Symmetry reduction to optimize a graph polynomial from queueing theory

Sven Polak

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Symmetry, Stability, and interactions with Computation

Start of story

Ellen Cardinaels

Sem Borst

Johan van Leeuwaarden

Power-of-two sampling in redundancy systems: The impact of assignment constraints, *Operations Research Letters* (2022)

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Power-of-two sampling in redundancy systems: The impact of assignment constraints, *Operations Research Letters* (2022)

Daniel Brosch

Monique Laurent

Andries Steenkamp

Optimizing Hypergraph-Based Polynomials Modeling Job-Occupancy in Queuing with Redundancy Scheduling, *SIAM Journal on Optimization* (2021)

Conjecture

For given integers $n, d \geq 2$, define $V := [n]$, $E := \{e \subseteq V \ : \ |e| = 2\}$, and

$$f_d := \sum_{(e_1,\ldots,e_d)\in E^d} \prod_{i=1}^{d} \frac{x_{e_i}}{|e_1 \cup \ldots \cup e_i|} \quad \in \mathbb{R}[x_e \mid e \in E].$$

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Set $m = |E| = \binom{n}{2}$. The standard simplex in $\mathbb{R}^m$ is

$$\Delta_m := \left\{ x \in \mathbb{R}^m \ : \ x \geq 0, \sum_{e\in E} x_e = 1 \right\}.$$

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Conjecture (Cardinaels, Borst, Van Leeuwaarden 2022)

The polynomial f_d attains its minimum over Δ_m at $x^* = \frac{1}{m}(1,\dots,1)$.

Lemma (Brosch, Laurent, Steenkamp 2021)

If f_d is convex over Δ_m, then f_d attains its minimum over Δ_m at x^*.

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2. They also consider

$$p_d(x) := \sum_{(e_1,\ldots,e_d)\in E^d} \frac{1}{|e_1 \cup \ldots \cup e_d|} x_{e_1} \cdots x_{e_d},$$

and prove its convexity, for all d and all edge sizes L.

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Strategy: prove that the Hessian $H(f_d)$ is positive semidefinite.
Symmetry reduction, properties of Hamming and Johnson schemes.

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Proof.

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Proof. For $\sigma \in S_n$ and $e = \{u, v\} \in E$, set $\sigma(e) = \{\sigma(u), \sigma(v)\}$. For $x \in \Delta_m$, set $\sigma(x) := (x_{\sigma(e)})_{e \in E} \in \Delta_m$. Then

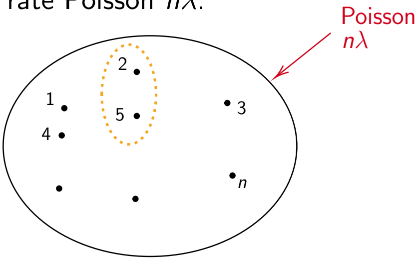
$$f_d(\sigma(x)) = f_d(x).$$

Then $f_d(x^*) = f_d\Big(\underbrace{\tfrac{1}{n!}\sum_{\sigma \in S_n} \sigma(x)}_{= x^*}\Big) \le \tfrac{1}{n!}\sum_{\sigma \in S_n} f_d(\sigma(x)) = f_d(x)$. □

Motivation

Parallel-servers system with redundancy.

Assume: n parallel servers, speed μ. Jobs arrive at rate Poisson $n\lambda$.



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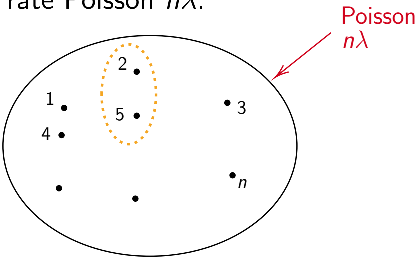

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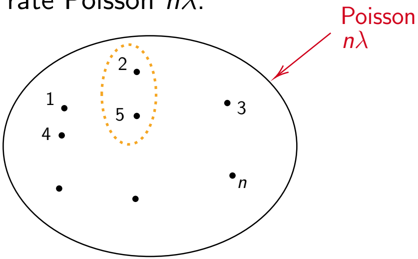

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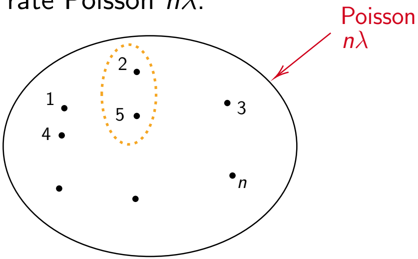

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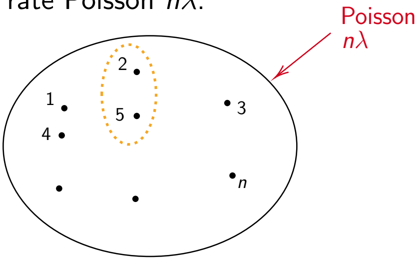

Link to f_d

System occupancy is minimized in the light-traffic regime when $x = x^*$.
(If f_d attains minimum over Δ_m at x^*.)

New contribution

Theorem (P., 2022)

The polynomial f_d is convex over Δ_m for $d \leq 9$, for all $n \geq 2$.

Proof strategy

First express Hessian as matrix polynomial in matrices $Q_{\gamma(n)}$. Then:

1. Find block-diagonalization of these $Q_{\gamma(n)}$.
2. Eliminate n from coefficients of block-diagonalized matrices. (crucial step to verify conjecture for all n, for fixed d).

Verify positive semidefiniteness of the resulting matrices (computer).

The Hessian

To show: f_d is convex over Δ_m. Equivalently:

$$H(f_d) := \left(\frac{\partial^2 f_d(x)}{\partial x_e \partial x_f} \right) \quad \text{is positive semidefinite over } \Delta_m.$$

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with $\mathbb{N}_d^m := \{(\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}_{\geq 0}^m \mid \sum_{i=1}^m \alpha_i = d\}$, and all $b_\alpha \in \mathbb{Q}$.

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$$H(f_d) = \sum_{\gamma\in\mathbb{N}_{d-2}^m} Q_\gamma x^\gamma, \quad \text{for some } Q_\gamma \in \mathbb{R}^{E\times E}.$$

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Observation

If $Q_\gamma \succeq 0$ for all $\gamma \in \mathbb{N}_{d-2}^m$, then $H(f_d) \succeq 0$ on Δ_m.

Detailed definition of Q_γ

Definition of b_α

$$b_\alpha := \sum_{\substack{(e_1,\ldots,e_d)\in E^d \\ \alpha_n(\underline{e})=\alpha}} \prod_{i=1}^{d} \frac{1}{|e_1 \cup \ldots \cup e_i|},$$

where $\alpha_n(\underline{e}) \in \mathbb{N}_d^m$ is such that

$$x_{e_1} \cdot \ldots \cdot x_{e_d} = x^{\alpha_n(\underline{e})}.$$

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Example of sequence $\alpha_n(\underline{e})$

$\underline{e} = (\{1,2\},\{2,3\},\{1,2\})$, gives $\alpha_3(\underline{e}) = (2,0,1)$.

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For $e \in E$, let v_e be the corresponding standard basis vector in $\mathbb{C}^E$. Then

$$H(f_d)(x) = \sum_{\gamma\in\mathbb{N}_{d-2}^m} x^\gamma Q_\gamma, \text{ where } (Q_\gamma)_{i,j} = \begin{cases} (\gamma_i+1)(\gamma_j+1)b_{\gamma+v_i+v_j} & \text{if } i \neq j, \\ (\gamma_i+1)(\gamma_i+2)b_{\gamma+2v_i} & \text{if } i = j. \end{cases}$$

The case $d = 3$

Note: $H(f_3) = \sum_{g \in E} Q_g x_g$. Want to show: $Q_g \succeq 0$ for all $g \in E$.
WMA: $g = e_1 := \{1, 2\}$. Write $A := Q_{e_1}$. Then:

$$A_{e,f} = \frac{1}{|e_1 \cup e \cup f|}\left(\frac{1}{|e_1 \cup e|} + \frac{1}{|e_1 \cup f|} + \frac{1}{|e \cup f|}\right)$$

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Let $S_{n-2} := \{\sigma \in S_n \mid \sigma(1) = 1,\ \sigma(2) = 2\}$.

Crucial property

Matrix A is invariant under S_{n-2}, meaning that
for $\sigma \in S_{n-2}$: $A_{\sigma(e),\sigma(f)} = A_{e,f}$ for all $e, f \in E$.

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$\Longrightarrow$ Can apply a symmetry reduction!

Centralizer algebra

Write $\left(\mathbb{C}^{E\times E}\right)^{S_{n-2}}$ for the set of S_{n-2}-invariant matrices in $\mathbb{C}^{E\times E}$.

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$$\left(\mathbb{C}^{E\times E}\right)^{S_{n-2}} := \left\{A \in \mathbb{C}^{E\times E} \mid A_{\sigma(e),\sigma(f)} = A_{e,f} \;\; \forall \sigma \in S_{n-2}, e, f \in E\right\}.$$

Centralizer algebra

$(\mathbb{C}^{E\times E})^{S_{n-2}}$ is called the centralizer algebra for the action of S_{n-2} on $\mathbb{C}^{E}$.

Artin-Wedderburn

By Artin-Wedderburn, the algebra $(\mathbb{C}^{E\times E})^{S_{n-2}}$ can be block-diagonalized.

General theory

Let G be a finite group acting on a $\mathbb{C}$-vector space V.
(Our case: $G := S_{n-2}$ and $V := \mathbb{C}^E$). Decompose V as:

$$V = \bigoplus_{i=1}^{k} \underbrace{\bigoplus_{j=1}^{m_i} V_{i,j}}_{=: V_i},$$

such that the $V_{i,j}$ are irreducible G-modules with $V_{i,j} \cong V_{i',j'}$ iff $i = i'$.

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Any set $\{U_1, \ldots, U_k\}$ obtained in this way is called representative for the action of G on V.

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Define the map

$$\Phi : \left(\mathbb{C}^{E \times E}\right)^{S_{n-2}} \to \bigoplus_{i=1}^{k} \mathbb{C}^{m_i \times m_i},$$
$$B \mapsto \bigoplus_{i=1}^{k} U_i^* B U_i$$

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Key fact

The map Φ is bijective, and $\forall B \in \left(\mathbb{C}^{E\times E}\right)^{S_{n-2}}$:
$B \succeq 0 \iff \Phi(B) \succeq 0$.

Application

So $A^{(n)} \in (\mathbb{C}^{E \times E})^{S_{n-2}}$ with

$$A^{(n)}{}_{e,f} = \frac{1}{|e_1 \cup e \cup f|}\left(\frac{1}{|e_1 \cup e|} + \frac{1}{|e_1 \cup f|} + \frac{1}{|e \cup f|}\right)$$

is positive semidefinite if and only if

$$\underbrace{\bigoplus_{i=1}^{3} U_i^{(n)*} A^{(n)} U_i^{(n)}}_{\text{constant size, coefficients dependent on } n} \succeq 0. \qquad (1)$$

Crucial extra step to verify positive semidefiniteness for all n

Can prove that matrices in (1) are positive semidefinite for all $n \geq 2$ if and only if 3 constant matrices are positive semidefinite.

(Proof technique: elementary matrix operations preserving positive semidefiniteness, limit argument.)

The case $d = 3$

Recall

$$A^{(n)}_{e,f} = \frac{1}{|e_1 \cup e \cup f|}\left(\frac{1}{|e_1 \cup e|} + \frac{1}{|e_1 \cup f|} + \frac{1}{|e \cup f|}\right), \quad \text{for } e, f \in \binom{[n]}{2}. \quad \text{Then (if } n \geq 6)$$

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$$U_1^{(n)} A^{(n)} U_1^{(n)} = \begin{pmatrix} \frac{3}{4} & \frac{7}{18}(n-2) & \frac{7}{18}(n-2) & \frac{1}{4}\binom{n-2}{2} \\ \frac{7}{18}(n-2) & \left(\frac{1}{4}n - \frac{13}{36}\right)(n-2) & \left(\frac{11}{48}n - \frac{17}{48}\right)(n-2) & \left(\frac{1}{6}n - \frac{5}{24}\right)\binom{n-2}{2} \\ \frac{7}{18}(n-2) & \left(\frac{11}{48}n - \frac{17}{48}\right)(n-2) & \left(\frac{1}{4}n - \frac{13}{36}\right)(n-2) & \left(\frac{1}{6}n - \frac{5}{24}\right)\binom{n-2}{2} \\ \frac{1}{4}\binom{n-2}{2} & \left(\frac{1}{6}n - \frac{5}{24}\right)\binom{n-2}{2} & \left(\frac{1}{6}n - \frac{5}{24}\right)\binom{n-2}{2} & \left(\frac{1}{16}n^2 - \frac{11}{48}n + \frac{1}{6}\right)\binom{n-2}{2} \end{pmatrix},$$

$$U_2^{(n)} A^{(n)} U_2^{(n)} = \begin{pmatrix} \frac{10}{36} & \frac{10}{48} & \frac{2}{16}(n-4) \\ \frac{10}{48} & \frac{10}{36} & \frac{2}{16}(n-4) \\ \frac{2}{16}(n-4) & \frac{2}{16}(n-4) & \left(\frac{1}{24}n - \frac{1}{8}\right)2(n-4) \end{pmatrix},$$

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When positive semidefinite for all $n \geq 6$?

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$$U_2^{(n)} A^{(n)} U_2^{(n)} = \begin{pmatrix} \frac{10}{36} & \frac{10}{48} & \frac{2}{16}(n-4) \\ \frac{10}{48} & \frac{10}{36} & \frac{2}{16}(n-4) \\ \frac{2}{16}(n-4) & \frac{2}{16}(n-4) & \left(\frac{1}{24}n - \frac{1}{8}\right) 2(n-4) \end{pmatrix},$$

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Positive semidefinite for all $n \geq 6$ if and only if

$$B_2 = \begin{pmatrix} \frac{5}{36} & \frac{5}{48} & \frac{1}{16} \\ \frac{5}{48} & \frac{5}{36} & \frac{1}{16} \\ \frac{1}{16} & \frac{1}{16} & \frac{1}{24} \end{pmatrix}, \quad B_3 = \frac{1}{24} \quad \text{are positive semidefinite.}$$

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Why? The matrix $U_2^{(n)} A^{(n)} U_2^{(n)}$ is PSD iff

$$\begin{pmatrix} \frac{10}{36} & \frac{10}{48} & \frac{2}{16} \\ \frac{10}{48} & \frac{10}{36} & \frac{2}{16} \\ \frac{2}{16} & \frac{2}{16} & \frac{2\left(\frac{1}{24}n - \frac{1}{8}\right)}{n-4} \end{pmatrix} = \begin{pmatrix} \frac{10}{36} & \frac{10}{48} & \frac{2}{16} \\ \frac{10}{48} & \frac{10}{36} & \frac{2}{16} \\ \frac{2}{16} & \frac{2}{16} & \frac{2}{24}\frac{n-3}{n-4} \end{pmatrix} = 2B_2 + \frac{1}{n-4}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2B_3 \end{pmatrix} \succeq 0$$

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Conclusion

$U_2^{(n)} A^{(n)} U_2^{(n)}$ and $U_3^{(n)} A^{(n)} U_3^{(n)}$ are PSD for all $n \geq 6$ if and only if B_2, B_3 are PSD.

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Conclusion

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Can be extended: $U_i^{(n)} A^{(n)} U_i^{(n)} \succeq 0$ for all $n \geq 6$ and $i \in [3]$ if and only if 3 constant matrices are PSD

Fix $k \in \mathbb{N}$. Let $S_{n-k} := \{\sigma \in S_n \mid \sigma(i) = i \text{ for } i = 1, \ldots, k\}$.

Auxiliary result to remove dependence on n (P., 2022)

Suppose that $(A^{(n)})_{n \geq k}$ is a sequence of symmetric matrices such that:

1. $A^{(n)} \in (\mathbb{R}^{\binom{[n]}{2} \times \binom{[n]}{2}})^{S_{n-k}}$ for each $n \geq k$,
2. For all $n', n \in \mathbb{N}$ with $k \leq n' \leq n$ and all $e_i, e_j \in \binom{[n']}{2}$, $A^{(n')}_{e_i,e_j} = A^{(n)}_{e_i,e_j}$,

There exist three constant matrices B_1, B_2, B_3 (size and coefficients independent of n), constructed from $A^{(k+4)}$, such that:

$$A^{(n)} \succeq 0 \;\; \forall n \geq k \quad \iff \quad B_1, B_2, B_3 \succeq 0.$$

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$$A^{(n)} \succeq 0 \;\; \forall n \geq k \quad \Longleftrightarrow \quad B_1, B_2, B_3 \succeq 0.$$

1. B_1 has order $(\binom{k}{2} + k + 1) \times (\binom{k}{2} + k + 1)$.
2. B_2 has order $(k+1) \times (k+1)$.
3. B_3 has order 1×1.

$$B_1 = \begin{array}{c|c|c|c} & \binom{k}{2} & k & 1 \\ \hline \binom{k}{2} & \left(a_{(e_i,e_j)}\right)_{e_i,e_j\in\binom{[k]}{2}} & \left(a_{(e_i,\{j,k+1\})}\right)_{e_i\in\binom{[k]}{2},\, j\in[k]} & \left(a_{(e_i,\{k+1,k+2\})}\right)_{e_i\in\binom{[k]}{2}} \\ \hline k & \left(a_{(e_i,\{j,k+1\})}\right)^{\mathsf{T}}_{e_i\in\binom{[k]}{2},\, j\in[k]} & (a_{(\{i,k+1\},\{j,k+2\})})_{i,j\in[k]} & (a_{(\{i,k+1\},\{k+2,k+3\})})_{i\in[k]} \\ \hline 1 & \left(a_{(e_i,\{k+1,k+2\})}\right)^{\mathsf{T}}_{e_i\in\binom{[k]}{2}} & (a_{(\{i,k+1\},\{k+2,k+3\})})^{\mathsf{T}}_{i\in[k]} & a_{(\{k+1,k+2\},\{k+3,k+4\})} \end{array},$$

$$B_2 = \begin{array}{c|c|c} & k & 1 \\ \hline k & \left(a_{(\{i,k+1\},\{j,k+1\})} - a_{(\{i,k+1\},\{j,k+2\})}\right)_{i,j\in[k]} & \left(a_{(\{i,k+1\},\{k+1,k+2\})} - a_{(\{i,k+1\},\{k+2,k+3\})}\right)_{i\in[k]} \\ \hline 1 & \left(a_{(\{i,k+1\},\{k+1,k+2\})} - a_{(\{i,k+1\},\{k+2,k+3\})}\right)^{\mathsf{T}}_{i\in[k]} & a_{(\{k+1,k+2\},\{k+1,k+3\})} - a_{(\{k+1,k+2\},\{k+3,k+4\})} \end{array},$$

$$B_3 = a_{(\{k+1,k+2\},\{k+1,k+2\})} - 2a_{(\{k+1,k+2\},\{k+1,k+3\})} + a_{(\{k+1,k+2\},\{k+3,k+4\})}$$

Here we write $a_{(e_i,e_j)}$ for the (e_i, e_j)-th entry of $A^{(k+4)}$, for $e_i, e_j \in \binom{[k+4]}{2}$.

How to verify conjecture for fixed d, for all n

Theorem (P., 2022)

The polynomial f_d is convex over Δ_m for $d \leq 9$, for all $n \geq 2$.

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Recall: $H(f_d) = \sum_{\gamma(n) \in \mathbb{N}^m_{d-2}} Q_{\gamma(n)} x^{\gamma(n)}$.

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3. For each $n \geq k$, set $\gamma(n) := \alpha_n(\underline{e}) \in \mathbb{N}^m_{d-2}$, and $A^{(n)} := \gamma(n)! Q_{\gamma(n)}$.

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d	multigraphs	λ_{min}
3	1	**0.00357563**
4	3	**0.00059703**
5	8	**0.00015202**
6	23	**0.00004653**
7	66	**0.00001583**
8	212	**0.00000576**
9	686	**0.00000220**

The case $d = 3$

One multigraph: $\underline{e} = (\{1,2\})$. Then $k = |\{1,2\}| = 2$, and

$$A^{(n)}_{e,f} = \frac{1}{|e_1 \cup e \cup f|}\left(\frac{1}{|e_1 \cup e|} + \frac{1}{|e_1 \cup f|} + \frac{1}{|e \cup f|}\right), \quad \text{for } e, f \in \binom{[n]}{2}. \quad \text{Then}$$

$$U^{(n)}_1 A^{(n)} U^{(n)}_1 = \begin{pmatrix} \frac{3}{4} & \frac{7}{18}(n-2) & \frac{7}{18}(n-2) & \frac{1}{4}\binom{n-2}{2} \\ \frac{7}{18}(n-2) & \left(\frac{1}{4}n - \frac{13}{36}\right)(n-2) & \left(\frac{11}{48}n - \frac{17}{48}\right)(n-2) & \left(\frac{1}{6}n - \frac{5}{24}\right)\binom{n-2}{2} \\ \frac{7}{18}(n-2) & \left(\frac{11}{48}n - \frac{17}{48}\right)(n-2) & \left(\frac{1}{4}n - \frac{13}{36}\right)(n-2) & \left(\frac{1}{6}n - \frac{5}{24}\right)\binom{n-2}{2} \\ \frac{1}{4}\binom{n-2}{2} & \left(\frac{1}{6}n - \frac{5}{24}\right)\binom{n-2}{2} & \left(\frac{1}{6}n - \frac{5}{24}\right)\binom{n-2}{2} & \left(\frac{1}{16}n^2 - \frac{11}{48}n + \frac{1}{6}\right)\binom{n-2}{2} \end{pmatrix},$$

$$U^{(n)}_2 A^{(n)} U^{(n)}_2 = \begin{pmatrix} \frac{10}{36} & \frac{10}{48} & \frac{2}{16}(n-4) \\ \frac{10}{48} & \frac{10}{36} & \frac{2}{16}(n-4) \\ \frac{2}{16}(n-4) & \frac{2}{16}(n-4) & \left(\frac{1}{24}n - \frac{1}{8}\right)2(n-4) \end{pmatrix},$$

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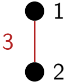
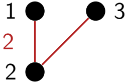
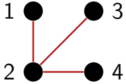
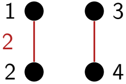
Positive semidefinite for all $n \geq 6$ if and only if

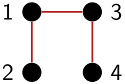
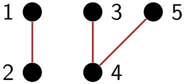
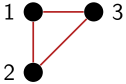
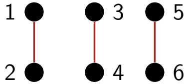
$$B_1 = \begin{pmatrix} \frac{3}{4} & \frac{7}{18} & \frac{7}{18} & \frac{1}{4} \\ \frac{7}{18} & \frac{1}{4} & \frac{11}{48} & \frac{1}{6} \\ \frac{7}{18} & \frac{11}{48} & \frac{1}{4} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{6} & \frac{1}{6} & \frac{1}{8} \end{pmatrix}, \quad B_2 = \begin{pmatrix} \frac{5}{36} & \frac{5}{48} & \frac{1}{16} \\ \frac{5}{48} & \frac{5}{36} & \frac{1}{16} \\ \frac{1}{16} & \frac{1}{16} & \frac{1}{24} \end{pmatrix}, \quad B_3 = \frac{1}{24} \quad \text{are positive semidefinite.}$$

This is the case as $\lambda_{\min}(B_1) \approx 0.00357563$, $\lambda_{\min}(B_2) \approx 0.00837652$.

For each sequence $\alpha \in \mathbb{N}_d^m$, set $\widehat{b}_\alpha := \alpha! b_\alpha$.

Recall that $A^{(n)} := \gamma(n)! Q_{\gamma(n)} = \left(\widehat{b}_{\gamma(n)+v_{e_i}+v_{e_j}}\right)_{i,j=1}^{m}$. Consider $d = 5$:

multigraph				
$\lambda_{\min}(B_1)$	0.00155469	0.00075588	0.00041988	0.00042023
$\lambda_{\min}(B_2)$	0.00916612	0.00401292	0.00226269	0.00226730
$\lambda_{\min}(B_3)$	0.06250000	0.03240741	0.02083333	0.02083333

multigraph				
$\lambda_{\min}(B_1)$	0.00038799	0.00023870	0.00066088	0.00015202
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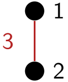
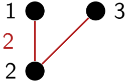
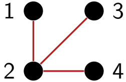
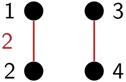
Results for $d = 5$

For each sequence $\alpha \in \mathbb{N}_d^m$, set $\widehat{b}_\alpha := \alpha! b_\alpha$.

Recall that $A^{(n)} := \gamma(n)! Q_{\gamma(n)} = \left(\widehat{b}_{\gamma(n)+v_{e_i}+v_{e_j}}\right)_{i,j=1}^m$. Consider $d = 5$:

multigraph				
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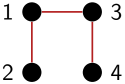
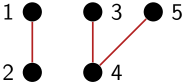
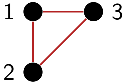
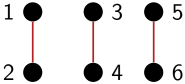
multigraph				
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Smallest eigenvalue at matrix B_1 of matching? Yes for all d we checked!

Why not verified for $d \geq 10$?

- If $d = 10$, the largest k is $2(d-2) = 16$.

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Interesting question II

How to prove the conjecture for L-uniform hypergraphs G, with $L > 2$?

Backup slides

Decomposition (case $k = 2$)

For $e \in E$, let v_e be the corresponding standard basis vector in $\mathbb{C}^E$. A decomposition of $\mathbb{C}^E$ into mutually orthogonal S_{n-2}-invariant subspaces is

$$\mathbb{C}^E = V_{1,1} \oplus W_2 \oplus W_3 \oplus W_4,$$

where

$$\begin{aligned}
V_{1,1} &:= \mathrm{span}\{v_{\{1,2\}}\} && \text{(dimension } 1\text{)},\\
W_2 &:= \mathrm{span}\{v_{\{1,j\}} \,|\, 3 \le j \le n\} && \text{(dimension } n-2\text{)},\\
W_3 &:= \mathrm{span}\{v_{\{2,j\}} \,|\, 3 \le j \le n\} && \text{(dimension } n-2\text{)},\\
W_4 &:= \mathrm{span}\{v_{\{i,j\}} \,|\, 3 \le i < j \le n\} && \text{(dimension } \tbinom{n-2}{2}\text{)}.
\end{aligned}$$

We decompose further:

$$W_2 = \operatorname{span}\{v_{\{1,j\}} \,|\, 3 \leq j \leq n\} = V_{1,2} \oplus V_{2,1},$$

where

$$V_{1,2} := \left\{\mu \sum_{i=3}^{n} v_{\{1,i\}} \mid \mu \in \mathbb{C}\right\} \qquad \text{(dimension 1)},$$

$$V_{2,1} := \left\{\sum_{i=3}^{n} c_i v_{\{1,i\}} \mid \ c \in \mathbb{C}^n : c_1 = 0, c_2 = 0, c^\mathsf{T}\mathbf{1} = 0\right\} \text{(dimension } n-3\text{)}.$$

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Similarly,

$$W_3 = \mathrm{span}\{v_{\{2,j\}} \,|\, 3 \le j \le n\} = V_{1,3} \oplus V_{2,2},$$

where

$$V_{1,3} := \left\{ \mu \sum_{i=3}^{n} v_{\{2,i\}} \mid \mu \in \mathbb{C} \right\} \qquad \text{(dimension 1)},$$

$$V_{2,2} := \left\{ \sum_{i=3}^{n} c_i v_{\{2,i\}} \mid \; c \in \mathbb{C}^n : c_1 = 0, c_2 = 0, c^{\mathsf{T}}\mathbf{1} = 0 \right\} \text{(dimension } n-3\text{)}.$$

$$W_3 = V_{1,4} \oplus V_{2,3} \oplus V_{3,1},$$

$$V_{1,4} := \left\{ \mu \sum_{3 \le i < j \le n}^{n} v_{\{i,j\}} \mid \mu \in \mathbb{C} \right\} \qquad \text{(dim 1)},$$

$$V_{2,3} := \left\{ \sum_{\substack{e=\{i,j\}: \\ 3 \le i < j \le n}} (c_i + c_j) v_{\{i,j\}} \mid \; c \in \mathbb{C}^n : c_1 = 0, c_2 = 0, c^{\mathsf{T}} \mathbf{1} = 0 \right\} \qquad \text{(dim } n-3\text{)},$$

$$V_{3,1} := \left\{ \sum_{\substack{e=\{i,j\}: \\ 3 \le i < j \le n}} \lambda_e v_e \mid \lambda_e \in \mathbb{C}, \forall k \in \{3, \ldots, n\} \; : \; \sum_{e : k \in e} \lambda_e = 0 \right\} \text{(dim } \tbinom{n-2}{2} - (n-2)\text{)}.$$

Can check: decompositions into mutually orthogonal S_{n-2}-invariant subspaces, sum of dimensions of all $V_{i,j}$ is equal to $|E| = \binom{n}{2}$.

Can check: decompositions into mutually orthogonal S_{n-2}-invariant subspaces, sum of dimensions of all $V_{i,j}$ is equal to $|E| = \binom{n}{2}$. Set $(m_1, m_2, m_3) = (4, 3, 1)$. Then

$$\mathbb{C}^E = \bigoplus_{i=1}^{3} \bigoplus_{j=1}^{m_i} V_{i,j}.$$

We count $\dim((\mathbb{C}^{E \times E})^{S_{n-2}}) = |(E \times E)/S_{n-2}|$.

$(\{1,2\},\{1,2\})$, $(\{1,2\},\{1,3\})$, $(\{1,2\},\{2,3\})$, $(\{1,2\},\{3,4\})$, $(\{1,3\},\{1,2\})$, $(\{1,3\},\{1,3\})$, $(\{1,3\},\{1,4\})$,

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We find $|(E \times E)/S_{n-2}| = 26$. Also $m_1^2 + m_2^2 + m_3^2 = 4^2 + 3^2 + 1^2 = 26$. So the $V_{i,j}$ indeed form a decomposition of V into *irreducible* S_{n-2}-submodules.

Representative set

Representative vectors are

$$u_{1,1} := v_{\{1,2\}}, \quad u_{1,2} := \sum_{i=3}^{n} v_{\{1,i\}}, \quad u_{1,3} := \sum_{i=3}^{n} v_{\{2,i\}}, \quad u_{1,4} := \sum_{3 \le i < j \le n}^{n} v_{\{i,j\}}$$

$$u_{2,1} := v_{\{1,3\}} - v_{\{1,4\}}, \quad u_{2,2} := v_{\{2,3\}} - v_{\{2,4\}}, \quad u_{2,3} := \sum_{i=5}^{n} (v_{\{3,i\}} - v_{\{4,i\}}),$$

$$u_{3,1} := v_{\{3,5\}} - v_{\{3,6\}} - v_{\{4,5\}} + v_{\{4,6\}}.$$