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Signature tensors and their representation theory

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November 15, 2023

1 Outline

1. Signature tensors of paths
2. Representation theory

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$$\sigma^k(X)_{i_1,\dots,i_k} = \int_0^1 \int_0^{t_k} \cdots \int_0^{t_3} \int_0^{t_2} \dot{X}_{i_1}(t_1) \cdots \dot{X}_{i_k}(t_k) dt_1 \cdots dt_k.$$

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In other words: $\sigma^1(X) = X(1) - X(0)$.

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- We will write

$$\widetilde{\mathcal{U}}_{d,k} = \{k\text{-th signatures of paths in } \mathbb{R}^d\} \subseteq (\mathbb{R}^d)^{\otimes k}.$$

1 The signature variety

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- For instance:

$$\mathcal{U}_{2,2} = \{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in (\mathbb{C}^2)^{\otimes 2} \mid \det \begin{pmatrix} a_{11} & \frac{a_{12}+a_{21}}{2} \\ \frac{a_{12}+a_{21}}{2} & a_{22} \end{pmatrix} = 0 \}.$$

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- The defining equations are easier if we change coordinates:

$$b_{11} = a_{11}, b_{12} = \frac{a_{12}+a_{21}}{2}, b_{22} = a_{22}, c = \frac{a_{12}-a_{21}}{2}.$$

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- Then the defining equation becomes

$$b_{11}b_{22} - b_{12}^2.$$

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  to tensors.
- Before we can do this, we need to get a better handle on the signature variety.

1 The tensor algebra

Let V be a vector space (think $V = \mathbb{C}^d$).

Definition

We define the tensor algebra

$$\mathbb{T}((V)) = \mathbb{C} \times V \times V^{\otimes 2} \times \cdots$$

Elements in $\mathbb{T}((V))$ will be denoted by formal power series

$$\mathcal{T} = \lambda + v + A + T + \cdots$$

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In particular we can define the *total signature*

$$\sigma(X) := 1 + \sigma^1(X) + \sigma^2(X) + \cdots \in \mathbb{T}((V)).$$

1 Chow's theorem

Theorem

The set of total signature tensors of piecewise linear paths is the image of the *free Lie algebra* $\mathrm{Lie}(V) \subset \mathbb{T}((V))$ under the *exponential map*

$$\begin{aligned} \exp : \mathbb{T}_0((V)) &\to \mathbb{T}((V)) \\ \mathcal{T} &\mapsto 1 + \mathcal{T} + \frac{1}{2!}\mathcal{T}^{\otimes 2} + \frac{1}{3!}\mathcal{T}^{\otimes 3} + \cdots \end{aligned}$$

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Definition

The *free Lie algebra* $\mathrm{Lie}(V)$ is the smallest vector subspace of $\mathbb{T}((V))$ that contains V and is closed under the commutator bracket $[\cdot, \cdot]$.

- $\mathrm{Lie}^k V = \mathrm{Lie}(V) \cap V^{\otimes k}$.
- In particular $\mathrm{Lie}^1 V = V$ and $\mathrm{Lie}^2 V = \bigwedge^2 V$.

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- The *Campbell-Baker-Hausdorff formula*:

$$\exp(v) \otimes \exp(w) = \exp(v + w + \frac{1}{2}[v, w] + \frac{1}{12}[v, [v, w]] + \cdots).$$

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Corollary

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Example: $k = 2$

If $\mathcal{T} = v + Q + T + \cdots \in \mathrm{Lie}(V)$, we have $\exp(\mathcal{T}) =$

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Expanding and taking the degree 2 part yields

$$\exp(\mathcal{T})_2 = \frac{1}{2} v \otimes v + Q.$$

1 Higher order signatures

The signature variety $\mathcal{U}_{d,3}$ consists of all tensors of the form

$$\frac{1}{6}v^{\otimes 3} + \frac{1}{2}(v \otimes Q + Q \otimes v) + T \tag{1}$$

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We can factor our map $\operatorname{Lie} V \to V^{\otimes 3}$:

$$\begin{array}{ccccc} \operatorname{Lie} V \to S^3(\operatorname{Lie}^1 V) & \oplus (\operatorname{Lie}^1 V \otimes \operatorname{Lie}^2 V) & \oplus \operatorname{Lie}^3 V & \cong & V^{\otimes 3} \\ v + Q + T + \cdots \mapsto (\quad v^{\otimes 3} \quad , & v \otimes Q \quad , & T \quad) & \mapsto & (1) \end{array}$$

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The red isomorphism is a consequence of the Poincaré-Birkhoff-Witt theorem.

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In suitable coordinates, this is a *monomial map*, given by all monomials of weighted degree 3:

$$\mathbb{C}^3 \oplus \mathbb{C}^3 \oplus \mathbb{C}^8 \to \mathbb{C}^{10} \oplus \mathbb{C}^9 \oplus \mathbb{C}^8 \cong (\mathbb{C}^3)^{\otimes 3}$$

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[CGM20]: study $\mathcal{U}_{d,k}$ using *toric geometry*.

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2. Representation theory

2 Thrall modules

Definition

For any partition λ of k, we define the *Thrall module*

$$W_\lambda(V) := \mathrm{Sym}^{a_1(\lambda)}(\mathrm{Lie}^1 V) \otimes \cdots \otimes \mathrm{Sym}^{a_k(\lambda)}(\mathrm{Lie}^k V),$$

where $a_i(\lambda)$ is the number of times the integer i occurs in λ.

- $W_{(3)}(V) = \mathrm{Lie}^3 V$,
- $W_{(2,1)}(V) = \mathrm{Lie}^2 V \otimes \mathrm{Lie}^1 V$,
- $W_{(1,1,1)}(V) = \mathrm{Sym}^3(\mathrm{Lie}^1 V)$.

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- The exponential map factors as

$$\mathrm{Lie}\, V \to \bigoplus_{\lambda \vdash k} W_\lambda(V) \xrightarrow{\cong} V^{\otimes k}.$$

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2 Thrall modules

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- Other famous decomposition of $V^{\otimes k}$ given by *Schur-Weyl duality*:
$$V^{\otimes k} \cong \bigoplus_{\lambda \vdash k} \mathbb{S}^\lambda V^{\oplus m_\lambda}.$$

  How are these related?

2 Thrall modules: $k = 3$

- We have the two decompositions

$$\begin{aligned} V^{\otimes 3} &= \operatorname{Sym}^3 V \oplus \bigwedge^3 V \oplus \big(\mathbb{S}_{(2,1)}(V)\big)^{\oplus 2} \\ &= W_{(1,1,1)}(V) \oplus W_{(2,1)}(V) \oplus W_{(3)}(V). \end{aligned}$$

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  - $W_{(3)}(V) \cong \mathrm{Lie}^3 V \cong \mathbb{S}_{(2,1)}(V)$.

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- Characters of the representations $W_\lambda(V)$ are known as *Gessel-Reutenauer symmetric functions*.
  - Can compute a_μ^λ by decomposing these into Schur functions.
  - Demonstration in Sage on mathrepo.

2 Thrall modules: general k

- Want to understand the multiplicities a_μ^λ in

$$W_\lambda(V) \cong \bigoplus_{\mu \vdash k} \mathbb{S}_\mu(V)^{\oplus a_\mu^\lambda}.$$

- Characters of the representations $W_\lambda(V)$ are known as *Gessel-Reutenauer symmetric functions*.
    - Can compute a_μ^λ by decomposing these into Schur functions.
    - Demonstration in Sage on mathrepo.
- *Thrall's problem*: find a combinatorial interpretation for a_μ^λ.

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  Geometric interpretation?

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ArXiv: 2308.11571 [AGRO+23]

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