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Title: Limit Theorems for the Epstein Zeta-Function

Abstract: In the talk, some results of joint works with A. Laurinčikas on the value-distribution of the Epstein zeta-function will be presented. That is, the generalized Bohr-Jessen type limit theorems in the sense of the weak convergence of probability measures on the complex plane will be considered, and the explicit forms of the limit measures will be given.

Let Q be a positive definite quadratic $n \times n$ matrix and $Q[\underline{x}] = \underline{x}^{\mathrm{T}} Q \underline{x}$ for $\underline{x} \in \mathbb{Z}^n$. The Epstein zeta-function $\zeta(s;Q)$, $s = \sigma + it$, is defined, for $\sigma > \frac{n}{2}$, by the series

$$\zeta(s;Q) = \sum_{\underline{x} \in \mathbb{Z}^n \setminus \{\underline{0}\}} (Q[\underline{x}])^{-s},$$

and can be continued analytically to the whole complex plane, except for a simple pole at the point $s = \frac{n}{2}$ with residue $\frac{\pi^{n/2}}{\Gamma(n/2)\sqrt{\det Q}}$. The function $\zeta(s;Q)$ was introduced by P. Epstein in [1], its value-distribution was investigated by various authors; for example, an extensive survey of the results for the function $\zeta(s;Q)$ is given in [2].

We will present the probabilistic limit theorems of continuous and discrete types for $\zeta(s;Q)$ with even $n \geq 4$ and integers $Q[\underline{x}]$. More precisely, it will be discussed that, for a special differentiable function $\varphi(t)$ with a monotonic derivative,

$$\frac{1}{T} \text{meas} \left\{ t \in [0,T] : \zeta(\sigma + i\varphi(t); Q) \in A \right\}, \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to an explicitly given probability measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ as $T \to \infty$ [3]. Here meas A denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}$, and $\mathcal{B}(\mathbb{C})$ – the Borel σ-field of the space $\mathbb{C}$. Furthermore, it will be showed that, for a certain increasing differentiable function $\varphi(t)$ with a continuous monotonic bounded derivative and with an additional condition for the sequence $\{\varphi(k)\}$, on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$, there exists an explicitly described

probability measure $P_{Q,\sigma}$ such that

$$\frac{1}{N}\#\left\{N \leqslant k \leqslant 2N : \zeta(\sigma + i\varphi(k); Q) \in A\right\}, \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to $P_{Q,\sigma}$ as $N \to \infty$ [4]. Also, a few examples of the function $\varphi(t)$ will be given.

References

[1] Epstein, P. Zur Theorie allgemeiner Zetafunktionen, *Math. Ann.,* **56**, 615–644, 1903.

[2] Nakamura, T., Pańkowski, Ł. On zeros and c-values of Epstein zeta-functions, *Šiauliai Math. Semin.,* **8**(16), 181–195, 2013.

[3] Laurinčikas, A., Macaitienė, R. A generalized Bohr–Jessen type theorem for the Epstein zeta-function. *Mathematics,* 10(12): 2042, 1–11, 2022.

[4] Laurinčikas, A., Macaitienė, R. A generalized discrete Bohr–Jessen type theorem for the Epstein zeta-function. *Mathematics,* 11(4): 799, 1–13, 2023.