Differential Categories

A Tutorial

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Quick Hello!

- Full Name: Jean-Simon Pacaud Lemay, please feel free to call me **JS**
- I'm from Canada, Quebec (so my first and main language is not english but french!)
- I'm a lecturer/assistant professor at **Macquarie University** (Sydney, Australia – so I'm very jetlagged!)
- I'm a category theorist, and I study:

  • **Differential Categories**
  • **Tangent Categories**
  • Differential Geometry, Algebraic Geometry, Differential Algebras
  • Traced Monoidal Categories
  • Restriction Categories
  • Other stuff...

If you find differential categories interesting and would like to chat/work together or even visit our category theory group at Macquarie: feel free to come to talk to me or reach out by email!

The Differential Category World: The Four Tomes

Differential Categories

Blute, Cockett, Seely - 2006

Cartesian Differential Categories

Blute, Cockett, Seely - 2009

Differential Restriction Categories

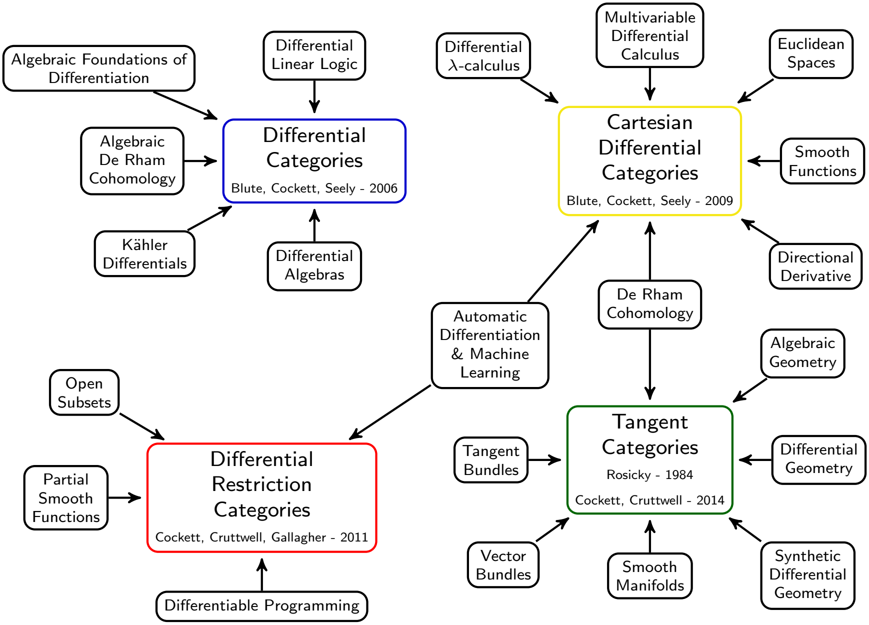
Cockett, Cruttwell, Gallagher - 2011

Tangent Categories

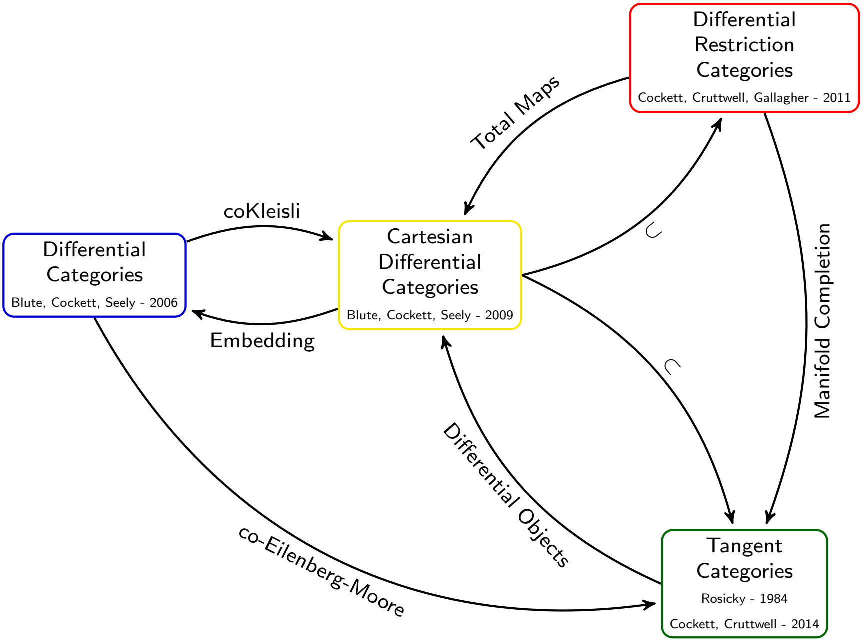
Rosicky - 1984

Cockett, Cruttwell - 2014

The Differential Category World: A Taster



The Differential Category World: It's all connected!



Today's story

First Half: **Differential Categories**

- Categorical foundations of the algebraic properties of differentiation (derivations, Kähler differentials, differential algebras, etc.)
- Categorical semantics of Differential Linear Logic

Second Half: **Cartesian Differential Categories**

- Categorical foundations of differential calculus over Euclidean spaces
- Categorical semantics of differential λ-calculus

Very briefly at the end: **Differential Restriction Categories** and **Tangent Categories**.

Differential Categories

- **Differential Categories**: Categorical semantics of Differential Linear Logic
- **Codifferential Categories**: Categorical foundations of the algebraic properties of differentiation (derivations, Kähler differentials, differential algebras, etc.)

Some introductory references:

Blute, R., Cockett, R., Seely, R.A.G. **Differential Categories** (2006)

Blute, R., Cockett, R., Seely, R.A.G., Lemay, J.-S. P. **Differential categories revisited**. (2019)

Ehrhard, T. **An introduction to differential linear logic: proof-nets, models and antiderivatives**. (2018)

Fiore, M. **Differential structure in models of multiplicative biadditive intuitionistic linear logic** (2007)

Differential Categories - Definition

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- An *additive* symmetric monoidal category,
- With a *differential* modality = coalgebra modality equipped with a deriving transformation

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Notation for Symmetric Monoidal Categories

For a symmetric monoidal category we denote:

- The underlying category as $\mathcal{L}$
- The monoidal product by $\otimes$
- The monoidal unit by I
- The symmetry isomorphism by $\sigma_{A,B} : A \otimes B \to B \otimes A$
- For simplicity we work in the *strict* setting, so the unitors and the associators of the monoidal product are identities.

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- For simplicity we work in the *strict* setting, so the unitors and the associators of the monoidal product are identities.
- Being *closed* does not necessarily play a role in the definition of differential categories. So we will not assume it here either...
- You don't need products either in the definition of differential categories... but we will add them in later!

Main Examples

Example

Let k be a field and and let VEC_k to be the category of all k-vector spaces and k-linear maps between them. VEC_k is symmetric monoidal category where:

- The monoidal structure is given by the tensor product of vector spaces $\otimes$ and the unit is k.

Example

Let REL be the category of sets and relations. Objects are sets X, and maps $R : X \to Y$ are subsets $R \subseteq X \times Y$. REL is a symmetric monoidal category where:

- The monoidal structure is given by the Cartesian product of sets.
  - Unit: $\{*\}$
  - Tensor product of objects $X \otimes Y := X \times Y$
  - Tensor product of relations $R \subset X \times Y$ and $S \subseteq A \times B$ is $R \otimes S := \{((x,a),(y,b)) \mid (x,y) \in R, (y,b) \in S\} \subseteq (X \times A) \times (Y \times B)$.

Differential Categories - Definition

A **differential category** is:

- An *additive* symmetric monoidal category,
- With a *differential* modality = **coalgebra modality** equipped with a deriving transformation

Coalgebra Modalities

- Coalgebra modalities help interpret the exponential modality ! in the categorical semantics of Linear Logic.
- But they don't capture the whole story, for that you need a *monoidal* coalgebra modality (also called a linear exponential modality). We will talk about these later.

- Coalgebra modalities help interpret the exponential modality ! in the categorical semantics of Linear Logic.
- But they don't capture the whole story, for that you need a *monoidal* coalgebra modality (also called a linear exponential modality). We will talk about these later.
- But why consider coalgebra modalities?

  Answer: Because you don't necessarily need the monoidal aspect to work with differentiation. And there any interesting examples of differential categories whose coalgebra modality is not monoidal...

Coalgebra Modality

A **coalgebra modality** ! on a symmetric monoidal category consists of:

- An endofunctor $! : \mathcal{L} \to \mathcal{L}$
- Four natural transformations:

$$\underset{\textit{Digging}}{\mathsf{p}_A : !A \to !!A} \qquad \underset{\textit{Dereliction}}{\mathsf{d}_A : !A \to A} \qquad \underset{\textit{Contraction}}{\mathsf{c}_A : !A \to !A \otimes !A} \qquad \underset{\textit{Weakening}}{\mathsf{w}_A : !A \to I}$$

such that:

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such that:

- $(!, \mathsf{p}, \mathsf{d})$ is a comonad
- $(!A, \mathsf{c}_A, \mathsf{w}_A)$ is a cocommutative comonoid
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The dual notion is called an **algebra modality** S, where in particular S is a monad and each $\mathsf{S}(A)$ is a commutative monoid.

Coalgebra Modality – Naive Intuition

Let us naively suppose that $I = \mathbb{R}$, and we have some notion of "smooth function" $A \to \mathbb{R}$, with $\mathcal{C}^\infty(A, \mathbb{R})$ be the set of smooth functions.

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- Elements of $!A$ can be thought of as *distributions*, so linear maps $\mathcal{C}^\infty(A, \mathbb{R}) \multimap \mathbb{R}$.
- For every $x \in A$, we have the Dirac distribution $\delta_x \in !A$ which evaluates a smooth function at x, $\delta_x(f) = f(x)$. Then the coalgebra modality structural maps are given by:

$$\mathsf{p}_A(\delta_x) = \delta_{\delta_x} \qquad \mathsf{d}_A(\delta_x) = x \qquad \mathsf{c}_A(\delta_x) = \delta_x \otimes \delta_x \qquad \mathsf{w}_A(\delta_x) = 1$$

- Dually, for an algebra modality, we think of $\mathsf{S}(A)$ as a subalgebra of smooth functions, $\mathsf{S}(A) \subseteq \mathcal{C}^\infty(A, \mathbb{R})$. The monad structure tells us how to compose smooth functions, while the monoid structure tells us how to multiply smooth functions.

Coalgebra Modality - Example I

Example

For a set X, let $\mathsf{M}X$ be the free commutative monoid over a set X, equivalently the free $\mathbb{N}$-module over X, or equivalently the set of finite multisets of X.

Explicitly, for a function $f : X \to \mathbb{N}$ define $\mathsf{supp}(f) := \{x \in X | \ f(x) \neq 0\}$. Then define $\mathsf{M}X$ as:

$$\mathsf{M}X = \{f : X \to \mathbb{N}| \ |\mathsf{supp}(f)| < \infty\}$$

The monoid structure on $\mathsf{M}X$ is defined by point-wise addition, $(f + g)(x) = f(x) + g(x)$, while the unit is $0 : X \to \mathbb{N}$ which maps everything to zero. For each $x \in X$, let $\eta_x : X \to \mathbb{N}$

$$\eta_x(y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

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$$\eta_x(y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

Then $!X = \mathsf{M}X$ induces a coalgebra modality on REL where:

$$\mathsf{d}_X = \{(\eta_x, x) | \ x \in X\} \subseteq \mathsf{M}X \times X \qquad \mathsf{p}_X = \left\{ (f, F) | \sum_{g \in \mathsf{supp}(F)} g = f \right\} \subseteq \mathsf{M}X \times \mathsf{MM}X$$

$$\mathsf{w}_X = \{(0, *)\} \subseteq \mathsf{M}X \times \{*\} \qquad \mathsf{c}_X = \{(f, (g, h)) \mid f = g + h\} \subseteq \mathsf{M}X \times (\mathsf{M}X \times \mathsf{M}X)$$

Coalgebra Modality - Example II

Example

A commutative monoid in VEC_k is a commutative k-algebra. Define the algebra modality Sym on VEC_k (so a coalgebra modality on VEC_k^{op}) as follows: for a k-vector space V let $\mathsf{Sym}(V)$ be the free commutative k-algebra over V, also known as the free symmetric algebra on V.

$$\mathsf{Sym}(V) := k \oplus V \oplus (V \otimes_{sym} V) \oplus \ldots = \bigoplus_{n \in \mathbb{N}} V \otimes_{sym} \ldots \otimes_{sym} V$$

where $\otimes_{sym}$ is the symmetrize tensor power of V.

If $X = \{x_1, x_2, \ldots\}$ is a basis of V, then $\mathsf{Sym}(V) \cong k[X]$.

Note that $k[X]$ is the free k-vector space over $\mathsf{M}X$. In particular for k^n, $\mathsf{Sym}(k^n) \cong k[x_1, \ldots, x_n]$.

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Then the algebra modality structure can be described in terms of polynomials as (which we extend by linearity):

$$\begin{aligned} \mathsf{d}_V : V &\to k[X] \\ x_i &\mapsto x_i \end{aligned} \qquad \begin{aligned} \mathsf{p}_V : k[\mathsf{M}X] &\to k[X] \\ P\left(p_1(\vec{x_1}), \ldots, p_n(\vec{x_n})\right) &\mapsto P\left(p_1(\vec{x_1}), \ldots, p_n(\vec{x_n})\right) \end{aligned}$$

$$\begin{aligned} \mathsf{w}_V : k &\to k[X] \\ 1 &\mapsto 1 \end{aligned} \qquad \begin{aligned} \mathsf{c}_V : k[X] \otimes k[X] &\to k[X] \\ p(\vec{x}) \otimes q(\vec{y}) &\mapsto p(\vec{x})q(\vec{y}) \end{aligned}$$

Therefore, p and d correspond to polynomial composition, while c and w correspond to polynomial multiplication.

Differential Categories - Definition

A **differential category** is:

- An **additive** symmetric monoidal category,
- With a *differential* modality = coalgebra modality equipped with a deriving transformation

Additive Symmetric Monoidal Categories - Definition

In short: an additive symmetric monoidal categories are symmetric commutative monoid enriched monoidal categories.

Definition

An **additive category** is a category $\mathcal{L}$ such that each hom-set $\mathcal{L}(A, B)$ is a commutative monoid with binary operation $+$ and zero 0, that is, we can add parallel maps $f + g$ and there is a zero map 0, and such that composition preserves the additive structure:

$$f \circ (g + h) \circ k = (f \circ g \circ k) + (f \circ h \circ k) \qquad f \circ 0 = 0 = 0 \circ f$$

Definition

An **additive symmetric monoidal category** is a symmetric monoidal category which is also an additive category, such that the tensor product $\otimes$ preserves the additive structure:

$$f \otimes (g + h) = f \otimes g + f \otimes h \qquad (f + g) \otimes h = f \otimes h + g \otimes h$$

$$f \otimes 0 = 0 \qquad 0 \otimes f = 0$$

Note that this definition does not assume biproducts or negatives.

Additive Symmetric Monoidal Categories - Examples

Example

VEC_k is additive symmetric monoidal category where:

- The sum of k-linear maps $f, g : V \to W$ is the standard pointwise sum of linear maps:

$$(f+g)(v) := f(v) + g(v)$$

- The zero maps $0 : V \to W$ are the k-linear maps which map everything to zero.

Example

REL is an additive symmetric monoidal category where:

- The sum of maps $R, S \subseteq X \times Y$ is their union $R + S := R \cup S \subseteq X \times Y$.
- The zero maps are the empty subsets $0 := \emptyset \subseteq X \times Y$.

The additive structure in both examples are induced from finite biproducts.

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Deriving Transformation

A **differential modality** ! on an additive symmetric monoidal category is a coalgebra modality ! equipped with a **deriving transformation** which is a natural transformation:

$$\overline{\partial}_A : !A \otimes A \to !A$$

satisfying five axioms based on the basic identities from differential calculus.

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IDEA: For a coalgebra modality, on distributions:

$$\overline{\partial}_A(\delta_x \otimes y) = \mathsf{D}_x[\_](y)$$

where for a smooth function f, $\mathsf{D}_x[f](y)$ is the derivative of f at point x in the direction of the vector y.

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For an algebra modality, the deriving transformation $\overline{\partial}_A : \mathsf{S}(A) \to \mathsf{S}(A) \otimes A$ is an actual derivation from algebra:

$$f(x) \mapsto f'(x) \otimes \mathsf{d}x$$

and so the five axioms are:

- Constant rule: $c' = 0$
- Product rule: $(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$
- Linear rule: $x' = 1$
- Chain rule: $(f \circ g)'(x) = f'(g(x))g'(x)$
- Interchange rule: $\frac{d^2 f(x,y)}{dxdy} = \frac{d^2 f(x,y)}{dydx}$

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But let's see some examples first!

Deriving Transformation - Examples

Example

In REL, for the set X the deriving transformation is the subset:

$$\overline{\partial}_X := \{((f, x), f + \eta_x) \mid \forall x \in X, f \in \mathsf{M}X\} \subset (\mathsf{M}X \times X) \times \mathsf{M}X$$

Example

Let V be a k-vector space with basis $X = \{x_1, x_2, \ldots\}$.

The deriving transformation can be described in terms of polynomials as follows:

$$\overline{\partial}_V : k[X] \to k[X] \otimes V$$

$$\mathsf{p}(x_1, \ldots, x_n) \mapsto \sum_{i=1}^{n} \frac{\partial \mathsf{p}(x_1, \ldots, x_n)}{\partial x_i} \otimes x_i$$

D.1 - Constant Rule

$$\begin{tikzcd} !A \otimes A \arrow[r, "\overline{\partial}_A"] \arrow[dr, "0"'] & !A \arrow[d, "\mathsf{w}_A"] \\ & A \end{tikzcd}$$

Example

In Vec_k, consider k^n, so $\mathsf{Sym}(k^n) \cong k[x_1, \dots, x_n]$.

For a constant polynomial $\mathsf{p}(x_1, \dots, x_n) = r$:

$$\sum_{i=1}^{n} \frac{\partial \mathsf{p}}{\partial x_i}(x_1, \dots, x_n) \otimes x_i = 0$$

D.2 - Product Rule

$$
\begin{tikzcd}
!A \otimes A \arrow[rr, "\overline{\partial}_A"] \arrow[d, "c\otimes 1"'] & & !A \arrow[d, "c_A"] \\
!A \otimes !A \otimes A \arrow[rr, "(1\otimes\overline{\partial}_A)+(\overline{\partial}_A\otimes 1)\circ(1\otimes\sigma)"'] & & !A \otimes !A
\end{tikzcd}
$$

Example

In Vec_k, consider k^n, so $\mathrm{Sym}(k^n) \cong k[x_1, \ldots, x_n]$.

For polynomials $p(x_1, \ldots, x_n)$ and $q(x_1, \ldots, x_n)$:

$$
\sum_{i=1}^{n} \frac{\partial pq}{\partial x_i}(x_1, \ldots, x_n) \otimes x_i
$$

$$
= \sum_{i=1}^{n} p(x_1, \ldots, x_n) \frac{\partial q}{\partial x_i}(x_1, \ldots, x_n) \otimes x_i + \sum_{i=1}^{n} \frac{\partial p}{\partial x_i}(x_1, \ldots, x_n) q(x_1, \ldots, x_n) \otimes x_i
$$

D.3 - Linear Rule

$$\begin{tikzcd} !A \otimes A \arrow[r, "\overline{\partial}_A"] \arrow[dr, "\mathsf{w}_A \otimes 1"'] & !A \arrow[d, "\mathsf{d}_A"] \\ & A \end{tikzcd}$$

Example

In Vec_k, consider k^n, so $\mathrm{Sym}(k^n) \cong k[x_1, \ldots, x_n]$.

For a monomial of degree 1, $\mathrm{p}(x_1, \ldots, x_n) = x_j$:

$$\sum_{i=1}^{n} \frac{\partial x_j}{\partial x_i}(x_1, \ldots, x_n) \otimes x_i = 1 \otimes x_j$$

D.4 - Chain Rule

$$\begin{tikzcd}
!A \otimes A \arrow[rr, "\overline{\partial}_A"] \arrow[d, "\mathsf{c}_A \otimes 1"'] & & !A \arrow[d, "\mathsf{p}_A"] \\
!A \otimes !A \otimes A \arrow[r, "\mathsf{p}_A \otimes \overline{\partial}_A"'] & !!A \otimes !A \arrow[r, "\overline{\partial}_{!A}"'] & !!A
\end{tikzcd}$$

Example

In Vec_k, consider k^n, so $\mathsf{Sym}(k^n) \cong k[x_1, \ldots, x_n]$.

For polynomials $p(x_1, \ldots, x_n)$ and $q(x)$:

$$\sum_{i=1}^{n} \frac{\partial q(p(x_1, \ldots, x_n))}{\partial x_i}(x_1, \ldots, x_n) \otimes x_i = \sum_{i=1}^{n} \frac{\partial q}{\partial x}(p(x_1, \ldots, x_n)) \frac{\partial q}{\partial x_i}(x_1, \ldots, x_n) \otimes x_i$$

D.5 - Interchange Rule

$$
\begin{tikzcd}
!A \otimes A \otimes A \arrow[rr, "\overline{\partial}_A"] \arrow[d, "1\otimes\sigma"'] & & !A \otimes A \arrow[d, "\overline{\partial}_A"] \\
!A \otimes A \otimes A \arrow[r, "\overline{\partial}_A"'] & !A \otimes A \arrow[r, "\overline{\partial}_A"'] & !A
\end{tikzcd}
$$

Example

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For a polynomial $p(x_1, \ldots, x_n)$:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\frac{\partial p}{\partial x_i}}{\partial x_j}(x_1, \ldots, x_n) \otimes x_j \otimes x_i = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\frac{\partial p}{\partial x_i}}{\partial x_j}(x_1, \ldots, x_n) \otimes x_i \otimes x_j$$

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Getting closer to Differential Linear Logic

Now let's consider differential categories with a **monoidal** differential modality.

In this setting, the differential structure can be described in terms of a **codereliction**.

Monoidal Coalgebra Modality

A **monoidal coalgebra modality** ! on a symmetric monoidal category is a coalgebra modality ! equipped with a natural transformation and a map:

$$\mu_{A,B} : !A \otimes !B \to !(A \otimes B) \qquad\qquad \mu_I : I \to !I$$

such that:

- ! is a symmetric monoidal functor
- p and d are monoidal transformations
- c and w are monoidal transformations (which is equivalent to saying that μ and μ_I are comonoid morphisms)
- c and w are !-coalgebra modalities.

An equivalent way of describing a monoidal coalgebra modality is as a comonad on a symmetric monoidal category such that the monoidal product becomes a product for its Eilenberg-Moore category (more generally, this can be described in terms of linear-non-linear adjunctions).

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When we have finite products or when we are in additive setting, we have other equivalent ways of defining monoidal coalgebra modalities...

Storage Modality

For a category with finite products, we denote:

- The binary product as $\times$ and the terminal object as $\top$
- The projection maps as $\pi_0 : A \times B \to A$ and $\pi_1 : A \times B \to B$

A **storage modality** ! on a symmetric monoidal category with finite products is a coalgebra modality ! such that the canonical maps:

$$!(A \times B) \xrightarrow{\mathsf{c}_{A \times B}} !(A \times B) \otimes !(A \times B) \xrightarrow{!(\pi_0) \otimes !(\pi_1)} !A \otimes !B$$

$$!\top \xrightarrow{\mathsf{w}_\top} I$$

are isomorphisms, so $!(A \times B) \cong !A \otimes !B$ and $!\top \cong I$, and these are called the Seely isomorphisms.

Lemma

For a symmetric monoidal category with finite products, to give a monoidal coalgebra modality is precisely the same thing as giving a storage modality. In other words, every monoidal coalgebra modality is a storage modality, and vice versa.

Additive Bialgebra Modality

An **additive bialgebra modality** on ! on an additve symmetric monoidal category is a coalgebra modality ! equipped with natural transformations:

$$\underbrace{\overline{\mathsf{c}}_A : !A \otimes !A \to !A}_{\textit{Cocontraction}} \qquad \underbrace{\overline{\mathsf{w}}_A : I \to !A}_{\textit{Coweakening}}$$

such that:

- $(!A, \overline{\mathsf{c}}_A, \overline{\mathsf{w}}_A)$ is a commutative monoid;
- $!A$ is a bimonoid;
- Some other compatibility relations involving the dereliction and some identities about bialgebra convolution.

IDEA: On Dirac distributions:

$$\overline{\mathsf{c}}_A(\delta_x \otimes \delta_y) = \delta_{x+y} \qquad \overline{\mathsf{w}}_A(1) = \delta_0$$

Lemma

For an additive symmetric monoidal category to give a monoidal coalgebra modality is precisely the same thing as giving an additive bialgebra modality (and if we also have finite (bi)products, also the same as a storage modality). In other words, every monoidal coalgebra modality is an additive bialgebra modality, and vice versa.

Blute, R., Cockett, R., Seely, R.A.G., Lemay, J.-S. P. **Differential categories revisited**. *(2019)*

Codereliction

In an additive symmetric monoidal category, for a monoidal coalgebra modality !, a **codereliction** is a natural transformation:

$$\overline{\mathsf{d}}_A : A \to A$$

such that the following diagrams commutes[1]:

[dC.1] Constant Rule:

$$\begin{tikzcd} A \arrow[r, "\overline{\mathsf{d}}_A"] \arrow[dr, "0"'] & !A \arrow[d, "\mathsf{w}_A"] \\ & k \end{tikzcd}$$

[dC.2] Product Rule:

$$\begin{tikzcd} A \arrow[r, "\overline{\mathsf{d}}_A"] \arrow[dr, "\overline{\mathsf{d}}_A \otimes \overline{\mathsf{w}}_A + \overline{\mathsf{w}}_A \otimes \overline{\mathsf{d}}_A"'] & !A \arrow[d, "\mathsf{c}_A"] \\ & !A \otimes !A \end{tikzcd}$$

[dC.3] Linear Rule:

$$\begin{tikzcd} A \arrow[r, "\overline{\mathsf{d}}_A"] \arrow[dr, equal] & !A \arrow[d, "\mathsf{d}_A"] \\ & A \end{tikzcd}$$

[dC.4′] Alternative Chain Rule:

$$\begin{tikzcd} A \arrow[rr, "\overline{\mathsf{d}}_A"] \arrow[d, "\overline{\mathsf{w}}_A \otimes \overline{\mathsf{d}}_A"'] & & !A \arrow[d, "\mathsf{p}_A"] \\ !A \otimes !A \arrow[r, "\mathsf{p}_A \otimes \overline{\mathsf{d}}_{!A}"'] & !!A \otimes !!A \arrow[r, "\overline{\mathsf{c}}_{!A}"'] & !!A \end{tikzcd}$$

[dC.m] Monoidal Rule:

$$\begin{tikzcd} !A \otimes B \arrow[r, "1_{!A} \otimes \overline{\mathsf{d}}_B"] \arrow[d, "\mathsf{d}_A \otimes 1_B"'] & !A \otimes !B \arrow[d, "\mu_{A,B}"] \\ A \otimes B \arrow[r, "\overline{\mathsf{d}}_{A \otimes B}"'] & !(A \otimes B) \end{tikzcd}$$

[1] **[dC.1]** and **[dC.2]** are redundant

Only one notion of differentiation

Theorem

For a monoidal coalgebra modality !, there is a bijective correspondence between deriving transformations and coderelictions. Explicitly:

- *From a deriving transformation we get a codereliction as follows:*

$$A \xrightarrow{\overline{\mathsf{w}}_A \otimes 1} !A \otimes A \xrightarrow{\overline{\partial}_A} !A$$

- *From a codereliction we get a deriving transformastion as follows:*

$$!A \otimes A \xrightarrow{1 \otimes \overline{\mathsf{d}}_A} !A \otimes !A \xrightarrow{\overline{\mathsf{c}}_A} !A$$

These constructions are inverses of each other.

Blute, R., Cockett, R., Seely, R.A.G., Lemay, J.-S. P. **Differential categories revisited**. *(2019)*

So codereliction is precisely given by linearization, which is evaluating the derivative at zero:

$$\overline{\mathsf{d}}_A(y) = \mathsf{D}_0[\_](y) \qquad\qquad \overline{\mathsf{d}}_A(f) = f'(0)x$$

Codereliction – Examples

Example

In REL, for the set X the codereliction is the subset:

$$\overline{\mathsf{d}}_X := \{(x, \eta_x) \mid \forall x \in X\} \subset X \times \mathsf{M}X$$

Example

Let V be a k-vector space with basis $X = \{x_1, x_2, \ldots\}$.

The codereliction can be described in terms of polynomials as follows:

$$\overline{\mathsf{d}}_V : k[X] \to V$$
$$\mathsf{p}(x_1, \ldots, x_n) \mapsto \sum_{i=1}^{n} \frac{\partial \mathsf{p}}{\partial x_i}(0, \ldots, 0) \otimes x_i$$

It picks out the degree 1 terms of $p(x_1, \ldots, x_n)$.

Codereliction for Free Exponential Modalities

A **free exponential modality** is a monoidal coalgebra modality[2] $!$ such that for each object A, $!A$ is a **cofree cocommutative comonoid** over A, that is, if C is a cocommutative comonoid then for every map $C \xrightarrow{f} A$, there exists a unique comonoid morphism which makes the following diagram commute:

$$\begin{tikzcd} C \arrow[r, dashed, "\exists!\ f^\flat"] \arrow[dr, "f"'] & !A \arrow[d, "\mathsf{d}_A"] \\ & A \end{tikzcd}$$

Proposition

Every free exponential modality on an additive symmetric monoidal category with finite (bi)products has a codereliction has a codereliction. (Every additive Lafont category is a differential category)

Lemay, J.-S. P. **Coderelictions for Free Exponential Modalities**. *(2021)*

Blute, R., Lucyshyn-Wright, R.B.B. and O'NeilL, K. **Derivations in codifferential categories**. *(2016)*

Example

Both of our examples so far are free exponential modalities.

[2]From the universal property, the monoidal coalgebra modality structure can be derived

Uniqueness of codereliction!

Proposition

For a monoidal coalgebra modality !, a codereliction (if it exists) is unique.

Uniqueness of Differentiation in Differential Categories. *Talk at Category Theory Octoberfest 2022, Slides: https://richardblute.files.wordpress.com/2022/10/lemay-ofest.pdf*

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OPEN QUESTION: For a non-monoidal coalgebra modality !, are deriving transformations unique?

- Looking for a proof of uniqueness
- Or, an example of a non-monoidal coalgebra modality ! with two distinct deriving transformations.

Please help!

Other examples of differential categories

Example

A $\mathcal{C}^\infty$**-ring** is commutative $\mathbb{R}$-algebra A such that for each for smooth map $f : \mathbb{R}^n \to \mathbb{R}$ there is a function $\Phi_f : A^n \to A$ and such that the Φ_f satisfy certain coherences between them. For a smooth manifold M, $\mathcal{C}^\infty(M) = \{f : M \to \mathbb{R} | \ f \text{ smooth}\}$ is a $\mathcal{C}^\infty$-ring.

For every $\mathbb{R}$-vector space V, there is a free $\mathcal{C}^\infty$-ring over V, $\mathsf{S}^\infty(V)$. This induces an algebra modality which has a deriving transformation. In particular, $\mathsf{S}^\infty(\mathbb{R}^n) = \mathcal{C}^\infty(\mathbb{R}^n)$, and:

$$\mathsf{d} : \mathcal{C}^\infty(\mathbb{R}^n) \to \mathcal{C}^\infty(\mathbb{R}^n) \otimes \mathbb{R}^n \qquad f \longmapsto \sum_i \frac{\partial f}{\partial x_i} \otimes x_i$$

This example is a non-monoidal coalgebra modality on $\mathsf{VEC}^{op}_{\mathbb{R}}$.

Cruttwell, G.S.H., Lemay, J.-S. P. and Lucyshyn-Wright, R.B.B. **Integral and differential structure on the free C^∞-ring modality**. (2019)

Example

- Fininiteness Spaces, Köthe spaces, etc.

Ehrhard, T. **An introduction to differential linear logic: proof-nets, models and antiderivatives**. (2018)

- Convenient vector spaces

Blute, R., Ehrhard, T. and Tasson, C. **A convenient differential category** (2012)

Differential Categories - Some Algebra you can do with them

Example

Derivations and Kähler differentials

Blute, R., Lucyshyn-Wright, R.B.B. and O'NeilL, K. **Derivations in codifferential categories**. (2016)

Example

Hochschild complex, de Rham complex, and (co)homology

O'Neill, K. **Smoothness in codifferential categories** (PhD Thesis) (2017)

Example

Differentials algebras

Lemay, J.-S.P. **Differential algebras in codifferential categories**. (2019)

Differential Categories - Smooth Maps

Every differential category has a notion of a *smooth map*.

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A smooth map $A \to B$ is a coKleisli map, that is, a map $!A \to B$.

Differential Categories - Smooth Maps

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Example

Let's consider our $\mathcal{C}^\infty$-ring codifferential category example, where $!(\mathbb{R}^n) := \mathcal{C}^\infty(\mathbb{R}^n)$.

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$$f : \mathbb{R}^n \to \mathbb{R} \qquad f \text{ is a smooth function}$$

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$$\frac{f : \mathbb{R}^n \to \mathbb{R} \qquad f \text{ is a smooth function}}{f \in \mathcal{C}^\infty(\mathbb{R}^n)}$$

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$$\dfrac{\dfrac{f : \mathbb{R}^n \to \mathbb{R} \qquad f \text{ is a smooth function}}{f \in \mathcal{C}^\infty(\mathbb{R}^n)}}{q_f : \mathbb{R} \to \mathcal{C}^\infty(\mathbb{R}^n) \qquad q_f \text{ linear map in } \mathsf{VEC}_{\mathbb{R}},\ q_f(1) = f}$$

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$$\begin{array}{c}
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\hline\hline
f \in \mathcal{C}^\infty(\mathbb{R}^n) \\
\hline\hline
q_f : \mathbb{R} \to \mathcal{C}^\infty(\mathbb{R}^n) \qquad q_f \text{ linear map in } \mathsf{VEC}_{\mathbb{R}},\ q_f(1) = f \\
\hline\hline
\mathcal{C}^\infty(\mathbb{R}^n) \to \mathbb{R} \qquad \text{map in } \mathsf{VEC}_{\mathbb{R}}^{op}
\end{array}$$

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\hline\hline
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\hline\hline
\mathcal{C}^\infty(\mathbb{R}^n) \to \mathbb{R} \qquad \text{map in } \mathsf{VEC}_{\mathbb{R}}^{op} \\
\hline\hline
!(\mathbb{R}^n) \to \mathbb{R}
\end{array}
$$

Differential Categories - Smooth Maps

Amongst the smooth maps we have:

- The constant maps:

$$!A \xrightarrow{\mathsf{w}} I \longrightarrow B$$

- The linear maps:

$$!A \xrightarrow{\mathsf{d}} A \longrightarrow B$$

- The product of smooth maps:

$$!A \xrightarrow{\mathsf{c}} !A \otimes !A \xrightarrow{f \otimes g} B \otimes C$$

- The composition of smooth maps:

$$!A \xrightarrow{\mathsf{p}} !!A \xrightarrow{!(f)} !B \xrightarrow{g} C$$

Differential Categories - Smooth Maps

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- The product of smooth maps:

$$!A \xrightarrow{\mathsf{c}} !A \otimes !A \xrightarrow{f \otimes g} B \otimes C$$

- The composition of smooth maps:

$$!A \xrightarrow{\mathsf{p}} !!A \xrightarrow{!(f)} !B \xrightarrow{g} C$$

The differential of a smooth map $f : !A \to B$ is then:

$$!A \otimes A \xrightarrow{\overline{\partial}} !A \xrightarrow{f} B$$

So the deriving transformation axioms describe differentiation of constants, identity maps, composition, etc. in the *coKleisli category*!

So next we will take a closer look at the coKleisli category of a differential category: **Cartesian Differential Categories!**

Cartesian Differential Categories

- Categorical foundations of differential calculus over Euclidean spaces
- Categorical semantics of differential λ-calculus

Some introductory references:

Blute, R., Cockett, R., Seely, R.A.G. **Cartesian Differential Categories** (2009)

Garner, R, and Lemay, J-S P. **Cartesian differential categories as skew enriched categories**.

Manzonetto, G. **What is a Categorical Model of the Differential and the Resource λ-Calculi?**. (2012)

A **Cartesian differential category** is:

- A Cartesian left additive category;
- With a differential combinator.

Cartesian Differential Categories - Definition

A **Cartesian differential category** is:

- A **Cartesian left additive category**;
- With a differential combinator.

Cartesian Left Additive Category - Definition

A **left additive category** is a category $\mathcal{C}$ which is *skew-enriched* over commutative monoids:

Campbell, A., 2018. **Skew-enriched categories**.

Explicitly, every homset is a commutative monoid, so we can add maps and have zero maps:

$$+ : \mathcal{C}(A, B) \times \mathcal{C}(A, B) \to \mathcal{C}(A, B) \qquad\qquad 0 \in \mathcal{C}(A, B)$$

such that composition preserves the addition in the following sense:

$$(f + g) \circ x = f \circ x + g \circ x \qquad\qquad 0 \circ x = 0$$

A map f is **additive** if:

$$f \circ (x + y) = f \circ x + f \circ y \qquad\qquad f \circ 0 = 0$$

A **Cartesian left additive category** (CLAC) is a left additive category with finite products such that all projection maps are additive.

Cartesian Left Additive Categories - Examples

Example

Every category with finite biproducts is a CLAC where every map is additive. For example, VEC_k is a CLAC.

Example

For any commutative semiring k, let Poly_k be the Lawvere theory of polynomials, that is, the category whose objects are $n \in \mathbb{N}$ and where a map $P : n \to m$ is a tuple of polynomials:

$$P = \langle p_1(\vec{x}), \dots, p_m(\vec{x}) \rangle \qquad p_i(\vec{x}) \in R[x_1, \dots, x_n]$$

Then Poly_k is a CLAC (where $n \times m = n + m$).

Example

Let SMOOTH be the category of smooth real functions, that is, the category whose objects are the Euclidean vector spaces $\mathbb{R}^n$ and whose maps are smooth function $F : \mathbb{R}^n \to \mathbb{R}^m$, which is actually an m-tuple of smooth functions:

$$F = \langle f_1, \dots, f_m \rangle \qquad f_i : \mathbb{R}^n \to \mathbb{R}$$

Then SMOOTH is a CLAC. Note that $\mathsf{Poly}_{\mathbb{R}}$ is a sub-CLAC of SMOOTH.

Cartesian Differential Categories - Definition

A **Cartesian differential category** is:

- A Cartesian left additive category;
- With a **differential combinator**.

Differential Combinator - Definition

A **differential combinator** on a Cartesian left additive category $\mathcal{C}$ is a combinator D, which is a family of functions $\mathcal{C}(A, B) \to \mathcal{C}(A \times A, B)$, which written as an inference rule:

$$\frac{f : A \to B}{\mathsf{D}[f] : A \times A \to B}$$

Before giving the axioms, let's look at some examples!

Differential Combinator - Main Example

Example

SMOOTH is a Cartesian differential category where the differential combinator is defined as the total derivative of a smooth function, which is given by the sum of partial derivatives.

Differential Combinator - Main Example

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For a smooth function $F = \langle f_1, \ldots, f_m \rangle : \mathbb{R}^n \to \mathbb{R}^m$, recall that the Jacobian matrix of F at vector $\vec{x} \in \mathbb{R}^n$ is the matrix $\mathbf{J}(F)(\vec{x})$ of size $m \times n$ whose coordinates are the partial derivatives of the f_i:

$$\mathbf{J}(F)(\vec{x}) := \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\vec{x}) & \frac{\partial f_1}{\partial x_2}(\vec{x}) & \ldots & \frac{\partial f_1}{\partial x_n}(\vec{x}) \\ \frac{\partial f_2}{\partial x_1}(\vec{x}) & \frac{\partial f_2}{\partial x_2}(\vec{x}) & \ldots & \frac{\partial f_2}{\partial x_n}(\vec{x}) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\vec{x}) & \frac{\partial f_m}{\partial x_2}(\vec{x}) & \ldots & \frac{\partial f_m}{\partial x_n}(\vec{x}) \end{bmatrix}$$

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So for a smooth function $F : \mathbb{R}^n \to \mathbb{R}^m$, its derivative $\mathsf{D}[F] : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ is then defined as:

$$\mathsf{D}[F](\vec{x}, \vec{y}) := \mathbf{J}(F)(\vec{x}) \cdot \vec{y} = \left\langle \sum_{i=1}^{n} \frac{\partial f_1}{\partial x_i}(\vec{x}) y_i, \ldots, \sum_{i=1}^{n} \frac{\partial f_m}{\partial x_i}(\vec{x}) y_i \right\rangle$$

In particular for smooth function $f : \mathbb{R}^n \to \mathbb{R}$:

$$\mathsf{D}[f](\vec{x}, \vec{y}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{x}) y_i$$

Cartesian Differential Categories - Other Main Examples

Example

Any category with finite biproduct $\oplus$ is a CDC, where for a map $f : A \to B$:

$$\mathsf{D}[f] := \; A \oplus A \xrightarrow{\quad \pi_1 \quad} A \xrightarrow{\quad f \quad} B$$

For example, VEC_k is a CDC where $\mathsf{D}[f](x, y) = f(y)$.

Example

POLY_k is a CDC where for a map $P : n \to m$ with $P = \langle p_1(\vec{x}), \ldots, p_m(\vec{x}) \rangle$, $\mathsf{D}[P] : n \times n \to m$ is:

$$\mathsf{D}[P] := \left\langle \sum_{i=1}^{n} \frac{\partial p_1(\vec{x})}{\partial x_i} y_i, \ldots, \sum_{i=1}^{n} \frac{\partial p_m(\vec{x})}{\partial x_i} y_i \right\rangle$$

where $\sum\limits_{i=1}^{n} \frac{\partial p_i(\vec{x})}{\partial x_i} y_i \in R[x_1, \ldots, x_n, y_1, \ldots, y_n]$. Note that $\mathsf{POLY}_{\mathbb{R}}$ is a sub-CDC of SMOOTH.

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$$\frac{f : A \to B}{\mathsf{D}[f] : A \times A \to B}$$

To help us with the axioms, we will use the following notation/proto-term logic:

$$\mathsf{D}[f](a, b) := \frac{\mathrm{d}f(x)}{\mathrm{d}x}(a) \cdot b$$

Example

The notation comes from SMOOTH: $\mathsf{D}[F](\vec{x}, \vec{y}) := \mathbf{J}(F)(\vec{x}) \cdot \vec{y}$.

Remark

There is a sound and complete term logic for Cartesian differential categories. In short: anything we can prove using the term logic, holds in any Cartesian differential category. So doing proofs in the term logic is super useful!

CD.1 - Additivity of Combinator & CD.2 - Additivity in Second Argument

- Additivity of Combinator:

$$\mathrm{D}[f+g] = \mathrm{D}[f] + \mathrm{D}[g] \qquad\qquad \mathrm{D}[0] = 0$$

$$\frac{\mathrm{d}f(x)+g(x)}{\mathrm{d}x}(a)\cdot b = \frac{\mathrm{d}f(x)}{\mathrm{d}x}(a)\cdot b + \frac{\mathrm{d}g(x)}{\mathrm{d}x}(a)\cdot b \qquad\qquad \frac{\mathrm{d}0}{\mathrm{d}x}(a)\cdot b = 0$$

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- Additivity in Second Argument

$$\mathrm{D}[f]\circ\langle a, b+c\rangle = \mathrm{D}[f]\circ\langle a, b\rangle + \mathrm{D}[f]\circ\langle a, c\rangle \qquad \mathrm{D}[f]\circ\langle x, 0\rangle = 0$$

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x}(a)\cdot(b+c) = \frac{\mathrm{d}f(x)}{\mathrm{d}x}(a)\cdot b + \frac{\mathrm{d}f(x)}{\mathrm{d}x}(a)\cdot c \qquad \frac{\mathrm{d}f(x)}{\mathrm{d}x}(a)\cdot 0 = 0$$

CD.3 - Identities + Projections & CD.4 - Pairings

- Identities + Projections

$$\mathsf{D}[1] = \pi_1 \qquad \mathsf{D}[\pi_i] = \pi_i \circ \pi_1$$

$$\frac{\mathrm{d}x}{\mathrm{d}x}(a) \cdot b = b \qquad \frac{\mathrm{d}x_i}{\mathrm{d}(x_0, x_1)}(a_0, a_1) \cdot (b_0, b_1) = b_i$$

CD.3 - Identities + Projections & CD.4 - Pairings

- Identities + Projections

$$\mathrm{D}[1] = \pi_1 \qquad\qquad \mathrm{D}[\pi_i] = \pi_i \circ \pi_1$$

$$\frac{\mathrm{d}x}{\mathrm{d}x}(a) \cdot b = b \qquad\qquad \frac{\mathrm{d}x_i}{\mathrm{d}(x_0, x_1)}(a_0, a_1) \cdot (b_0, b_1) = b_i$$

- Pairings

$$\mathrm{D}[\langle f, g\rangle] = \langle \mathrm{D}[f], \mathrm{D}[g]\rangle$$

$$\frac{\mathrm{d}\langle f(x), g(x)\rangle}{\mathrm{d}x}(a) \cdot b = \left\langle \frac{\mathrm{d}f(x)}{\mathrm{d}x}(a) \cdot b, \frac{\mathrm{d}g(x)}{\mathrm{d}x}(a) \cdot b \right\rangle$$

Example

In SMOOTH, if $F = \langle f_1, \ldots, f_n\rangle$, then $\mathrm{D}[F](\vec{x}, \vec{y}) := \langle \mathrm{D}[f_1](\vec{x}, \vec{y}), \ldots, \mathrm{D}[f_n](\vec{x}, \vec{y})\rangle$.

CD.5 - Chain Rule

Chain Rule:

$$\mathrm{D}[g \circ f] = \mathrm{D}[g] \circ \langle f \circ \pi_0, \mathrm{D}[f] \rangle$$

$$\frac{\mathrm{d}g(f(x))}{\mathrm{d}x}(a) \cdot b = \frac{\mathrm{d}g(x)}{\mathrm{d}x}(f(a)) \cdot \left(\frac{\mathrm{d}f(x)}{\mathrm{d}x}(a) \cdot b \right)$$

CD.6 - Linearity in Second Argument & CD.7 - Symmetry

$$\dfrac{\dfrac{f : A \to B}{\mathrm{D}[f] : A \times A \to B}}{\mathrm{D}\left[\mathrm{D}[f]\right] : (A \times A) \times (A \times A) \to B}$$

CD.6 - Linearity in Second Argument & CD.7 - Symmetry

$$\cfrac{\cfrac{f : A \to B}{\mathrm{D}[f] : A \times A \to B}}{\mathrm{D}\left[\mathrm{D}[f]\right] : (A \times A) \times (A \times A) \to B}$$

- Linearity in Second Argument

$$\mathrm{D}\left[\mathrm{D}[f]\right] \circ \langle a, 0, 0, b \rangle = \mathrm{D}[f] \circ \langle a, b \rangle$$

$$\frac{\mathrm{d} \frac{\mathrm{d}f(x)}{\mathrm{d}x}(y) \cdot z}{\mathrm{d}(y, z)}(a, 0) \cdot (0, b) = \frac{\mathrm{d}f(x)}{\mathrm{d}x}(a) \cdot b$$

CD.6 - Linearity in Second Argument & CD.7 - Symmetry

$$\dfrac{\dfrac{f : A \to B}{\mathrm{D}[f] : A \times A \to B}}{\mathrm{D}\left[\mathrm{D}[f]\right] : (A \times A) \times (A \times A) \to B}$$

- Linearity in Second Argument

$$\mathrm{D}\left[\mathrm{D}[f]\right] \circ \langle a, 0, 0, b\rangle = \mathrm{D}[f] \circ \langle a, b\rangle$$

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- Symmetry

$$\mathrm{D}\left[\mathrm{D}[f]\right] \circ \langle\langle a, b\rangle, \langle c, d\rangle\rangle = \mathrm{D}\left[\mathrm{D}[f]\right] \circ \langle\langle a, c\rangle, \langle b, d\rangle\rangle$$

$$\frac{\mathrm{d}\frac{\mathrm{d}f(x)}{\mathrm{d}x}(y) \cdot z}{\mathrm{d}(y, z)}(a, b) \cdot (c, d) = \frac{\mathrm{d}\frac{\mathrm{d}f(x)}{\mathrm{d}x}(y) \cdot z}{\mathrm{d}(y, z)}(a, c) \cdot (b, d)$$

More on these axioms soon!

Cartesian Differential Categories - Definition

A **Cartesian differential category** is:

- A Cartesian left additive category;
- With a differential combinator.

$$\frac{f : A \to B}{\mathrm{D}[f] : A \times A \to B}$$

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Before we give some more examples: let's see what we can do within a CDC!

Partial Derivatives I

Suppose we have a map $f : A \times B \to C$ and we only want to differentiate with respect to A.

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Partial Derivatives I

Suppose we have a map $f : A \times B \to C$ and we only want to differentiate with respect to A.

We can zero out in $\mathrm{D}[f] : (A \times B) \times (A \times B) \to C$ to obtain a partial derivative!

Define the partial derivative $\mathrm{D}_0[f] : (A \times B) \times A \to C$ as follows:

$$\mathrm{D}_0[f] := \ (A \times B) \times A \xrightarrow{(1_A \times 1_B) \times \langle 1_A, 0 \rangle} (A \times B) \times (A \times B) \xrightarrow{\mathrm{D}[f]} C$$

$$\mathrm{D}_0[f](a, b, c) := \frac{\mathrm{d}f(x, b)}{\mathrm{d}x}(a) \cdot c := \frac{\mathrm{d}f(x, y)}{\mathrm{d}(x, y)}(a, b) \cdot (c, 0)$$

Similarly, define the partial derivative $\mathrm{D}_1[f] : (A \times B) \times B \to C$ as follows:

$$\mathrm{D}_1[f] := \ (A \times B) \times B \xrightarrow{(1_A \times 1_B) \times \langle 0, 1_B \rangle} (A \times B) \times (A \times B) \xrightarrow{\mathrm{D}[f]} C$$

$$\mathrm{D}_1[f](a, b, d) := \frac{\mathrm{d}f(a, y)}{\mathrm{d}y}(b) \cdot d := \frac{\mathrm{d}f(x, y)}{\mathrm{d}(x, y)}(a, b) \cdot (0, d)$$

You can also do this with maps $f : A_0 \times \ldots \times A_n \to B$.

Partial Derivatives II

A consequence of **[CD.7]** symmetry rule is that for $f : A \times B \to C$, doing the partial derivative with respect to A then B is the same as doing the partial derivative with respect to B then A.

$$\frac{\mathrm{d}\frac{\mathrm{d}f(x,y)}{\mathrm{d}y}(b) \cdot d}{\mathrm{d}x}(a) \cdot c = \frac{\mathrm{d}\frac{\mathrm{d}f(x,y)}{\mathrm{d}x}(a) \cdot c}{\mathrm{d}y}(b) \cdot d$$

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[CD.2] Additivity in the second argument tells us that for $f : A \times B \to C$, $\mathrm{D}[f]$ is the sum of the partial derivatives:

$$\begin{aligned}\frac{\mathrm{d}f(x,y)}{\mathrm{d}(x,y)}(a,b)\cdot(c,d) &= \frac{\mathrm{d}f(x,y)}{\mathrm{d}(x,y)}(a,b)\cdot((c,0)+(0,d))\\ &= \frac{\mathrm{d}f(x,y)}{\mathrm{d}(x,y)}(a,b)\cdot(c,0) + \frac{\mathrm{d}f(x,y)}{\mathrm{d}(x,y)}(a,b)\cdot(0,d)\\ &= \frac{\mathrm{d}f(x,b)}{\mathrm{d}x}(a)\cdot c + \frac{\mathrm{d}f(a,y)}{\mathrm{d}y}(b)\cdot d\end{aligned}$$

Example

For a smooth map $f : \mathbb{R}^n \to \mathbb{R}$, $\mathrm{D}[f]$ is the sum of its partial derivatives:

$$\mathrm{D}[f] : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \qquad \mathrm{D}[f](\vec{v},\vec{w}) := \mathbf{J}(f)(\vec{v})\cdot\vec{w} = \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(\vec{v})w_i$$

Linear Maps I

In a Cartesian differential category, there is a natural notion of **linear maps**. A map $f : A \to B$ is said to be linear if:

$$\mathsf{D}[f] := \; A \times A \xrightarrow{\pi_1} A \xrightarrow{f} B \qquad\qquad \frac{\mathrm{d}f(x)}{\mathrm{d}x}(a) \cdot b = f(b)$$

Example

- In a category with finite biproducts, every map is linear (by definition!).
- In POLY_k, $P = \langle p_1, \dots, p_m \rangle$ is linear if each $p_i \in k[x_1, \dots, x_n]$ is a polynomial of degree 1, that is, a sum of the form $p_i = \sum_{j=1}^{n} a_j x_j$.
- In $\mathrm{SMOOTH}_{\mathbb{R}}$, a smooth function $F : \mathbb{R}^n \to \mathbb{R}^m$ is linear in the Cartesian differential sense precisely when it is $\mathbb{R}$-linear in the classical sense:

  $$F(s\vec{x} + t\vec{y}) = sF(\vec{x}) + tF(\vec{y})$$

  for all $s, t \in \mathbb{R}$ and $\vec{x}, \vec{y} \in \mathbb{R}^n$.

- Linear $\Rightarrow$ Additive, but not necessarily the converse!

  (But in the above examples: Additive = Linear)
- Identity maps and projection maps are linear by CD.3

Linear Maps II

A map $f : A \times B \to C$ can also be linear in its second argument if it is linear with respect to its partial derivative:

$$\mathsf{D}_1[f] := \ (A \times B) \times B \xrightarrow{\pi_0 \times 1} A \times B \xrightarrow{f} C \qquad \frac{\mathrm{d}f(a,y)}{\mathrm{d}y}(b) \cdot c = f(a,c)$$

The linearity in the second argument rule, CD.6, says that for any $f : A \to B$, $\mathsf{D}[f]$ is linear in its second argument:

$$\frac{\mathrm{d}\frac{\mathrm{d}f(x)}{\mathrm{d}x}(a) \cdot y}{\mathrm{d}y}(b) \cdot c = \frac{\mathrm{d}f(x)}{\mathrm{d}x}(a) \cdot c$$

Example

For a smooth map $f : \mathbb{R}^n \to \mathbb{R}$, $\mathsf{D}[f]$ is linear in its second argument:

$$\mathsf{D}[f] : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \qquad \mathsf{D}[f](\vec{v}, \vec{w}) := \mathbf{J}(f)(\vec{v}) \cdot \vec{w} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{v}) w_i$$

Cartesian Closed Differential Categories

For a Cartesian closed category, we denote:

- The internal hom by $[A, B]$
- The evaluation map by $\mathsf{ev}_{A,B} : A \times [A, B] \to B$
- The curry of map $f : C \times A \to B$ by $\lambda(f) : A \to [C, B]$

A **Cartesian closed differential category** is a Cartesian differential category which is Cartesian closed such that the evaluation map $\mathsf{ev}_{A,B} : [A, B] \times A \to B$ is linear in its internal hom-argument, which is equivalent to saying that for every map $f : C \times A \to B$, the derivative of its curry $\lambda(f) : A \to [C, B]$ is equal to the curry of the partial of f:

$$\mathsf{D}[\lambda(f)] = \lambda\left(\mathsf{D}[f]\right) \qquad\qquad \frac{\mathsf{d}\lambda x.f(x,u)}{\mathsf{d}u}(a) \cdot b = \lambda x. \frac{\mathsf{d}f(x,u)}{\mathsf{d}u}(a) \cdot b$$

Example

Every model of the differential λ-calculus induces a Cartesian closed differential category. Conversely, every Cartesian closed differential category gives rises to a model of the differential λ-calculus.

Bucciarelli, A., Ehrhard, T. and Manzonetto. G. **Categorical models for simply typed resource calculi.** (2010)

Manzonetto, G. **What is a Categorical Model of the Differential and the Resource λ-Calculi?**. (2012)

J.R.B. Cockett, R. and Gallagher, J. **Categorial models of the differential λ-calculus** (2019)

Cartesian Differential Categories - Other Examples

Example

Bauer, Johnson, Osborne, Riehl, and Tebbe (BJORT) constructed an Abelian functor calculus model of a Cartesian differential category. Brenda will talk about this on Thursday!

Bauer, K., Johnson, B., Osborne, C., Riehl, E. and Tebbe, A. **Directional derivatives and higher order chain rules for abelian functor calculus.** (2018)

Example

There is a couniversal construction of Cartesian differential categories, known as the Faa di Bruno construction, that is, for every Cartesian left additive category there is a cofree Cartesian differential category over it.

Cockett, J.R.B. and Seely, R.A.G. **The Faa di bruno construction**. (2011)

Garner, R, and Lemay, J-S P. **Cartesian differential categories as skew enriched categories**.

Lemay, J-S P. **A Tangent Category Alternative to the Faa di Bruno Construction**.

Lemay, J-S P. **Properties and Characterisations of Cofree Cartesian Differential Categories**.

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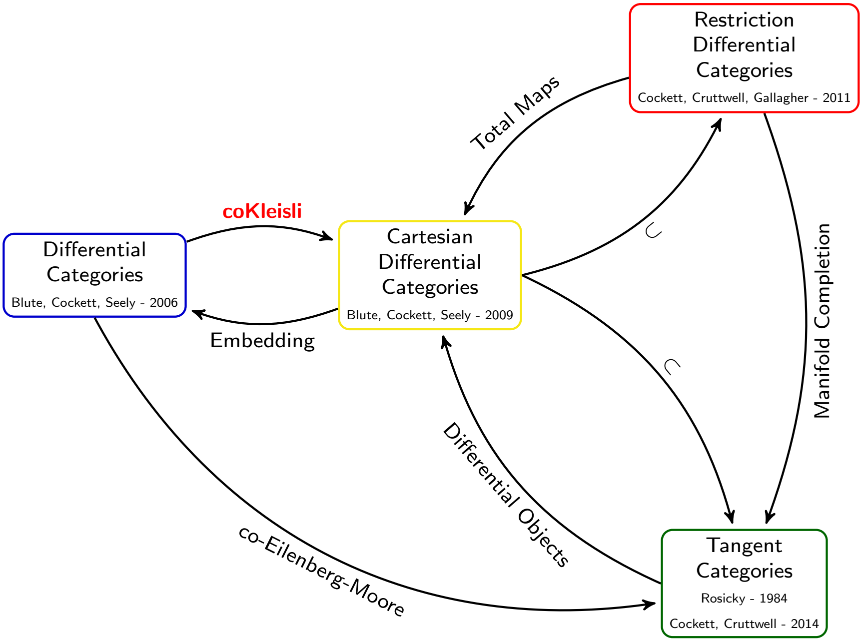
Another important source of CDC comes from differential categories!

The Differential Category World: It’s all connected!



The coKleisli Category of a Differential Category I

Let $\mathcal{L}$ be a differential category with differential modality ! and finite products.

The coKleisli Category of a Differential Category I

Let $\mathcal{L}$ be a differential category with differential modality ! and finite products.

Let $\mathcal{L}_!$ be the coKleisli category and we are going to use interpretation brackets $[\![-]\!]$.

$$\frac{f : A \to B \text{ in } \mathcal{L}_!}{\overline{[\![f]\!] : !A \to B}}$$

$$[\![1]\!] = \; !A \xrightarrow{\quad \mathsf{d} \quad} A$$

$$[\![fg]\!] = \; !A \xrightarrow{\mathsf{p}} !!A \xrightarrow{!([\![f]\!])} !B \xrightarrow{[\![g]\!]} C$$

So how do we make $\mathcal{L}_!$ into a Cartesian differential category?

The coKleisli Category of a Differential Category II

For the product structure:

- On objects, $A \times B$
- Projections:

$$[\![\pi_i]\!] := \;!(A_0 \times A_1) \xrightarrow{\quad\mathsf{d}\quad} A_0 \times A_1 \xrightarrow{\quad\pi_i\quad} A_i$$

For a comonad on a category with finite products, the coKleisli category has finite products.

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For a comonad on a category with finite products, the coKleisli category has finite products.

For the additive structure:

- The sum of maps: $[\![f + g]\!] := [\![f]\!] + [\![g]\!]$
- Zero maps: $[\![0]\!] := 0$

For a comonad on a category with finite biproducts, the coKleisli category is a Cartesian left additive category.

The coKleisli Category of a Differential Category III

Recall that earlier we defined the differential of $[\![f]\!] : !A \to B$ as:

$$!A \otimes A \xrightarrow{\mathsf{d}} !A \xrightarrow{[\![f]\!]} B$$

The coKleisli Category of a Differential Category III

Recall that earlier we defined the differential of $[\![f]\!] : !A \to B$ as:

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But this is not a coKleisli map!

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But this is not a coKleisli map!

The differential combinator $[\![\mathrm{D}[f]]\!] : !(A \times A) \to B$ is defined as follows:

$$!(A \times A) \xrightarrow{\mathsf{c}} !(A \times A) \otimes !(A \times A) \xrightarrow{!(\pi_0) \otimes !(\pi_1)} !A \otimes !A \xrightarrow{1 \otimes \mathsf{d}} !A \otimes A \xrightarrow{\overline{\partial}} !A \xrightarrow{[\![f]\!]} B$$

Theorem

For a differential category with finite products, its coKleisli category is a Cartesian differential category.

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Theorem

For a differential category with finite products, its coKleisli category is a Cartesian differential category.

Every coKleisli map of the form $!A \xrightarrow{\mathsf{d}_A} A \to B$ is linear – so every map from the base category induces a linear map.

This is an if and only if when one has the Seely isomorphisms, in other words, for a differential storage category $\mathcal{L}$, the subcategory of linear maps of $\mathcal{L}_!$ is isomorphic to $\mathcal{L}$.

Some examples

Example

Consider the differential category VEC_k^{op} with $!(V) = \mathsf{Sym}(V)$. Then POLY_k is a sub-CDC of the coKleisli category $\left(\mathsf{VEC}_k^{op}\right)_{\mathsf{Sym}}$.

Example

Consider the differential category $\mathsf{VEC}_{\mathbb{R}}^{op}$ with $!(V) = \mathsf{S}^\infty(V)$. Then SMOOTH is a sub-CDC of the coKleisli category $\left(\mathsf{VEC}_{\mathbb{R}}^{op}\right)_{\mathsf{S}^\infty}$.

More explicit examples are described in:

Bucciarelli, A. and Ehrhard, T. and Manzonetto, G. **Categorical models for simply typed resource calculi**.

which include the relational model and the finiteness space model.

The other direction: Cartesian differential storage categories

Blute, R., Cockett, J.R.B. and Seely, R.A., 2015. **Cartesian differential storage categories**.

"... it was not obvious how to pass from Cartesian differential categories back to monoidal differential categories.This paper provides natural conditions under which the linear maps of a Cartesian differential category form a monoidal differential category. ... The purpose of this paper is to make precise the connection between the two types of differential categories. "

Main idea: While not every Cartesian differential category is the coKleisli category of a differential category, **Cartesian differential storage categories** are precisely the coKleisli categories of differential categories.

Theorem

For a differential storage category, it's coKleisli category is a Cartesian differential storage category. Conversely, for a Cartesian differential storage category, its category of linear maps form a differential storage category.

The other direction: Embedding

Garner, R, and Lemay, J-S P. **Cartesian differential categories as skew enriched categories**.

Theorem

Every Cartesian differential category embeds into the coKleisli category of a differential category.

What can we do with Cartesian differential categories?

- Study and solve differential equations, and also study exponential functions, trigonometric functions, hyperbolic functions, etc.

Cockett, R., Cruttwel, G., Lemay, J-S. P., **Differential equations in a tangent category I: Complete vector fields, flows, and exponentials**.

Lemay, J-S.P., **Exponential Functions for Cartesian Differential Categories**.

- Linearization, Jacobians and gradients:

Cockett, R., Lemay, J-S.P., **Linearizing Combinators**.

Lemay, J-S.P., **Jacobians and Gradients for Cartesian Differential Categories**.

- Foundations for automatic differentiation and machine learning algorithms via **reverse differentiation**.

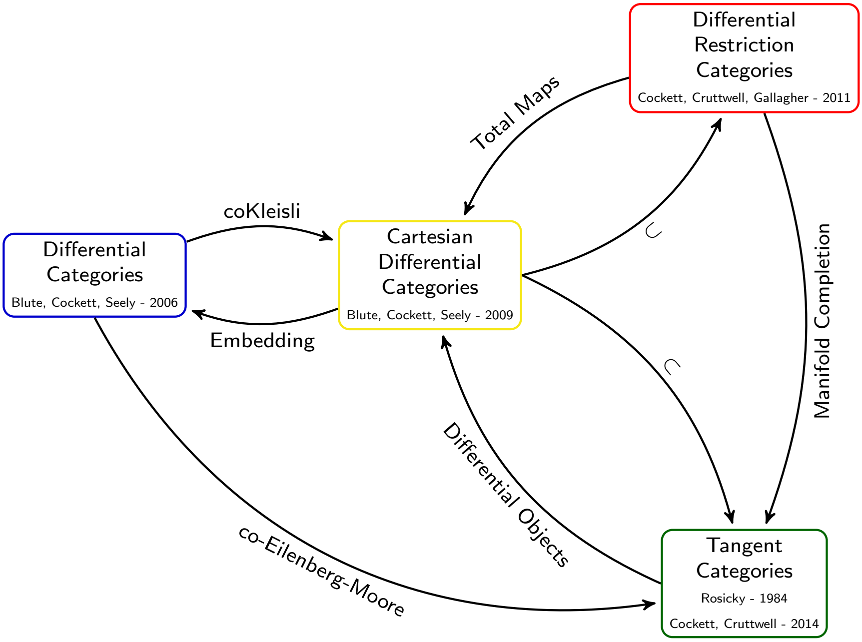
Cockett, R., Cruttwell, G., Gallagher, J., Lemay, J.-S. P., MacAdam, B., Plotkin, G., & Pronk, D. (2020). **Reverse derivative categories.**

Wilson, P., & Zanasi, F. Reverse **Derivative Ascent: A Categorical Approach to Learning Boolean Circuits.**

Cruttwell, G., Gallagher, J., & Pronk, D. **Categorical semantics of a simple differential programming language.**

Cruttwell, G., Gavranovic, B., Ghani, N., Wilson, P., & Zanasi, F. **Categorical Foundations of Gradient-Based Learning.**

The Differential Category World: It's all connected!



A quick word on Differential Restriction Categories

A **restriction category** is a category equipped with a restriction operator

$$\frac{f : A \to B}{\overline{f} : A \to A}$$

where you should think of $\overline{f}$ as capturing the domain of definition of f. Restriction categories allow us to work with partially defined functions.

Lack, S., and Cockett, R. **Restriction Categories (I - III)**.

A **differential restriction category** is NAIVELY a Cartesian differential category with a restriction operator such that the differential operator and restriction operator are compatible.

Cockett, R., Cruttwell, G., and Gallagher, J. **Differential Restriction Categories**.

Example

- The category of smooth functions defined on open subsets is a differential restriction category.
- Any Cartesian differential category is a differential restriction category where $\overline{f} = 1$, so every map is total.
- Conversly, the subcategory of maps such that $\overline{f} = 1$ in a differential restriction category is a Cartesian differential category.

A quick word on Tangent Categories

Tangent Categories:

- Formalize differential calculus on smooth manifold and their tangent bundles
- Formalize notions from differential geometry, algebraic geometry, synthetic differential geometry, etc.

Briefly a tangent category is a category $\mathbb{X}$ equipped with an endofunctor $\mathsf{T} : \mathbb{X} \to \mathbb{X}$, where for an object A we think of $\mathsf{T}(A)$ as the tangent bundle of A – and some natural transformations that capture the essential properties of the tangent bundle of smooth manifolds.

J. Rosický **Abstract tangent functors** (1984)

R. Cockett, G. Cruttwell **Differential structure, tangent structure, and SDG** (2014)

R. Garner **An embedding theorem for tangent categories** (2018)

Example

- The category of finite dimensional smooth manifolds, SMAN is a tangent category where the tangent bundle functor maps a smooth manifold M to its usual tangent bundle $\mathsf{T}(M)$.
- The category of commutative rings, CRING, is a tangent category with the tangent functor which maps a commutative ring R to its ring of dual numbers $\mathsf{T}(R) = R[\epsilon]$.

Tangent Categories – Relation to Differential Categories

Proposition

Every Cartesian differential category $\mathbb{X}$ *with differential combinator* D *is a tangent category:*

$$\mathsf{T}(A) = A \times A \qquad\qquad \mathsf{T}(f)(a,b) = \left\langle f(a), \frac{\mathrm{d}f(x)}{\mathrm{d}x}(a) \cdot b \right\rangle$$

Conversely, the subcategory of **differential objects** *of a tangent category is a CDC.*

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Conversely, the subcategory of **differential objects** *of a tangent category is a CDC.*

Theorem (Cockett, Lemay, Lucyshyn-Wright)

- *The opposite category of the coEilenberg-Moore category of a differential category is a tangent category.*
- *If we have enough limits, the coEilenberg-Moore category of a differential category (representable) tangent category.*

R. Cockett, R., Lemay, J-S. P., Lucyshyn-Wright, R. **Tangent Categories from the Coalgebras of Differential Categories.**

The Differential Category World: It's all connected!

Hope you enjoyed it!
Thanks for listening!
Merci!