Specht ideals for S_n and B_n

S. Debus

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Motivation (Solving symmetric systems of equations)

- $f_1, \ldots, f_m$ symmetric n-variate polynomials, $\deg f_i = d_i$, $d = \max_i d_i$

$$f_1 = 0, \ldots f_m = 0$$

has a solution in $\mathbb{R}^n$ iff there is a solution with $\leq \max\{d, 2\}$ different coordinates.

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- What about the multiplicity structure of solutions $a = (a_1, \ldots, a_n)$?

- $(\mu_1, \ldots, \mu_\ell)$ with $\mu_i = \#\{j \in [n] : a_j = a^i\}$

- If $\min\{d_1, \ldots, d_m\}$ is sufficiently smaller than n, then Specht ideals can help to give an answer [MRV'21]

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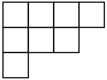
- Can be constructed using Specht polynomials. [Specht 1937]
- $\pi(n) = \#\{\lambda \vdash n\}$ many irred. repr. of S_n & $\sum_{k=0}^{n} \pi(k)\pi(n-k)$ many B_n irred. repr.

Partitions and diagrams

- A partition $\lambda \vdash n$ can be represented by a diagram of shape λ: For $(4, 3, 1) \vdash 8$,



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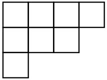
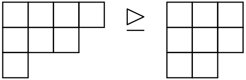


- λ dominates μ if for every k, $\sum_{j=1}^{k} \lambda_j \geq \sum_{j=1}^{k} \mu_j$:



- The poset of partitions of n with respect to the dominance order is a lattice.

Partitions, tableaux and Specht ideals

- A Young tableau of shape $\lambda \vdash n$ is a diagram T of shape λ filled-in with all the integers from 1 to n.

$$\begin{array}{|c|c|c|c|}\hline 4 & 2 & 6 & 1 \\ \hline 8 & 7 & 5 \\ \cline{1-3} 3 \\ \cline{1-1}\end{array}$$

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$$\mathrm{sp}_T = (X_4 - X_8)(X_4 - X_3)(X_8 - X_3)(X_2 - X_7)(X_6 - X_5)$$

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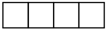
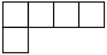
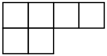
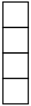
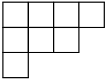
- The (S_n)-Specht ideal associated with λ is:

$$I_\lambda^{\mathrm{sp}} := (\mathrm{sp}_T : T \text{ of shape } \lambda) \subset \mathbb{R}[X_1, \ldots, X_n]$$

- The (S_n)-Specht variety associated with λ is:

$$V_\lambda := \{a \in \mathbb{R}^n : f(a) = 0 \ \forall f \in I_\lambda^{\mathrm{sp}}\}$$

Specht varieties

λ	V_λ
	$\emptyset$
	$\{(a,\ldots,a) : a \in \mathbb{R}\}$
	$S_n \cdot \{(a,\ldots,a,b) : a,b \in \mathbb{R}\}$
	$S_n \cdot \{(a,a,b,c)\}$
	?

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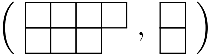
- A bipartition Λ of n can be represented by a bidiagram, i.e., a pair of diagrams: For $((4,3),(1,1))$



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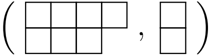

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$$\left(\begin{array}{|c|c|c|c|} \hline 4 & 10 & 6 & 1 \\ \hline 8 & 7 & 5 & \\ \cline{1-3} \end{array} , \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 9 & \\ \cline{1-1} \end{array} \right)$$

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- The $(B_n\text{-})$Specht polynomial $\mathrm{sp}_{(T,S)}$ associated with (T,S) is

$$\mathrm{sp}_{(T,S)} = \mathrm{sp}_T(X_1^2,\ldots,X_n^2)\cdot \mathrm{sp}_S(X_1^2,\ldots,X_n^2)\prod_{j\in S} X_j$$

For instance, $(X_4^2 - X_8^2)(X_{10}^2 - X_7^2)(X_6^2 - X_5^2)(X_2^2 - X_9^2)X_2X_3X_9$.

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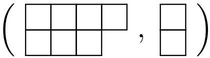

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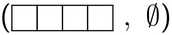
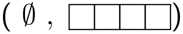
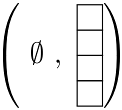
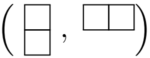
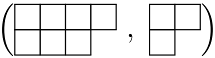
- The (B_n)-Specht ideals and Specht varieties $I^{\mathrm{sp}}_{(\lambda,\mu)}, V_{(\lambda,\mu)}$

B_n Specht varieties

(λ, μ)	$V_{(\lambda,\mu)}$
	$\emptyset$
	$S_n \cdot \{(a, b, c, 0)\}$
	$B_n \cdot \{(a, a, b, c)\} \cup \{(a, b, c, 0)\}$
	$B_n \cdot \{(a, b, 0, 0)\} \cup \{(a, a, a, b)\}$
	?

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- Theorem: [Haiman, Woo'05], [Moustrou, Riener, Verdure'21]
  Let $\lambda, \mu \vdash n$ be partitions. Then,

$$\mu \trianglelefteq \lambda \Leftrightarrow I_\lambda^{\mathrm{sp}} \supseteq I_\mu^{\mathrm{sp}} \Leftrightarrow V_\lambda \subseteq V_\mu.$$

- $H_\mu := \{x \in \mathbb{R}^n : \mathrm{Stab}_{S_n}(x) \cong S_\mu\}$ denotes the S_n-orbit set of μ, e.g.
  $H_{(3,2)} = S_n \cdot \{(a, a, a, b, b) \in \mathbb{R}^5 : a \neq b\}$

- Theorem: [HW'05],[MRV'21]

$$V_\lambda = \left(\biguplus_{\mu \trianglelefteq \lambda} H_\mu \right)^c = \biguplus_{\mu \ntrianglelefteq \lambda} H_\mu,$$

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- Dominance order on $(\lambda_1, \mu_1, \lambda_2, \mu_2, \ldots), (\lambda'_1, \mu'_1, \lambda'_2, \mu'_2, \ldots)$
- Bidominance order defines a poset $(\mathrm{BP}_n, \trianglelefteq)$.

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- The bipartitions $((2,1,1),(3,1)) \trianglelefteq ((3,2),(2,1))$ are comparable:

$$3 \geq 2, \quad 5 \geq 5, \quad 7 \geq 6, \quad 8 \geq 7, \quad 8 \geq 8.$$

$((2),(1,1))$ and $(\emptyset,(4))$ are not comparable, since $2 > 0$ but $3 < 4$.

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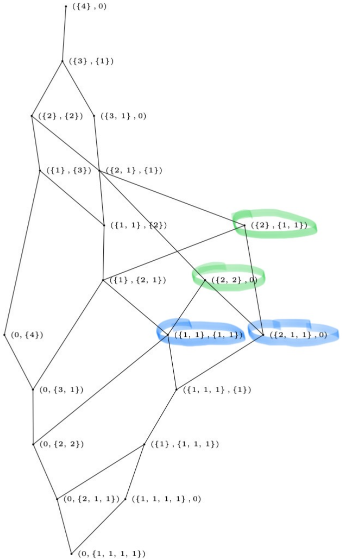
- In general $(\mathrm{BP}_n, \trianglelefteq)$ is not a lattice.

Hasse diagram of $(\mathrm{BP}_4, \trianglelefteq)$



Bipartitions, Specht ideals and Specht varieties

Theorem (DMRV)

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- Not known if $I^{\mathrm{sp}}_{(\lambda,\mu)}$ is radical.
- Classify the covering cases in the poset $(\mathrm{BP}_n, \trianglelefteq)$
- Constructively prove $(\lambda', \mu') \trianglelefteq (\lambda, \mu) \Rightarrow I^{\mathrm{sp}}_{(\lambda',\mu')} \subseteq I^{\mathrm{sp}}_{(\lambda,\mu)}$.
- Prove $V_\Lambda \subset V_\Omega \Rightarrow \Omega \trianglelefteq \Lambda$ by counting #boxes in bidiagrams.

$V^c_{(\vartheta,\omega)} \subset V^c_{(\lambda,\mu)} \Rightarrow (\vartheta,\omega) \trianglelefteq (\lambda,\mu)$

$$v = (\underbrace{1,\dots,1}_{\vartheta_1+\omega_1}, \underbrace{2,\dots,2}_{\vartheta_2+\omega_2}, \underbrace{3,\dots,3}_{\vartheta_3+\omega_3}\dots), w = (\underbrace{0,\dots,0}_{\vartheta_1}, \underbrace{1,\dots,1}_{\omega_1+\vartheta_2}, \underbrace{2,\dots,2}_{\omega_2+\vartheta_3},\dots).$$

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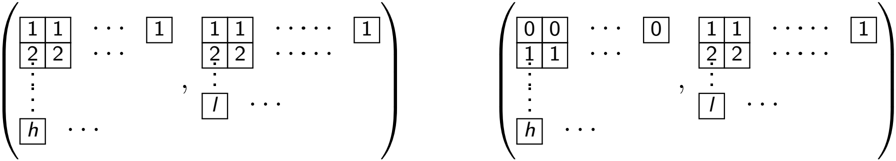

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$$\sum_{j=1}^{k}(\vartheta_j + \omega_j) = \#1's + \ldots + \#k's \leq \sum_{j=1}^{k}(\lambda_j + \mu_j),$$

$$\sum_{j=1}^{k-1}(\vartheta_j + \omega_j) + \vartheta_k = \#0's + \#1's + \ldots + \#(k-1)'s \leq \sum_{j=1}^{k-1}(\lambda_j + \mu_j) + \lambda_k.$$

Orbit types I

- Up to permutation, every $x \in \mathbb{R}^n$ is of the form

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- How to define an orbit type for B_n, e.g., for $(0, 0, 2, -2, 1, 2)$?

Orbit types I

- Up to permutation, every $x \in \mathbb{R}^n$ is of the form

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- Up to sign switching and permutation, every $x \in \mathbb{R}^n$ is of the form

$$x = (\underbrace{0, \ldots, 0}_{t}, \underbrace{x_1, \ldots, x_1}_{\lambda_1}, \underbrace{x_2, \ldots, x_2}_{\lambda_2}, \ldots, \underbrace{x_k, \ldots, x_k}_{\lambda_k})$$

with $x_i > 0$ p.w. distinct.

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  with $x_i > 0$ p.w. distinct.
- Guess: $((t), \lambda)$ or $(\lambda, (t))$

Orbit types II

- Define as cuts:

$$\begin{array}{|c|c|c|c|}\hline 3 & -3 & 3 & -3 \\ \hline 2 & 2 & 2 & 2 \\ \hline 5 & -5 & -5 \\ \cline{1-3} 0 & 0 \\ \cline{1-2} -1 & 1 \\ \cline{1-2} 9 \\ \cline{1-1} \end{array} \quad \to \quad \left(\begin{array}{|c|c|}\hline 3 & 3 \\ \hline 2 & 2 \\ \hline 5 & 5 \\ \hline 0 & 0 \\ \hline 1 & 1 \\ \hline 9 \\ \cline{1-1} \end{array} \; , \; \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 2 & 2 \\ \hline 5 \\ \cline{1-1} \end{array} \right)$$

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- *t-cut* of λ:

$$\text{cut}(\lambda, t) = ((t, \cdots, t, \lambda_j, \cdots, \lambda_m), (\lambda_1 - t, \cdots, \lambda_{j-1} - t))$$

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$$\Omega(z) = \text{cut}(\Lambda(z^2), t_z)$$

where $\mathcal{S}_{\Lambda(z^2)} \simeq_{\mathcal{S}_n} Stab(z^2)$, $t_z = \#0\text{'s in } z$

Orbit types II

- Define as cuts:

3	−3	3	−3
2	2	2	2
5	−5	−5	
0	0		
−1	1		
9			

$\rightarrow$

$$\left(\begin{array}{cc} 3 & 3 \\ 2 & 2 \\ 5 & 5 \\ 0 & 0 \\ 1 & 1 \\ 9 & \end{array}, \begin{array}{cc} 3 & 3 \\ 2 & 2 \\ 5 & \end{array}\right)$$

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where $\mathcal{S}_{\Lambda(z^2)} \simeq_{\mathcal{S}_n} Stab(z^2)$, $t_z = \#0$'s in z
- Not any bipartition has non-empty orbit type

$$\mu_i > 0 \implies \lambda_{i+1} = \lambda_1$$

Orbit types III

- Let $0 < a < b < c \in \mathbb{R}$. For $n = 3$ the orbit types are

$H_{(\lambda,\mu)}$	(λ, μ)
$(0, 0, 0)$	$((3), \emptyset)$
$(\pm a, 0, 0)$	$((2, 1), \emptyset)$
$(\pm a, \pm b, 0)$	$((1, 1, 1), \emptyset)$
$(\pm a, \pm a, 0)$	$((1, 1), (1))$
$(\pm a, \pm a, \pm a)$	$(\emptyset, (3))$
$(\pm a, \pm a, \pm b)$	$(\emptyset, (2, 1))$
$(\pm a, \pm b, \pm c)$	$(\emptyset, (1, 1, 1))$

while $((2), (1)), ((1), (2)), ((1), (1, 1))$ have empty orbit type.

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$(\pm a, \pm a, \pm a)$	$(\emptyset, (3))$
$(\pm a, \pm a, \pm b)$	$(\emptyset, (2,1))$
$(\pm a, \pm b, \pm c)$	$(\emptyset, (1,1,1))$

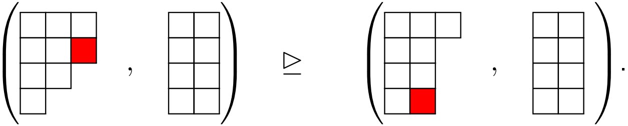
while $((2),(1)), ((1),(2)), ((1),(1,1))$ have empty orbit type.

Theorem (DMRV)

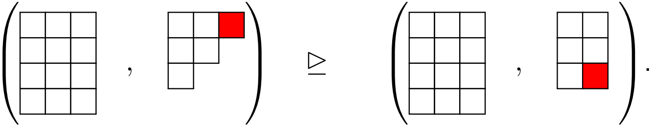
$$V_{(\lambda,\mu)} = \biguplus_{(\lambda',\mu') \not\trianglelefteq (\lambda,\mu)} H_{(\lambda',\mu')} = \left(\biguplus_{(\lambda',\mu') \trianglelefteq (\lambda,\mu)} H_{(\lambda',\mu')} \right)^c$$

The poset of bipartitions I

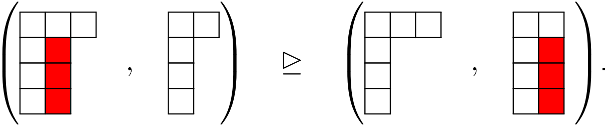
- A covering of type (1):



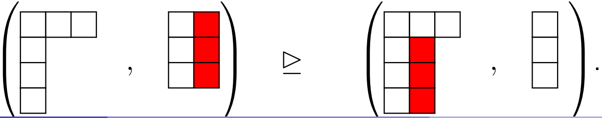
- A covering of type (2):



- A covering of type (3):



- A covering of type (4):



The poset of bipartitions II

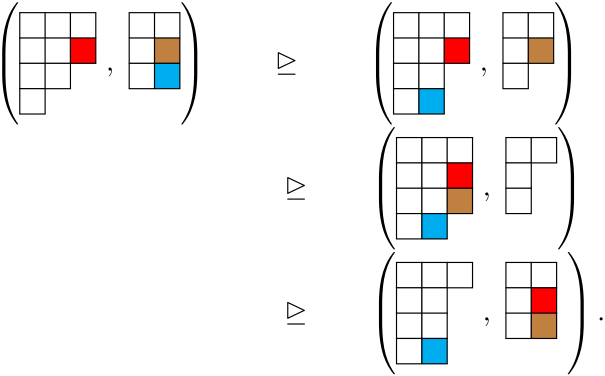
Theorem (DMRV)

Let $(\lambda,\mu),(\lambda',\mu')\in \mathrm{BP}_n$, $i=\min\{j\in[n]:(\lambda_j,\mu_j)\neq(\lambda'_j,\mu'_j)\}$. *Then,* (λ,μ) *covers* (λ',μ') *if and only if one of the following statements is true:*

1. $\mu=\mu'$, λ *covers* λ' *with respect to the dominance order on partitions with* $\lambda'_i=\lambda_i-1$, *and for* k *such that* $\lambda'_k=\lambda_k+1$, *we have* $\mu_{i-1}=\mu_i=\ldots=\mu_k$;
2. $\lambda=\lambda'$, μ *covers* μ' *with respect to the dominance order on partitions, with* $\mu'_i=\mu_i-1$, *and for* k *such that* $\mu'_k=\mu_k+1$, *we have* $\lambda_i=\lambda_{i+1}=\ldots=\lambda_{k+1}$;
3. $\lambda\neq\lambda'$, $\mu\neq\mu'$ *and* $\lambda_i>\lambda'_i$. *If* k *is maximal with* $\lambda_i=\lambda_k$, *then* $\mu_i=\mu_k$, $(\lambda'_j,\mu'_j)=(\lambda_j-1,\mu_j+1)$ *for any integer* $i\leq j\leq k$, *and* $(\lambda'_j,\mu'_j)=(\lambda_j,\mu_j)$ *otherwise;*
4. $\lambda\neq\lambda'$, $\mu\neq\mu'$, $\lambda_i=\lambda'_i$ *(and therefore* $\mu_i>\mu'_i$*). If* k *is maximal with* $\mu_i=\mu_k$, *then* $\lambda_{i+1}=\lambda_{k+1}$, $(\mu'_j,\lambda'_{j+1})=(\mu_j-1,\lambda_{j+1}+1)$ *for any integer* $i\leqslant j\leqslant k$ *and there is equality otherwise.*

The poset of bipartitions III

A non-example for a situation where condition (1) is not fully satisfied



Applications to B_n-invariant ideals

- $I \subset \mathbb{R}[X]$ be a B_n-invariant ideal.
- "Some conditions" $\implies I^{\text{sp}}_{(\lambda,\mu)} \subset I$ and $V(I) \subset V_{(\lambda,\mu)}$.
- $n = 4, f := X_2X_3(X_1^2 - 1) \in I \subset I^{\text{sp}}_{((1,1),(2))}$

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- $n = 4, f := X_2X_3(X_1^2 - 1) \in I \subset I^{\text{sp}}_{((1,1),(2))}$

  $V_{((1,1),(2))} = B_4 \cdot \{(0,0,a,a),(0,0,a,b),(a,a,a,a),(0,0,0,a),(0,0,0,0)\}$

  $V(B_4 \cdot f) = \{(\pm 1, \pm 1, \pm 1, \pm 1),(0,0,0,a),(0,0,0,0)\}.$

- Advantage: Know that many variables are zero.

Outlook

Question

Are the Specht ideals $I^{\mathrm{sp}}_{(\lambda,\mu)}$ *radical and do the Specht polynomials* $\mathcal{G}_{(\lambda,\mu)} := \{\mathrm{sp}_T : \mathit{shape}(T) \in \bigcup_{(\lambda',\mu') \trianglelefteq (\lambda,\mu)} (\lambda', \mu')\}$ *form a universal Gröbner basis?*

- True for S_n [Haiman, Woo'05] and [Murai, Ohsugi, Kohji'21].
- Proofs do not seem to transfer to type B.

Thank you for your attention!

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