic DiLL operations:

$$\begin{aligned} D_{f(0)}(g)(D_0(f)(v)) &= \delta_{f(0)} * D_{(-)}(D_0(f)(v)) \\ &= (f; \bar{w}; p) * \bar{d}; (\bar{d}; f) \\ \bar{d}; p &= (\bar{w}; p) \otimes (\bar{d}; \bar{d}); \bar{c} \end{aligned}$$

Cut-elimination between promotion and co-dereliction

$$\frac{\frac{f : (x : !A \vdash f(x)) : B}{\delta_f : x \mapsto \delta_{f(x)} : !A \vdash !B} \text{p} \quad \frac{\Gamma \vdash v : A}{\Gamma \vdash D_0(-)(v) : !A} \bar{\text{d}}}{\Gamma \vdash D_0(\delta_f)(v) : !B} \text{cut}$$

$\rightsquigarrow$

$$\frac{\frac{(x : !A \vdash f(x)) : B \quad \frac{\Gamma \vdash v : A}{\Gamma \vdash D_0(-)(v) : !A} \bar{\text{d}}}{\Gamma \vdash D_0(f)(v) : B} \text{cut} \quad \frac{\overline{\vdash \delta_0 : !A}}{\overline{\vdash f(0)) : B}} \bar{\text{w}} \quad \frac{\overline{\vdash \delta_{f(0)} : !B}}{\overline{\vdash \delta_{f(0)} : !B}} \text{p}}{\Gamma \vdash D_0(\delta_f)(v) = \delta_{f(0)} * D_{(-)}(D_0(f)(v)) : !B} \bar{\text{c}}$$

The chain-rule with contexts:

$$id_{!A} \otimes \bar{\text{d}}_A; \bar{\text{c}}_A; \text{p}_A = \text{c}_A \otimes \bar{\text{d}}_A; id_{!A} \otimes \bar{\text{c}}_A; \text{p}_A \otimes \bar{\text{d}}_{!A}; \bar{\text{c}}_{!A}$$

Sum-up on the Syntax

- ▶ A **symmetrization** of LL exponential rules,
- ▶ Which magically gives us everything we need to compute basic **differentials at higher-order**,
- ▶ That **challenges the resources** interpretation,
- ▶ First and maybe better expressed in proof-nets, and categories.



Thomas Ehrhard, Laurent Regnier. Differential interaction nets. Theoretical Computer Science, Elsevier. 2006



Michele Pagani, The Cut-Elimination Theorem for Differential Nets with Boxes, (TLCA 2009)



Paolo Tranquilli, Confluence of Pure Differential Nets with Promotion, CSL 2009:



Thomas Ehrhard. A semantical introduction to differential linear logic. 2011.

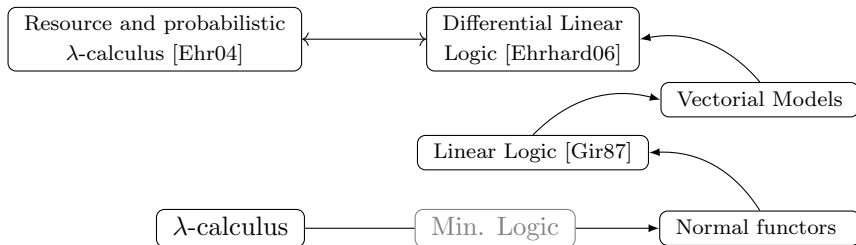
Denotational Semantics



It's a maths world.

Reverse Denotational Semantics

Programs	Logic	Semantics
$\text{fun } (x:A) \rightarrow (t:B)$	Proof of $A \vdash B$	$f : A \rightarrow B.$
Types	Formulas	Objects
Execution	Cut-elimination	Equality



The Relational Model

This historical model of Linear Logic and Differential Linear Logic expresses perfectly the notion of resources and linear argument at stakes.

- ▶ Formulas are interpreted by sets $\{a_1, \dots, a_n\}$,
- ▶ Proofs are interpreted relations $\llbracket A \vdash B \rrbracket = R \subseteq A \times B$,
- ▶ $!A := \mathfrak{M}_f(A)$ is the set of *finite multi-sets* of A ,
- ▶ $d_A = \{(\{a\}, a) \mid a \in A\}$ and $\bar{d}_A = \{(a, \{a\},) \mid a \in A\}$,
- ▶ $c_A = \{(m_1 \cup m_2, (m_1, m_2)) \mid m_1, m_2 \in \mathfrak{M}_f(A)\}$ and $\bar{c}_A = \{(m_1 \cup m_2, (m_1, m_2)) \mid m_1, m_2 \in \mathfrak{M}_f(A)\}$
- ▶ $w_A = \{(\emptyset, *)\}$ and $\bar{w}_A = \{(ast, \emptyset)\}$
- ▶ $p_A = \{(m_1 \cup \dots \cup m_n, [m_1, \dots, m_n]) \mid n \in \mathbb{N}, m_i \in !A\}$

See Guy Mccusker's talk on Tuesday!

Köthe spaces: it's all about the sum

- Köthe spaces := sequences spaces studied in functional analysis for their good *duality* properties, a.k.a perfect sequences spaces.
- The primary source of inspiration to build DiALL

For $E \subset \mathbb{R}^{\mathbb{N}}$: $E^{\perp} := \{\alpha \in \mathbb{R}^{\mathbb{N}} \mid \forall \lambda \in E, \sum_n |\lambda_n \alpha_n| < \infty\}$.

Definition

- A **perfect sequence space** is the data (X, E_X) of a subset $X \subset \mathbb{N}$ and $E_X \subset \mathbb{K}^X$ such that $E_X^{\perp\perp} = E_X$.
- The space $E \multimap F$ of **linear continuous maps** from E_X to F_Y correspond to the subset $\mathbb{K}^{X \times Y}$ of all M such that the sum:

$$\sum_{i,j} M_{i,j} x_i y'_j$$

is absolutely converging for all $x \in E$ and $y' \in F^{\perp}$.

There are topological notions at stakes in Köthe spaces.

Differentiable maps in Köthe spaces

Exponents. If μ is a finite multiset of X and $x \in E$, we write:

$$x^\mu = \prod_n x_n^{\mu(n)}.$$

Power series We define the set of scalar entire maps $E \Rightarrow \mathbb{K}$ as the vector space of matrices $M \in \mathbb{K}^{\mathcal{M}(X)}$ such that for all $x \in E$, the following sum converges absolutely:

$$f(x) = \sum_{\mu \in \mathcal{M}(X)} M_\mu x^\mu.$$

Distributions

$$!E := (E \Rightarrow \mathbb{K})^\perp.$$

Power series are differentiable:

$$\bar{d}_E(x) : (M : E \Rightarrow \mathbb{K}) \mapsto \sum_{a \in X} M_{\{a\}} x_a.$$



T. Ehrhard. On Köthe sequence spaces and linear logic. MSCS, 2002.



T. Ehrhard. Finiteness spaces. MSCS. 2005.

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$$f(x) = \sum_{\mu \in \mathcal{M}(X)} M_\mu x^\mu.$$

Distributions

$$!E := (E \Rightarrow \mathbb{K})^\perp.$$

Power series are differentiable:

$$\mathbf{d}_X(m, x) := \delta_{m, [x]} \quad \bar{\mathbf{d}}_X := (x, m) := \delta_{m, [x]}$$

$$\mathbf{c}_X(m, (m_1, m_2)) = \delta_{m, m_1 \sqcup m_2} \quad \bar{\mathbf{c}}_X((m_1, m_2), m) = \binom{|m_1| + |m_2|}{|m_1|} \delta_{m, m_1 \sqcup m_2}$$



T. Ehrhard. On Köthe sequence spaces and linear logic. MSCS, 2002.



T. Ehrhard. Finiteness spaces. MSCS. 2005.

Convenient vector spaces

Convenient vector spaces has been studied by functional analysts to provide a *infinite dimensional point of view on analysis*.

They form a smooth model of **Intuitionistic** DiLL.

- Vector spaces E endowed with a *bornology* $\mathcal{B}$ making them Mackey-Complete.

$$\forall B \in \mathcal{B}, E_B = \{\lambda x | x \in B\} \text{ is complete for } \|x\| = \inf\{\lambda | \frac{x}{\lambda} \in B\}$$

- Proofs are interpreted by *linear bounded maps*.
- Functions $f : E \rightarrow F$ are smooth when they are smooth when precomposed by all smooth curves

$$\forall c \in \mathcal{C}^\infty(\mathbb{R}, E), f \circ c \in \mathcal{C}^\infty(\mathbb{R}, F)$$

Convenient vector spaces

Exponentials: discretization instead of approximation

$$!E = \overline{\langle \delta_x | x \in E \rangle}$$

Interpreting (Intuitionistic) DiLL

$$d_E : \delta_x \in !E \mapsto x \quad \bar{d} : v \mapsto \lim_{t \rightarrow 0} \frac{\delta_{tv} - \delta_0}{t}$$

$$\bar{c} : \delta_x \otimes \delta_y \in !E \otimes !E \mapsto \delta_{x+y} \quad c : \delta_x \mapsto \delta_x \otimes \delta_x$$

$$p : \delta_x \rightarrow \delta_{\delta_x}$$



P.W. Michor, A. Kriegl, The Convenient Setting of Global Analysis, 1997



R. Blute, T. Ehrhard, C. Tasson. A Convenient Differential Category. Cahier de Topologie et Géométrie Différentielle Catégoriques, 2011.



M.K., C. Tasson. Mackey-Complete Spaces as a Quantitative model of DiLL, MSCS, 2018



Y. Dabrowski, M. K., Models of Linear Logic based on the Schwartz ε -product, TAC, 2020

The Differential Lambda-Calculus

$$D \left(\text{crocodile} \right) \cdot s$$

Differential λ -calculus

A more general version of resources calculus, without the multisets in the syntax but using partial derivatives intuitions.

$D(\lambda x.t)$ is the **linearization** of $\lambda x.t$, it substitute x linearly, and then it remains a term t' where x is free.

Syntax:

$$\begin{aligned}\Lambda^d : S, T, U, V &::= 0 \mid s \mid s + T \\ \Lambda^s : s, t, u, v &::= x \mid \lambda x.s \mid sT \mid \mathbf{D}s \cdot t\end{aligned}$$

Operational Semantics:

$$\begin{aligned}(\lambda x.s)T &\rightarrow_{\beta} s[T/x] \\ \mathbf{D}(\lambda x.s) \cdot t &\rightarrow_{\beta_D} \lambda x. \frac{\partial s}{\partial x} \cdot t\end{aligned}$$

where $\frac{\partial s}{\partial x} \cdot t$ is the **linear substitution** of x by t in s .

Partial Derivatives

$$D(\lambda x.s) \cdot t \rightarrow_{\beta_D} \lambda x. \frac{\partial s}{\partial x} \cdot t$$

Consider $f : \mathbb{R}^n \rightarrow \mathbb{R}$, and $(e_i)_i$ the canonical basis in $\mathbb{R}^n$:

$$\frac{\partial f}{\partial e_i} \cdot (v) := D_{-} f(v \cdot e_i)$$

- In Differential Lambda-Calculus we will have to operate the linear substitution (e.g. the partial derivative) *before* operating the non-linear substitution (e.g. fixing the point in which we want to differentiate the function).

Back to the resources

Linearity is about resources: A proof/program is *linear* iff it uses only once its hypotheses/argument.

Linear	Non-linear
$A \vdash A \vee B$	$A \vdash A \wedge A$
$\lambda f \lambda x. fxx$	$\lambda x. \lambda f. fxx$

Differentiation is about making a λ -term linear :

$\rightsquigarrow$ about making a λ -term have a linear usage of its arguments.

$$\lambda x \lambda f. fxx \rightsquigarrow ?$$

Back to the resources

Linearity is about resources: A proof/program is *linear* iff it uses only once its hypotheses/argument.

Linear	Non-linear
$A \vdash A \vee B$	$A \vdash A \wedge A$
$\lambda f \lambda x. f x x$	$\lambda x. \lambda f. f x x$

Differentiation is about making a λ -term linear :

$\rightsquigarrow$ about making a λ -term have a linear usage of its arguments.

$$D(\lambda x \lambda f. f x x) \cdot v := \lambda x. \lambda f. v x + ?$$

Back to the resources

Linearity is about resources: A proof/program is *linear* iff it uses only once its hypotheses/argument.

Linear	Non-linear
$A \vdash A \vee B$	$A \vdash A \wedge A$
$\lambda f \lambda x. f x x$	$\lambda x. \lambda f. f x x$

Differentiation is about making a λ -term linear :

$\rightsquigarrow$ about making a λ -term have a linear usage of its arguments.

$$D(\lambda x \lambda f. f x x) \cdot v := \lambda x. \lambda f. v x + \lambda x. \lambda f. D x v$$

The linear substitution ...

... which is not exactly a substitution

$$\frac{\partial y}{\partial x} \cdot t = \begin{cases} t & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\partial}{\partial x}(tu) \cdot s = \left(\frac{\partial t}{\partial x} \cdot s\right)u + (Dt \cdot \left(\frac{\partial u}{\partial x} \cdot s\right))u$$

$$\frac{\partial}{\partial x}(\lambda y.s) \cdot t = \lambda y. \frac{\partial s}{\partial x} \cdot t$$

$$\frac{\partial}{\partial x}(Ds \cdot u) \cdot t = D\left(\frac{\partial s}{\partial x} \cdot t\right) \cdot u + Ds \cdot \left(\frac{\partial u}{\partial x} \cdot t\right)$$

$$\frac{\partial 0}{\partial x} \cdot t = 0$$

$$\frac{\partial}{\partial x}(s + u) \cdot t = \frac{\partial s}{\partial x} \cdot t + \frac{\partial u}{\partial x} \cdot t$$

- ▶ $\frac{\partial s}{\partial x} \cdot t$ represents s where x is linearly (i.e. one time) substituted by t .
- ▶ Contrarily to maths, the linear variable has to be substituted first.

Symmetries in DiLL



DiLL est dans Laplace

Do you remember the Laplace transformation?

$$\mathcal{L} : f \mapsto x \mapsto \int_0^\infty f(t) e^{-xt} dt$$

DiLL est dans Laplace

Do you remember the Laplace transformation?

$$\mathcal{L} : f \mapsto x \mapsto \int_0^\infty f(t) e^{-x t} dt$$

That's not very higher-order

$$\mathcal{L} : \begin{cases} !E & \rightarrow ?E \\ \phi & \mapsto ((\ell : E') \mapsto \phi((y : E) \mapsto e^{<\ell|y>})) \end{cases}$$

The **Laplace Transformation** is the reason behind the symmetry of DiLL:

$$\mathcal{L}(\overline{w}, \overline{c}, \overline{d}) = w, c, d$$

And p ... ?



M. K. and J.-S. Pacaud Lemay, Laplace Distributors and Laplace Transformations for Differential Categories, FSCD 2024

The missing rule of Differential Linear Logic

Digging $p : !A \rightarrow !!A$:

- ▶ $p; d = id.$
- ▶ $p; c = c; p \otimes p$
- ▶ $\bar{d}; p = \bar{w} \otimes \bar{d}; p \otimes \bar{d}; \bar{c}$

Co-digging $\bar{p} : !!A \rightarrow !A$: $g : !A \Rightarrow !A$

- ▶ $\bar{d}; \bar{p} = id \quad D_0(g) = id.$
- ▶ $\bar{c}; \bar{p} = \bar{p} \otimes \bar{p}; \bar{c} \quad g(x + y) = g(x) * g(y)$
- ▶ $\bar{p}; d = c; \bar{p} \otimes d; w \otimes d$ It works!

The co-digging is an exponential function acting on distributions:

$$\bar{p} : \delta_\phi \mapsto \sum_n \frac{1}{n!} \phi^{*n}$$

The monadic rules:

$$\bar{d}; \bar{p} = id \quad \forall v, \bar{p}(\delta_{D_0(-)(v)}) = \delta_v \quad \forall v, \forall f, \sum_n \frac{1}{n!} D_0^{(n)} f(v) = f(v)$$

The **co-digging** characterizes **Taylor approximation** through its monadic rules.



M. K. and J.-S. Pacaud Lemay, Taylor as a monad in models of DiLL, LICS 2023

Thank you for listening ! Questions ?