Symmetry in Trigonometric Optimization

Tobias Metzlaff
University of Kaiserslautern–Landau

Symmetry, Stability, and interactions with Computation
Luminy 2023

The goal of trigonometric optimization is to find the global minimum of a function $\mathbb{R}^n \to \mathbb{R}$ such as

$$
\begin{aligned}
&-1 + 2/3\,(2\, \cos(2\,\pi\,x)\, \cos((-2\,x-2\,y)\,\pi)^2\, \cos(2\,\pi\,y) + 2\, \cos(2\,\pi\,x)\, \cos(2\,\pi\,y)^2\, \cos((-2\,x-2\,y)\,\pi)\\
&+2\, \cos(2\,\pi\,x)^2\, \cos(2\,\pi\,y)\, \cos((-2\,x-2\,y)\,\pi) + \cos(2\,\pi\,y)^2\, \cos((-2\,x-2\,y)\,\pi)^2 + \sin(2\,\pi\,x)^2\\
&\sin((-2\,x-2\,y)\,\pi)^2 + \cos(2\,\pi\,x)^2\, \cos((-2\,x-2\,y)\,\pi)^2 + \sin(2\,\pi\,x)^2\, \sin(2\,\pi\,y)^2 + \cos(2\,\pi\,x)^2\, \cos(2\,\pi\,y)^2\\
&-\sin(2\,\pi\,y)\, \sin((-2\,x-2\,y)\,\pi) - \cos(2\,\pi\,y)\, \cos((-2\,x-2\,y)\,\pi) - \sin(2\,\pi\,x)\, \sin((-2\,x-2\,y)\,\pi)\\
&-\cos(2\,\pi\,x)\, \cos((-2\,x-2\,y)\,\pi) - \sin(2\,\pi\,x)\, \sin(2\,\pi\,y) - \cos(2\,\pi\,x)\, \cos(2\,\pi\,y) + \sin(2\,\pi\,y)^2\\
&\sin((-2\,x-2\,y)\,\pi)^2 + 2\, \cos(2\,\pi\,x)\, \cos(2\,\pi\,y)\, \sin(2\,\pi\,x)\, \sin((-2\,x-2\,y)\,\pi) + 2\, \cos(2\,\pi\,x)\\
&\cos(2\,\pi\,y)\, \sin(2\,\pi\,x)\, \sin(2\,\pi\,y) + 2\, \cos(2\,\pi\,y)\, \cos((-2\,x-2\,y)\,\pi)\, \sin(2\,\pi\,y)\, \sin((-2\,x-2\,y)\,\pi)\\
&+2\, \sin(2\,\pi\,x)\, \sin((-2\,x-2\,y)\,\pi)\, \cos(2\,\pi\,y)\, \cos((-2\,x-2\,y)\,\pi) + 2\, \cos(2\,\pi\,x)\, \cos((-2\,x-2\,y)\,\pi)\\
&\sin(2\,\pi\,y)\, \sin((-2\,x-2\,y)\,\pi) + 2\, \cos(2\,\pi\,x)\, \cos((-2\,x-2\,y)\,\pi)\, \sin(2\,\pi\,x)\, \sin((-2\,x-2\,y)\,\pi)\\
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&\cos((-2\,x-2\,y)\,\pi) + 2\, \cos(2\,\pi\,x)\, \cos(2\,\pi\,y)\, \sin(2\,\pi\,y)\, \sin((-2\,x-2\,y)\,\pi) + 2\, \sin(2\,\pi\,x)^2\, \sin(2\,\pi\,y)\\
&\sin((-2\,x-2\,y)\,\pi) + 2\, \sin(2\,\pi\,x)\, \sin(2\,\pi\,y)^2\, \sin((-2\,x-2\,y)\,\pi) + 2\, \sin(2\,\pi\,x)\, \sin((-2\,x-2\,y)\,\pi)^2\, \sin(2\,\pi\,y)).
\end{aligned}
$$

By exploiting symmetry, one can often simplify the problem:

Here, we can rewrite the function as a polynomial $\boxed{6\,z^2 - 2\,z - 1}$!

Content

1. Trigonometric polynomials with symmetry
2. Using Chebyshev polynomials
3. Using symmetry adapted bases

[2] with *Evelyne Hubert*, *Philippe Moustrou* and *Cordian Riener* and supported by European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie Actions, grant agreement 813211 (POEMA).

[3] supported by Deutsche Forschungsgemeinschaft transregional collaborative research centre (SFB–TRR) 195 "Symbolic Tools in Mathematics and their Application".

Trigonometric polynomials with symmetry

Trigonometric optimization

Let $\Omega = \mathbb{Z}\,\omega_1 \oplus \ldots \oplus \mathbb{Z}\,\omega_n \leqslant \mathbb{R}^n$ be a lattice and $\langle \cdot, \cdot \rangle$ be the Euclidean scalar product.

$$\mathfrak{e}^{\mu}\,\mathfrak{e}^{\nu} = \mathfrak{e}^{\mu+\nu}$$
$$\mathfrak{e}^{\mu}\,\mathfrak{e}^{-\mu} = \mathfrak{e}^{0}$$

The algebra of trigonometric polynomials

For $\mu \in \Omega$, define $\mathfrak{e}^{\mu} : \mathbb{R}^n \to \mathbb{C}$ with

$$\mathfrak{e}^{\mu}(u) := \exp(-2\pi \mathrm{i}\, \langle \mu, u \rangle)$$

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For $f = \sum\limits_{\mu} f_{\mu}\,\mathfrak{e}^{\mu} \in \mathbb{R}[\Omega]$ with $f_{\mu} = f_{-\mu} \in \mathbb{R}$, find $f^* := \min\limits_{u \in \mathbb{R}^n} f(u)$.

Example ($\Omega = \mathbb{Z}$)

$$f(u) = 2\,\cos(2\,\pi\,u) = \underbrace{\exp(2\,\pi\,\mathrm{i}\,u)}_{=\mathfrak{e}^{-1}(u)} + \underbrace{\exp(-2\,\pi\,\mathrm{i}\,u)}_{=\mathfrak{e}^{1}(u)} \quad \Rightarrow \quad f^* = -2$$

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Crystallographic symmetry groups

Let $\mathcal{W} \leqslant \mathrm{O}_n(\mathbb{R})$ be a finite orthogonal group w.r.t. $\langle\cdot,\cdot\rangle$ and let Ω be a **$\mathcal{W}$–lattice**, that is, for $A \in \mathcal{W}$, $\mu \in \Omega$, we have $A\,\mu \in \Omega$.

Example ($\Omega = \mathbb{Z} = -\mathbb{Z}$, $\mathcal{W} = \{\pm 1\}$)

$$f(u) := 2\cos(2\,\pi\,u) \quad \Rightarrow \quad f(u) = f(-u)$$

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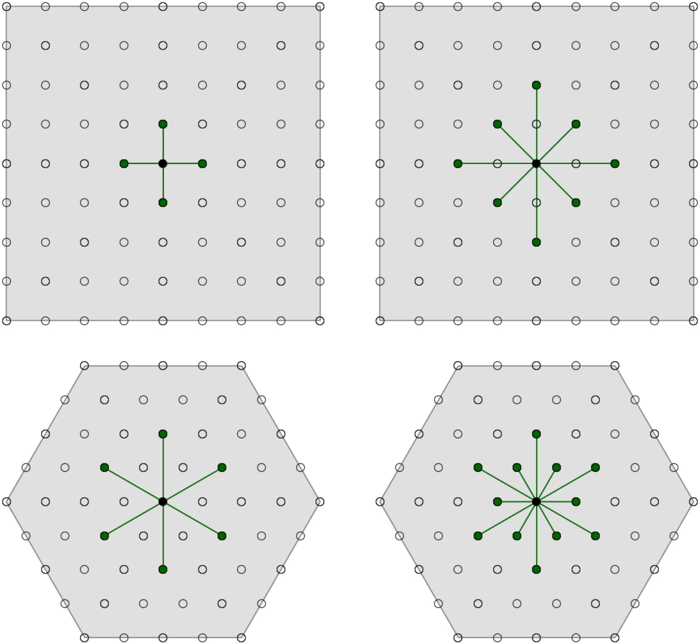
Lattices with crystallographic symmetry



1st approach: Using Chebyshev polynomials

Univariate Chebyshev polynomials

For $\mu \in \mathbb{Z}$, define $T_\mu \in \mathbb{R}[z]$, such that

$$T_\mu(\cos(2\pi u)) = \cos(2\pi \mu u).$$

(Food for thought: Why does this define a polynomial uniquely?)

Then

$$f(u) := 2\cos(2\pi u) = 2\,T_1(\cos(2\pi u))$$

and we have

$$f^* = \min_{u\in\mathbb{R}} f(u) = \min_{z\in \mathrm{im}(\cos(2\pi u))} 2\,T_1(z) = \min_{1-z^2\geq 0} 2z = -2.$$

We require two ingredients:

1. Generalization of cosine functions and Chebyshev polynomials
2. Description of the image of the cosine functions

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The generalized cosine functions

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$$\mathfrak{c}_\mu(u) := \frac{1}{|\mathcal{W}|} \sum_{A \in \mathcal{W}} \mathfrak{e}^{A\mu}(u).$$

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Bourbaki's Theorem

If $\mathcal{W}$ is generated by reflections, then

- the $\mathfrak{c}_{\omega_1}, \ldots, \mathfrak{c}_{\omega_n}$ are algebraically independent and
- $\mathbb{R}[\Omega]^{\mathcal{W}} = \mathbb{R}[\mathfrak{c}_{\omega_1}, \ldots, \mathfrak{c}_{\omega_n}]$ is a polynomial algebra.

The generalized Chebyshev polynomial associated to $\mu \in \Omega$

$T_\mu \in \mathbb{R}[z] = \mathbb{R}[z_1, \ldots, z_n]$, so that $T_\mu(\mathfrak{c}_{\omega_1}(u), \ldots, \mathfrak{c}_{\omega_n}(u)) = \mathfrak{c}_\mu(u)$.

Example ($\Omega = \mathbb{Z}$)

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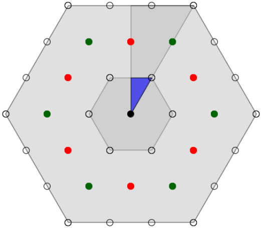
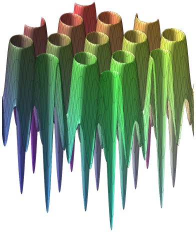
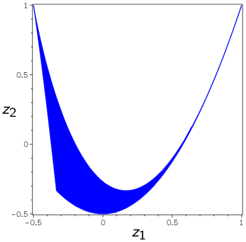
Rewriting the trigonometric optimization problem







Example ($\Omega = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ hexagonal lattice, $\mathcal{W} = \mathfrak{D}_{2\cdot 6}$)

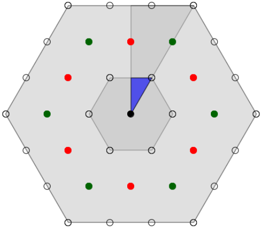
For $S := \mathcal{W}\,\{2\omega_1, \omega_2\}$ and $f_{2\omega_1} := 1$, $f_{\omega_2} := 2$, we have

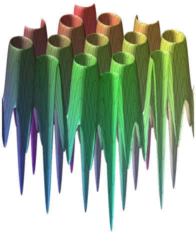
$$\inf_{u\in\mathbb{R}^2} \sum_{\mu\in S} f_\mu\, \mathfrak{c}_\mu(u) = \inf_{z\in\mathcal{T}} T_{2\omega_1}(z) + 2\,T_{\omega_2}(z) = \inf_{z\in\mathcal{T}} 6\,z_1^2 - 2\,z_1 - 1 = -\frac{7}{6}$$

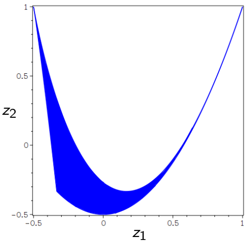
The new feasible region is

$$\mathcal{T} := \mathrm{im}(\mathfrak{c}) = \{\mathfrak{c}(u) := (\mathfrak{c}_{\omega_1}(u), \ldots, \mathfrak{c}_{\omega_n}(u)) \mid u \in \mathbb{R}^n\}.$$

Rewriting the trigonometric optimization problem







Example ($\Omega = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ hexagonal lattice, $\mathcal{W} = \mathfrak{D}_{2\cdot 6}$)

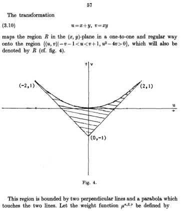
For $S := \mathcal{W}\,\{2\omega_1, \omega_2\}$ and $f_{2\omega_1} := 1$, $f_{\omega_2} := 2$, we have

$$\inf_{u \in \mathbb{R}^2} \sum_{\mu \in S} f_\mu\, \mathfrak{c}_\mu(u) = \inf_{z \in \mathcal{T}} T_{2\omega_1}(z) + 2\, T_{\omega_2}(z) = \inf_{z \in \mathcal{T}} 6\, z_1^2 - 2\, z_1 - 1 = -\frac{7}{6}$$

The new feasible region is

$$\mathcal{T} := \mathrm{im}(\mathfrak{c}) = \{\mathfrak{c}(u) := (\mathfrak{c}_{\omega_1}(u), \dots, \mathfrak{c}_{\omega_n}(u)) \mid u \in \mathbb{R}^n\}.$$

Appearances of $\mathcal{T}$ in the literature



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The transformation

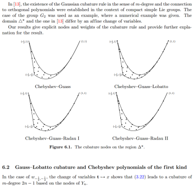
(3.10) $u=x+y,\ v=xy$

maps the region R in the (x,y)-plane in a one-to-one and regular way onto the region $\{(u,v)\,|\,-v-1<u<v+1,\ u^2-4v>0\}$, which will also be denoted by R (cf. fig. 4).

Fig. 4.

This region is bounded by two perpendicular lines and a parabola which touches the two lines. Let the weight function $\mu^{\alpha,\beta,\gamma}$ be defined by

Koornwinder'74



In [13], the existence of the Gaussian cubature rule in the sense of m-degree and the connection to orthogonal polynomials were established in the context of compact simple Lie groups. The case of the group G_2 was used as an example, where a numerical example was given. The domain $\triangle^*$ and the one in [13] differ by an affine change of variables.

Our results give explicit nodes and weights of the cubature rule and provide further explanation for the result.

Figure 6.1. The cubature nodes on the region $\triangle^*$.

6.2 Gauss–Lobatto cubature and Chebyshev polynomials of the first kind

In the case of $w_{-\frac{1}{2},-\frac{1}{2}}$, the change of variables $\mathbf{t}\mapsto x$ shows that (3.22) leads to a cubature of m-degree $2n-1$ based on the nodes of Y_n.

Xu'10

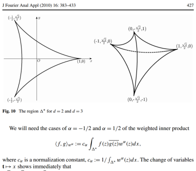


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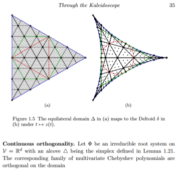
Fig. 10 The region Δ^* for $d=2$ and $d=3$

We will need the cases of $\alpha=-1/2$ and $\alpha=1/2$ of the weighted inner product

$$\langle f,g\rangle_{w^\alpha} := c_\alpha \int_{\Delta^*} f(z)\overline{g(z)}w^\alpha(z)dx,$$

where c_α is a normalization constant, $c_\alpha := 1/\int_{\Delta^*} w^\alpha(z)dx$. The change of variables $\mathbf{t}\mapsto x$ shows immediately that

Xu'12



Through the Kaleidoscope 35

Figure 1.5 The equilateral domain Δ in (a) maps to the Deltoid δ in (b) under $t\mapsto z(t)$.

Continuous orthogonality. Let Φ be an irreducible root system on $\mathcal{V}=\mathbb{R}^d$ with an alcove $\triangle$ being the simplex defined in Lemma 1.21. The corresponding family of multivariate Chebyshev polynomials are orthogonal on the domain

Munthe–Kaas'12

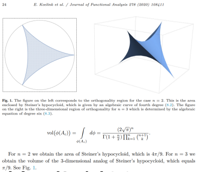


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Fig. 1. The figure on the left corresponds to the orthogonality region for the case $n=2$. This is the area enclosed by Steiner's hypocycloid, which is given by an algebraic curve of fourth degree (8.2). The figure on the right is the three-dimensional region of orthogonality for $n=3$ which is determined by the algebraic equation of degree six (8.3).

$$\mathrm{vol}(\phi(A_c)) = \int_{\phi(A_c)} d\phi = \frac{(2\sqrt{\pi})^n}{\Gamma(1+\frac{n}{2})\prod_{k=1}^n\binom{n+1}{k}}.$$

For $n=2$ we obtain the area of Steiner's hypocycloid, which is $4\pi/9$. For $n=3$ we obtain the volume of the 3-dimensional analog of Steiner's hypocycloid, which equals $\pi/9$. See Fig. 1.

Koelink'20

Describing $\mathcal{T}$ for the irreducible cases

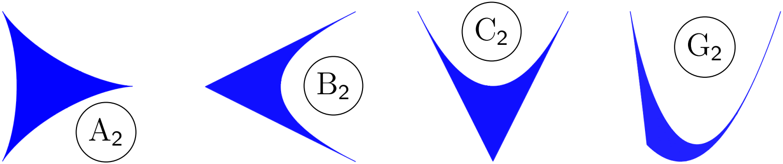
Main result (Hubert, M, Riener'22)

For groups $\mathcal{W}$ of type A_{n-1}, B_n, C_n, D_n or G_2,
we construct a Hankel matrix polynomial $H \in \mathbb{R}[z]^{n\times n}$, such that

$$\mathcal{T} = \{z \in \mathbb{R}^n \mid H(z) \succeq 0\}$$

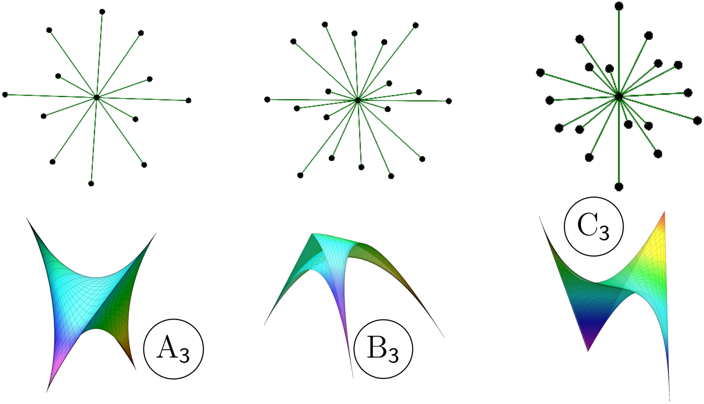
and give a closed formula in the Chebyshev basis:

$$H = \begin{pmatrix} (T_0 - T_{2\omega_1})/2 & (T_{\omega_1} - T_{3\omega_1})/4 & (T_0 - T_{4\omega_1})/8 & \cdots \\ (T_{\omega_1} - T_{3\omega_1})/4 & (T_0 - T_{4\omega_1})/8 & (2T_{\omega_1} - T_{3\omega_1} - T_{5\omega_1})/16 & \cdots \\ (T_0 - T_{4\omega_1})/8 & (2T_{\omega_1} - T_{3\omega_1} - T_{5\omega_1})/16 & (2T_0 + T_{2\omega_1} - 2T_{4\omega_1} - T_{6\omega_1})/32 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$



Describing $\mathcal{T}$ for the irreducible cases

$$H = \begin{pmatrix} (T_0 - T_{2\omega_1})/2 & (T_{\omega_1} - T_{3\omega_1})/4 & (T_0 - T_{4\omega_1})/8 & \cdots \\ (T_{\omega_1} - T_{3\omega_1})/4 & (T_0 - T_{4\omega_1})/8 & (2T_{\omega_1} - T_{3\omega_1} - T_{5\omega_1})/16 & \cdots \\ (T_0 - T_{4\omega_1})/8 & (2T_{\omega_1} - T_{3\omega_1} - T_{5\omega_1})/16 & (2T_0 + T_{2\omega_1} - 2T_{4\omega_1} - T_{6\omega_1})/32 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$



From trigonometric to polynomial optimization

Let $\mathcal{W}$ be a reflection group, Ω a $\mathcal{W}$–lattice and $f \in \mathbb{R}[\Omega]^{\mathcal{W}}$.

Rewriting to a polynomial optimization problem

We seek $f^* = \min\limits_{z \in \mathcal{T}} \sum\limits_{\mu} f_\mu \, T_\mu(z) = \min\limits_{H(z) \succeq 0} \sum\limits_{\mu} f_\mu \, T_\mu(z)$.

- (Lasserre'01) moment/sums of squares hierarchy for polynomial optimization problems with scalar constraints, based on Putinar's Positivstellensatz'93.
- (Henrion, Lasserre'06) ... with matrix constraints, based on the Hol–Scherer Positivstellensatz'05.

We want to use the natural setup of the problem in the Chebyshev basis.

Matrix SOS reinforcement

$$
\begin{aligned}
f^* \;&= \min \quad \sum_{\mu} f_\mu \, T_\mu(z) \\
&\quad\;\; \text{s.t.} \quad z \in \mathbb{R}^n, \; H(z) \succeq 0 \\[1em]
&= \max \quad r \\
&\quad\;\; \text{s.t.} \quad r \in \mathbb{R}, \; \forall\, H(z) \succeq 0 : \\
&\qquad\qquad \sum_{\mu} f_\mu \, T_\mu(z) - r \geq 0.
\end{aligned}
$$

Write $Q \in \mathrm{SOS}(\mathbb{R}[z]^{n \times n})$, if

$\exists\, Q_1, \ldots, Q_k \in \mathbb{R}[z]^n$, s.t.

$$Q(z) = \sum_{i=1}^{k} Q_i(z)\, Q_i(z)^t$$

$$
\begin{aligned}
&\geq \max \quad r \\
&\quad\;\; \text{s.t.} \quad r \in \mathbb{R}, \; q \in \mathrm{SOS}(\mathbb{R}[z]), \; Q \in \mathrm{SOS}(\mathbb{R}[z]^{n \times n}), \\
&\qquad\qquad \sum_{\mu} f_\mu \, T_\mu - r = q + \mathrm{tr}(H\, Q)
\end{aligned}
$$

For computations, restrict q, Q to finite space ($d \in \mathbb{N}$)

$$\mathcal{F}_d := \langle T_\mu \mid \langle \mu, \rho_0^\vee \rangle \leq d \rangle_{\mathbb{R}}$$

$$T_\mu \, T_\nu = \sum_{\langle \omega, \rho_0^\vee \rangle \leq \langle \mu + \nu, \rho_0^\vee \rangle} t_\omega \, T_\omega$$

If $T_\mu \in \mathcal{F}_{d_1}$ and $T_\nu \in \mathcal{F}_{d_2}$, then $T_\mu \, T_\nu \in \mathcal{F}_{d_1 + d_2}$.

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&\quad \text{s.t.} \quad z \in \mathbb{R}^n,\ H(z) \succeq 0 \\
&= \max \quad r \\
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&\quad \text{s.t.} \quad r \in \mathbb{R},\ q \in \mathrm{SOS}(\mathbb{R}[z]),\ Q \in \mathrm{SOS}(\mathbb{R}[z]^{n\times n}), \\
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&= \max \quad r \\
&\quad \text{s.t.} \quad r \in \mathbb{R},\ \forall\, H(z) \succeq 0 : \\
&\qquad\qquad \sum_{\mu} f_\mu \, T_\mu(z) - r \geq 0. \\[1em]
&\geq \max \quad r \\
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\geq \quad & \max \quad r \\
& \text{s.t.} \quad r \in \mathbb{R},\ q \in \mathrm{SOS}(\mathbb{R}[z]),\ Q \in \mathrm{SOS}(\mathbb{R}[z]^{n \times n}), \\
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$$
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$$

$$
T_\mu \, T_\nu = \sum_{\langle \omega, \rho_0^\vee \rangle \leq \langle \mu + \nu, \rho_0^\vee \rangle} t_\omega \, T_\omega
$$

If $T_\mu \in \mathcal{F}_{d_1}$ and $T_\nu \in \mathcal{F}_{d_2}$, then $T_\mu \, T_\nu \in \mathcal{F}_{d_1 + d_2}$.

Matrix SOS reinforcement

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f^* &= \min && \sum_\mu f_\mu\, T_\mu(z) \\
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&= \max && r \\
&\quad \text{s.t.} && r \in \mathbb{R},\ \forall\, H(z) \succeq 0 : \\
& && \sum_\mu f_\mu\, T_\mu(z) - r \geq 0. \\
&\geq \max && r \\
&\quad \text{s.t.} && r \in \mathbb{R},\ q \in \mathrm{SOS}(\mathbb{R}[z]),\ Q \in \mathrm{SOS}(\mathbb{R}[z]^{n\times n}), \\
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If $T_\mu \in \mathcal{F}_{d_1}$ and $T_\nu \in \mathcal{F}_{d_2}$, then $T_\mu\, T_\nu \in \mathcal{F}_{d_1+d_2}$.

Semi–definite lower bounds

SOS hierarchy for trigonometric polynomials with $\mathcal{W}$–symmetry

For $d \in \mathbb{N}$ sufficiently large and $\mathcal{F}_d = \langle T_\mu \mid \langle \mu, \rho_0^\vee \rangle \leq d \rangle_\mathbb{R}$, we have

$$f^* \geq f^d_{\mathrm{Cheby}} := \begin{array}{rl} \max & r \\ \text{s.t.} & r \in \mathbb{R},\ q \in \mathrm{SOS}(\mathcal{F}_d),\ Q \in \mathrm{SOS}(\mathcal{F}_{d-n}^{n\times n}), \\ & \sum\limits_{\mu} f_\mu\, T_\mu - r = q + \mathrm{tr}(H\, Q). \end{array}$$

Then $f^d_{\mathrm{Cheby}} \leq f^{d+1}_{\mathrm{Cheby}}$ and $\lim\limits_{d\to\infty} f^d_{\mathrm{Cheby}} = f^*$.

Translation to an SDP $\to$ MAPLE

Compute $A_0,\ A_\mu \in \mathrm{Sym}^d$, such that

$$f^d_{\mathrm{Cheby}} = \begin{array}{rl} \max & f_0 - \mathrm{tr}(A_0\, X) \\ \text{s.t.} & X \in \mathrm{Cheby}^d_{\succeq 0},\ \forall\, 0 \neq \mu : \\ & \mathrm{tr}(A_\mu\, X) = f_\mu. \end{array}$$

Matrix size:

$$N^d_{\mathrm{Cheby}} := \dim(\mathcal{F}_d) + n \dim(\mathcal{F}_{d-n})$$

Semi–definite lower bounds

SOS hierarchy for trigonometric polynomials with $\mathcal{W}$–symmetry

For $d \in \mathbb{N}$ sufficiently large and $\mathcal{F}_d = \langle T_\mu \mid \langle \mu, \rho_0^\vee \rangle \leq d \rangle_\mathbb{R}$, we have

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Translation to an SDP $\to$ Maple

Compute $A_0,\ A_\mu \in \mathrm{Sym}^d$, such that

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Semi–definite lower bounds

SOS hierarchy for trigonometric polynomials with $\mathcal{W}$–symmetry

For $d \in \mathbb{N}$ sufficiently large and $\mathcal{F}_d = \langle T_\mu \mid \langle \mu, \rho_0^\vee \rangle \leq d \rangle_\mathbb{R}$, we have

$$f^* \geq f^d_{\mathrm{Cheby}} := \begin{array}{rl} \max & r \\ \text{s.t.} & r \in \mathbb{R},\ q \in \mathrm{SOS}(\mathcal{F}_d),\ Q \in \mathrm{SOS}(\mathcal{F}_{d-n}^{n \times n}), \\ & \sum\limits_\mu f_\mu\, T_\mu - r = q + \mathrm{tr}(H\, Q). \end{array}$$

Then $f^d_{\mathrm{Cheby}} \leq f^{d+1}_{\mathrm{Cheby}}$ and $\lim\limits_{d \to \infty} f^d_{\mathrm{Cheby}} = f^*$.

Translation to an SDP → MAPLE

Compute A_0, $A_\mu \in \mathrm{Sym}^d$, such that

$$f^d_{\mathrm{Cheby}} = \begin{array}{rl} \max & f_0 - \mathrm{tr}(A_0\, X) \\ \text{s.t.} & X \in \mathrm{Cheby}^d_{\succeq 0},\ \forall\, 0 \neq \mu : \\ & \mathrm{tr}(A_\mu\, X) = f_\mu. \end{array}$$

Matrix size:

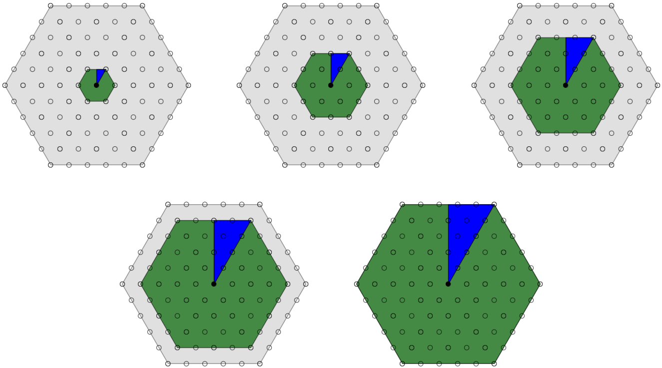
$$N^d_{\mathrm{Cheby}} := \dim(\mathcal{F}_d) + n \dim(\mathcal{F}_{d-n})$$

Comparison with the dense approach

SOHS hierarchy for trigonometric polynomials without symmetry

For $f = \sum_\mu f_\mu \, \mathfrak{e}^\mu \in \mathbb{R}[\Omega]$ with $f_\mu = f_{-\mu} \in \mathbb{R}$, find $f^* := \min_{u \in \mathbb{R}^n} f(u)$.

(Dumitrescu'07) $f^d_{\mathrm{dense}} := \max\{r \in \mathbb{R} \,|\, f - r \in \mathrm{SOHS}(d)\} \to$ SDP.



2nd approach: Using symmetry adapted bases

Translation to matrix form

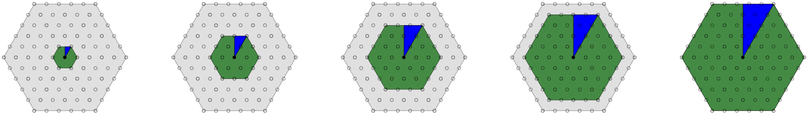
Remark

In the 1st approach, we

1. used symmetry and subsequently
2. applied a sums of squares reinforcement.

Now, we do the same thing in reverse order.

Denote $\Omega_d := \{\mu \in \Omega \mid \langle \mu, \rho_0^\vee \rangle \leq d\}$.



If $f \in \mathbb{R}[\Omega]$ is supported on Ω_{2d}, then we can write

$$f(u) = \overline{\mathbf{E}_d(u)}^t \,\mathbf{mat}(f)\, \mathbf{E}_d(u),$$

where

1. $\mathbf{E}_d(u)$ is the vector of all $\mathfrak{e}^\mu(u)/\sqrt{|\Omega_d|}$ with $\mu \in \Omega_d$ and
2. $\mathbf{mat}(f) \in \mathbb{R}^{\Omega_d \times \Omega_d}$ is a symmetric matrix independent of u.

Translation to matrix form

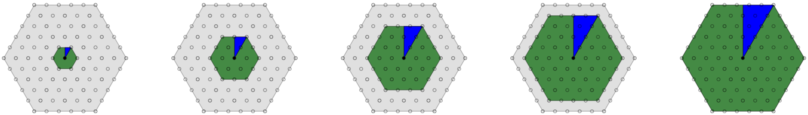
Remark

In the 1st approach, we

1. used symmetry and subsequently
2. applied a sums of squares reinforcement.

Now, we do the same thing in reverse order.

Denote $\Omega_d := \{\mu \in \Omega \,|\, \langle \mu, \rho_0^\vee \rangle \leq d\}$.



If $f \in \mathbb{R}[\Omega]$ is supported on Ω_{2d}, then we can write

$$f(u) = \overline{\mathbf{E}_d(u)}^t \, \mathbf{mat}(f) \, \mathbf{E}_d(u),$$

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1. $\mathbf{E}_d(u)$ is the vector of all $\mathfrak{e}^\mu(u)/\sqrt{|\Omega_d|}$ with $\mu \in \Omega_d$ and
2. $\mathbf{mat}(f) \in \mathbb{R}^{\Omega_d \times \Omega_d}$ is a symmetric matrix independent of u.

Translation to matrix form

Example

$$f := \underbrace{\mathfrak{e}^2 + \mathfrak{e}^{-2}}_{=2\,\cos(4\pi u)} - 2\,(\underbrace{\mathfrak{e}^1 + \mathfrak{e}^{-1}}_{=2\,\cos(2\pi u)}) + 3$$

$$= \underbrace{\begin{pmatrix} \mathfrak{e}^{-1} & 1 & \mathfrak{e}^1 \end{pmatrix} / \sqrt{3}}_{=\overline{\mathbf{E}_1}^t} \underbrace{\begin{pmatrix} 3 & -3 & 3 \\ -3 & 3 & -3 \\ 3 & -3 & 3 \end{pmatrix}}_{=\mathbf{mat}(f)} \underbrace{\begin{pmatrix} \mathfrak{e}^1 & 1 & \mathfrak{e}^{-1} \end{pmatrix} / \sqrt{3}}_{=\mathbf{E}_1}$$

is supported on $\Omega_2 = \{-2, -1, 0, 1, 2\}$.

Remark

We may always assume that $\mathbf{mat}(f)$ is a symm. Toeplitz matrix:

$$\begin{pmatrix} a & b & c \\ d & a & b \\ e & d & a \end{pmatrix} \in \mathrm{Toep}_1 \quad \xRightarrow{f_\mu = f_{-\mu}} \quad b = d,\; c = e$$

Translation to matrix form

Example

$$f := \underbrace{\mathfrak{e}^2 + \mathfrak{e}^{-2}}_{=2\cos(4\pi u)} - 2\,(\underbrace{\mathfrak{e}^1 + \mathfrak{e}^{-1}}_{=2\cos(2\pi u)}) + 3$$

$$= \underbrace{\begin{pmatrix}\mathfrak{e}^{-1} & 1 & \mathfrak{e}^1\end{pmatrix}/\sqrt{3}}_{=\overline{\mathbf{E}_1}^t} \underbrace{\begin{pmatrix} 3 & -3 & 3 \\ -3 & 3 & -3 \\ 3 & -3 & 3 \end{pmatrix}}_{=\mathbf{mat}(f)} \underbrace{\begin{pmatrix}\mathfrak{e}^{1} & 1 & \mathfrak{e}^{-1}\end{pmatrix}/\sqrt{3}}_{=\mathbf{E}_1}$$

is supported on $\Omega_2 = \{-2, -1, 0, 1, 2\}$.

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Matrix action

The action of $\mathcal{W}$ on Toeplitz matrices

$$\begin{array}{rcl} \mathcal{W} \times \mathrm{Toep}_{\boldsymbol{d}} & \to & \mathrm{Toep}_{\boldsymbol{d}}, \\ (A, \mathbf{X} = (\mathbf{X}_{\mu\,\nu})) & \mapsto & A \star \mathbf{X} := (\mathbf{X}_{A^{-1}\mu\, A^{-1}\nu}). \end{array}$$

We have $f \in \mathbb{R}[\Omega]^{\mathcal{W}}$ if and only if $\mathbf{mat}(f) \in \mathrm{Toep}_{\boldsymbol{d}}^{\mathcal{W}}$.

Example ($\mathcal{W} = \{\pm 1\}$, $\Omega_1 = \{-1, 0, 1\}$)

$$\begin{array}{c c} & \begin{array}{ccc} 1 & 0 & -1 \end{array} \\ \begin{array}{c} 1 \\ 0 \\ -1 \end{array} & \begin{pmatrix} a & b & c \\ d & a & b \\ e & d & a \end{pmatrix} \end{array} \xrightarrow{(-1)\star\ldots} \begin{array}{c c} & \begin{array}{ccc} 1 & 0 & -1 \end{array} \\ \begin{array}{c} 1 \\ 0 \\ -1 \end{array} & \begin{pmatrix} a & d & e \\ b & a & d \\ c & b & a \end{pmatrix} \end{array}$$

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Induced action by permutation representation

For $A \in \mathcal{W}$, let $\vartheta(A) \in \mathrm{O}(\mathbb{R}^{\Omega_d})$ be the permutation matrix with $(\vartheta(A)\,\mathbf{x})_\mu = \mathbf{x}_{A^{-1}\mu}$ whenever $\mathbf{x} \in \mathbb{R}^{\Omega_d}$. For $\mathbf{X} \in \mathrm{Toep}_d$, we have

$$A \star \mathbf{X} = \vartheta(A)\,\mathbf{X}\,\vartheta(A)^t.$$

Isotypic decomposition

$$\mathbb{R}^{\Omega_d} = \bigoplus_{i=1}^{h} \left(\bigoplus_{j=1}^{m_i} V_{i\,j} \right)$$

$V_{i\,1} \cong \ldots \cong V_{i\,m_i}$ irred. ϑ–submodules
$m_i \in \mathbb{N}$ multiplicity
$d_i := \dim(V_{i\,j})$

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Block diagonalization

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There is a $\mathbf{T} \in \mathrm{O}(\mathbb{R}^{\Omega_d})$ that transforms any $\mathbf{X} \in \mathrm{Toep}_d^{\mathcal{W}}$ into

$$\mathbf{T}^t\,\mathbf{X}\,\mathbf{T} = \begin{pmatrix} \boxed{\mathbf{X}_1} & & \\ & \ddots & \\ & & \boxed{\mathbf{X}_h} \end{pmatrix} \quad \text{with} \quad \mathbf{X}_i = \begin{pmatrix} \boxed{\tilde{\mathbf{X}}_i} & & \\ & \ddots & \\ & & \boxed{\tilde{\mathbf{X}}_i} \end{pmatrix},$$

where each $\mathbf{X}_i$ consists of d_i identical blocks $\tilde{\mathbf{X}}_i$ of size $m_i \times m_i$.

Example

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Example ($\mathcal{W} = \{\pm 1\}$, $\Omega_1 = \{-1, 0, 1\} \subseteq \mathbb{Z}$)

$$\vartheta(1) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \vartheta(-1) := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\mathbb{R}^3 = \left(\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \rangle_{\mathbb{R}} \oplus \langle \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \rangle_{\mathbb{R}} \right) \oplus \langle \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \rangle_{\mathbb{R}}$$

$$\mathbf{mat}(f) = \begin{pmatrix} 3 & -3 & 3 \\ -3 & 3 & -3 \\ 3 & -3 & 3 \end{pmatrix}, \mathbf{T}^t\, \mathbf{mat}(f)\, \mathbf{T} = \begin{pmatrix} 3 & -3\sqrt{2} & 0 \\ -3\sqrt{2} & 6 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Example

Isotypic decomposition

$$\mathbb{R}^{\Omega_d} = \bigoplus_{i=1}^{h} \left(\bigoplus_{j=1}^{m_i} V_{i\,j} \right)$$

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Example ($\mathcal{W} = \mathfrak{S}_3$, Ω the hexagonal lattice $\subseteq \mathbb{R}^2$)

```
julia> using Oscar;
julia> W = symmetric_group(3);
julia> X = character_table(W);
julia> E = elements(W);
julia> [[X[i](E[s]) for s in 1:length(E)] for i in 1:length(X)]
```

This gives us the character table:

$$\chi = \begin{array}{c c} & \begin{array}{cccccc} \mathrm{id} & s_1 & s_2 & s_1\,s_2 & s_2\,s_1 & s_1\,s_2\,s_1 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & -1 & -1 & 1 & 1 & -1 \\ 2 & 0 & 0 & -1 & -1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

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$$\chi = \begin{array}{cc} & \begin{array}{cccccc} \mathrm{id} & s_1 & s_2 & s_1\,s_2 & s_2\,s_1 & s_1\,s_2\,s_1 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{cccccc} 1 & -1 & -1 & 1 & 1 & -1 \\ 2 & 0 & 0 & -1 & -1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right] \end{array}$$

Fixing the order $d \in \mathbb{N}$ and solving for the multiplicities in

$$\mathrm{tr}(\vartheta(s)) = m_1\,\chi_1(s) + m_2\,\chi_2(s) + m_3\,\chi_3(s)$$

yields

	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$
m_1	0	1	3	6	10	15
m_2	2	6	12	20	30	42
m_3	3	6	10	15	21	28

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We have $\Omega_{d=1} = \{0, -\omega_1, \omega_1 - \omega_2, \omega_2, -\omega_2, \omega_2 - \omega_1, \omega_1\}$.
The to be expected block structure is

$$\begin{pmatrix} \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} & & \\ & \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} & \\ & & \begin{bmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix} \end{pmatrix}$$

Comparison in terms of matrix sizes

Symmetry reduction

The property $f - r \in \mathrm{SOHS}(d)$ can be parametrized with positive semidefinite Toeplitz matrices $\in \mathrm{Toep}_d$. In the SDP

$$\boxed{\text{(Dumitrescu'07)} \quad f_{\mathrm{dense}}^{d} := \sup\{r \in \mathbb{R} \mid f - r \in \mathrm{SOHS}(d)\}}$$

we may restrict to invariant (block diagonal) matrices $\in \mathrm{Toep}_d^{\mathcal{W}}$.

Number of nonzero matrix entries (d = order of the hierarchy)

- dense: $|\Omega_d|^2$
- Chebyshev: $\frac{|\Omega_d|^2 + n^2\,|\Omega_{d-D}|^2}{|\mathcal{W}|^2}$
- Symmetry adapted basis: $\sum_{i=1}^{h} (m_i^{(d)}\, d_i)^2$

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Thanks for your attention.

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