Stable cohomology of line bundles on flag varieties

Claudiu Raicu

(joint work with Keller VandeBogert)

Symmetry, Stability, and interactions with Computation

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Reduced homology of the simplex

$$\mathbf{k} = \text{a field}, \qquad d \geq 0 \text{ an integer, and write}$$

$$[d] = \{1, \cdots, d\}.$$

Consider the complex $C_\bullet = C_\bullet(d)$, with

$$C_t = \bigoplus_{J \in \binom{[d]}{t}} \mathbf{k} \cdot e_J \simeq \mathbf{k}^{\oplus \binom{d}{t}}, \quad t = 0, \cdots, d,$$

and differential

$$\partial(e_{j_1, \cdots, j_t}) = \sum (-1)^{i-1} e_{j_1, \cdots, \widehat{j_i}, \cdots, j_t}.$$

For $d = 3$, we get

$$0 \longrightarrow \mathbf{k} \xrightarrow{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}} \mathbf{k}^3 \xrightarrow{\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}} \mathbf{k}^3 \xrightarrow{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}} \mathbf{k} \longrightarrow 0$$

Exercise. $C_\bullet$ is exact (for all $d > 0$ and all $\mathbf{k}$).

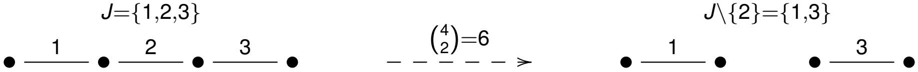
What if we rescale entries?

Exercise. Study how the homology depends on $\mathbf{k}$:

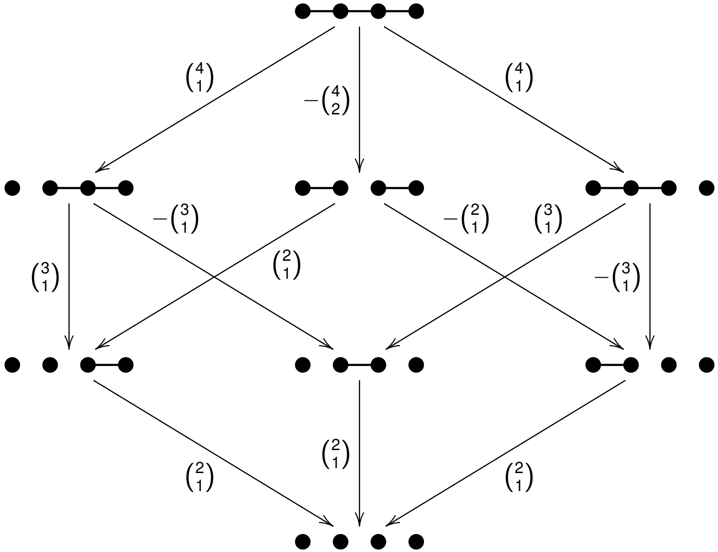
$$0 \longrightarrow \mathbf{k} \xrightarrow{\begin{bmatrix} 4 \\ -6 \\ 4 \end{bmatrix}} \mathbf{k}^3 \xrightarrow{\begin{bmatrix} 3 & 2 & 0 \\ -3 & 0 & 3 \\ 0 & -2 & -3 \end{bmatrix}} \mathbf{k}^3 \xrightarrow{\begin{bmatrix} 2 & 2 & 2 \end{bmatrix}} \mathbf{k} \longrightarrow 0$$

To construct such a complex, think of elements of $[d]$ as edge labels

- each $J \subseteq [d]$ gives a disjoint union of intervals.
- removing an element j from J breaks exactly one interval, of size ($:=$ number of vertices) w, into two intervals of size w' and $w - w'$.
- construct $\tilde{C}_\bullet = \tilde{C}_\bullet(d)$ from $C_\bullet$ by replacing ± 1 with $\pm\binom{w}{w'}$.



An arithmetic Koszul complex



Integers $\equiv 0, 1 \bmod p$

For a prime $p > 0$, enumerate non-negative integers $\equiv 0, 1 \bmod p$:

$$0, 1, p, p+1, 2p, 2p+1, \cdots$$

If m is in the list above, write $|m|_p$ for its position (p-index):

$$\text{if } m = pa + b, \text{ with } b \equiv 0, 1 \bmod p, \text{ then } |m|_p = 2a + b.$$

For $p = 3$:

m	0	1	2	3	4	5	6	7	8	9	10	11	$\cdots$
$\|m\|_p$	0	1		2	3		4	5		6	7		$\cdots$

For a tuple $\alpha = (\alpha_0, \cdots, \alpha_k)$, with $\alpha_i \equiv 0, 1 \bmod p$, we write

$$|\alpha|_p = \sum_{i=0}^{k} |\alpha_i|_p, \text{ and let}$$

$$A_{p,d} = \{\alpha = (\alpha_0, \cdots, \alpha_k) : \sum \alpha_i \cdot p^i = d, \ \alpha_i \equiv 0, 1 \bmod p\}.$$

Examples: $A_{3,8} = \emptyset$, $\quad A_{3,9} = \{(0,0,1), (0,3), (6,1), (9)\}$, and

$$|(0,0,1)|_3 = 1, \ |(0,3)|_3 = 2, \ |(6,1)|_3 = 5, \ |(9)|_3 = 6.$$

The homology of $\tilde{C}_\bullet(d)$ in characteristic $p > 0$

Theorem (R–VandeBogert)

Suppose that $\mathrm{char}(\mathbf{k}) = p > 0$, *and write*

$$P_d(t) := \sum_{i \geq 0} \dim_{\mathbf{k}} H_i(\tilde{C}_\bullet(d)) \cdot t^i$$

for the generating function of the homology of $\tilde{C}_\bullet(d)$. *We have*

$$P_d(t) = \sum_{\alpha \in A_{p,d+1}} t^{d+1-|\alpha|_p}.$$

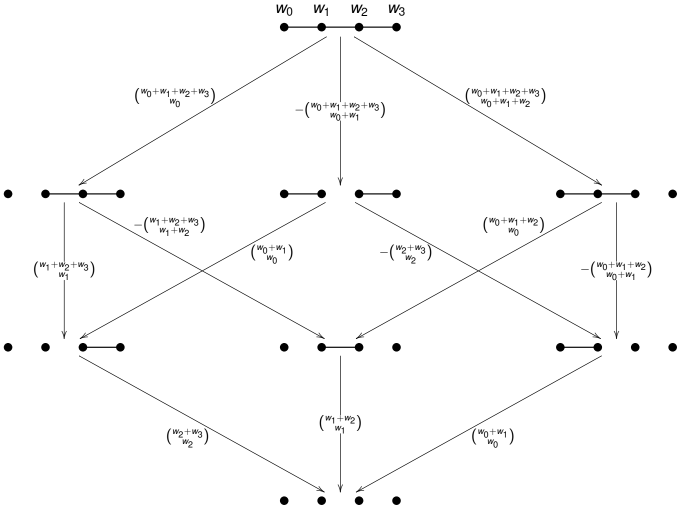
Examples:

$d = 0$: $P_0(t) = 1.$

$d = 1, p = 2$: $P_1(t) = 1 + t.$

$d = 8, p = 3$: $P_8(t) = t^{9-6} + t^{9-5} + t^{9-2} + t^{9-1} = t^3 + t^4 + t^7 + t^8.$

A weighted generalization: $C_\bullet(w_0, w_1, \cdots, w_d)$



If all $w_i = 0$, get the complex computing the reduced homology of the simplex!

Problem

Compute the homology of $C_\bullet(w_0, \cdots, w_d)$ over a field $\mathbf{k}$, $\mathrm{char}(\mathbf{k}) = p > 0$

Theorem (R–VandeBogert)

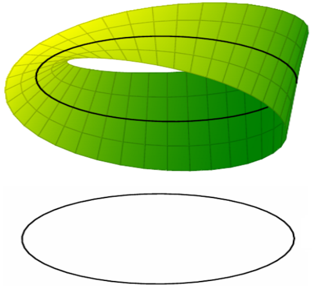
Can give simple recursion when $w_1 = \cdots = w_d = 1$, w_0 is arbitrary.

Flag varieties

Let $V \simeq \mathbf{k}^n$ be a vector space of dimension n.
Fl_n = the flag variety parametrizing complete flags of quotients

$$V_\bullet : \quad V \twoheadrightarrow V_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow V_1 \twoheadrightarrow 0, \text{ where } \dim(V_i) = i.$$

- The assignment $p = [V_\bullet] \mapsto V_i$ defines a tautological quotient bundle $\mathcal{Q}_i$.
- Similarly, $p = [V_\bullet] \mapsto \ker(V_i \twoheadrightarrow V_{i-1})$ defines a tautological line bundle $\mathcal{L}_i$.
- Indentifying $\mathrm{Fl}_n = \mathrm{GL}_n / B$, there is a natural action of GL_n on the cohomology of tautological bundles on Fl_n.



Open Problem

Describe the cohomology groups ($H^0, H^1, \cdots$) for line bundles on Fl_n.

Classification of line bundles and Kempf vanishing

GL_n-equivariant line bundles on Fl_n are indexed by weights $\lambda \in \mathbb{Z}^n$

$$\mathcal{O}(\lambda) = \mathcal{L}_1^{\otimes \lambda_1} \otimes \mathcal{L}_2^{\otimes \lambda_2} \otimes \cdots \otimes \mathcal{L}_n^{\otimes \lambda_n}.$$

The fundamental weights are

$$\omega_i = (1, \cdots, 1, 0, \cdots, 0) \longleftrightarrow \mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_i = \det(\mathcal{Q}_i), \quad i = 1, \cdots, n.$$

If we write $H^i(\lambda)$ for $H^i(Fl_n, \mathcal{O}(\lambda))$ then

$$H^0(\omega_i) = \bigwedge^i \mathbf{k}^n, \quad \text{and} \quad H^j(\omega_i) = 0 \quad \text{for } j > 0.$$

We also have $H^j(\lambda + \omega_n) = H^j(\lambda) \otimes \bigwedge^n \mathbf{k}^n \simeq H^j(\lambda)$, and

$$H^0(\lambda) \neq 0 \Longleftrightarrow \lambda \in \mathbb{Z}_{\geq 0}\langle \omega_1, \cdots, \omega_{n-1} \rangle + \mathbb{Z} \cdot \omega_n \Longleftrightarrow \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n,$$

in which case we say that λ is a dominant weight.

Theorem (Kempf '76, Haboush '80, Andersen '80)

If λ is a dominant weight then

$$H^j(\lambda) = 0 \text{ for all } j > 0.$$

The Borel–Weil–Bott theorem

Theorem (Borel–Weil–Bott)

Suppose that $\mathrm{char}(\mathbf{k}) = 0$*, and let* $\lambda \in \mathbb{Z}^n$*.*

(a) *There exists at most one value of* k *such that* $H^k(\lambda) \neq 0$*.*

(b) *If* $\lambda_i - i = \lambda_j - j$ *for some* $i \neq j$*, then*

$$H^k(\lambda) = 0 \text{ for all } k.$$

(c) *When* $H^k(\lambda) \neq 0$*, it is an irreducible* GL_n*-representation.*

Example of (b): If $d \geq 1$, $n \geq d + 2$ and

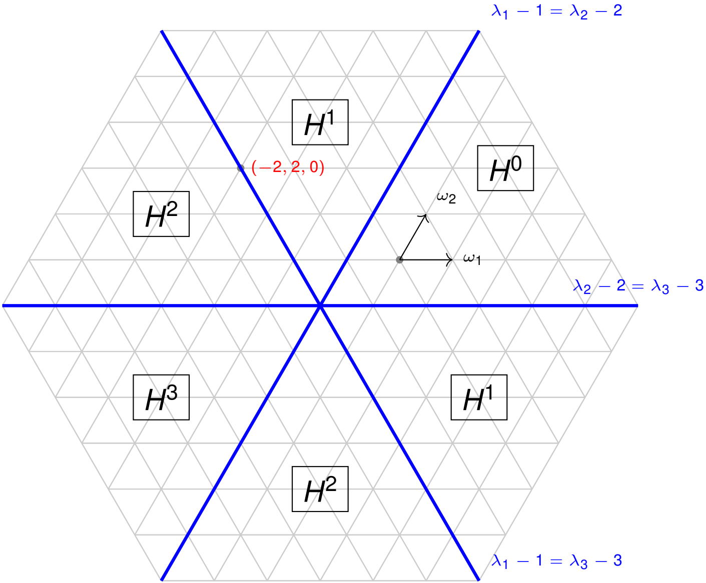
$$\lambda = (-d - 1, d + 1, 0, 0, \cdots, 0) \in \mathbb{Z}^n.$$

then $H^k(\lambda) = 0$ for all k (e.g. $d = 1$, $n = 3$, we get $\lambda = (-2, 2, 0)$).

Proof: We have $\lambda_1 - 1 = \lambda_{d+2} - (d + 2)$.

Borel–Weil–Bott for Fl_3 (char($\mathbf{k}$) = 0)

Let $\omega_1 = (1, 0, 0)$ and $\omega_2 = (1, 1, 0)$ viewed inside $\mathbb{Z}^3/\mathbb{Z}(1, 1, 1)$.



Some examples in $\mathrm{char}(\mathbf{k}) = p > 0$

We write $V = \mathbf{k}^n$, $\det(V) = \bigwedge^n V$, and focus on non-zero cohomology.

Example ($n = 2$): We have $\mathrm{Fl}_2 \simeq \mathbf{P}^1$ and (for any p)

$$H^1(-d-1, d+1) = \mathrm{D}^{2d}\, V \otimes (\det V^\vee)^d.$$

Example (Mumford): If $p = 2$, $\lambda = (-2, 2, 0, \cdots)$, then for $n \geq 3$

$$H^1(\lambda) = H^2(\lambda) = \mathbf{k}.$$

Example: If $\lambda = (-3, 3, 0, \cdots)$, for $n = 3$ one has $H^1(-3, 3, 0) = \mathbf{k}$ and a short exact sequence

$$0 \longrightarrow F^3 V \otimes \det(V^\vee) \longrightarrow H^2(-3, 3, 0) \longrightarrow \mathbb{W}_{(2,1)} V \otimes \det(V^\vee) \longrightarrow 0.$$

For $n \geq 4$ we have

$$H^1(\lambda) = H^2(\lambda) = \mathbf{k}.$$

Stabilization theorem

Requires the notion of a polynomial functor (no duals!), such as:

$$\mathrm{Sym}^d,\ \mathrm{D}^d,\ \bigwedge^r,\ \mathbb{S}_\mu,\ \mathbb{W}_\mu,\ F^p.$$

Notation: if $\lambda \in \mathbb{Z}^r$ then we write $|\lambda| = \lambda_1 + \cdots + \lambda_r$ and

$$\lambda^{[n]} = (\lambda_1, \cdots, \lambda_r, 0, \cdots, 0) \in \mathbb{Z}^n \quad \text{for} \quad n \geq r.$$

Theorem (R–VandeBogert)

There exists a polynomial functor $\mathcal{P}^j_\lambda$ of degree $|\lambda|$ such that

$$H^j\left(\lambda^{[n]}\right) = \mathcal{P}^j_\lambda(\mathbf{k}^n) \quad \text{for } n \gg 0 \quad (n \geq i - \lambda_i \text{ for all } i).$$

In particular we have for $n \gg 0$:

- *If $|\lambda| < 0$ then $H^j\left(\lambda^{[n]}\right) = 0$.*
- *If $|\lambda| = 0$ then $H^j\left(\lambda^{[n]}\right)$ is a trivial GL_n-representation.*

Example. If $\lambda = \omega_i$ then $\mathcal{P}^0_\lambda = \bigwedge^i$. If $\lambda = (d)$ then $\mathcal{P}^0_\lambda = \mathrm{Sym}^d$.

Stable cohomology for $\lambda = (-d-1, d+1)$

If $\lambda \in \mathbb{Z}^r$ with $|\lambda| = 0$ then we write

$$h^j_{st}(\lambda) = \dim_{\mathbf{k}} H^j\left(\lambda^{[n]}\right) \qquad \text{for } n \gg 0.$$

Theorem (R-VandeBogert)

If $d \geq 0$ and $\mathrm{char}(\mathbf{k}) = p > 0$ then for $\lambda = (-d-1, d+1)$ we have

$$h^j_{st}(\lambda) = \dim_{\mathbf{k}} H_{d+1-j}(\tilde{C}_\bullet(d)).$$

In particular

$$\sum_{j\geq 0} h^j_{st}(\lambda)\cdot t^j = \sum_{\alpha\in A_{p,d+1}} t^{|\alpha|_p}.$$

Example. If $p = 2$ then we have

$$\sum_{j\geq 0} h^j_{st}(-2,2)\cdot t^j = t + t^2,$$

$$\sum_{j\geq 0} h^j_{st}(-6,6)\cdot t^j = t^2 + 2t^3 + t^4 + t^5 + t^6.$$

Polynomial functors of the cotangent bundle on $\mathbf{P}$

There is a related stabilization phenomenon on projective space:

Theorem (R–VandeBogert)

If $\mathcal{P}$ is a polynomial functor of degree d, and Ω denotes the cotangent bundle on projective space $\mathbf{P} = \mathbb{P}^{n-1}$ then

$$H^j(\mathbf{P}, \mathcal{P}\Omega) \text{ stabilizes for } n \geq d+1$$

to a trivial GL_n-representation denoted $H^j_{st}(\mathcal{P}\Omega)$, of dimension $h^j_{st}(\mathcal{P}\Omega)$.

E.g. $h^j_{st}(\mathrm{Sym}^d\,\Omega) = h^j_{st}(-d, d) \quad h^j_{st}(\mathbb{S}_\mu\Omega) = h^j_{st}(-|\mu|, \mu_1, \mu_2, \cdots)$

One has a Künneth formula and invariance under Frobenius:

Theorem (R–VandeBogert)

1. $H^j_{st}\left(\mathcal{P}'(\Omega) \otimes \mathcal{P}''(\Omega)\right) = \bigoplus_{j'+j''=j} H^{j'}_{st}\left(\mathcal{P}'(\Omega)\right) \otimes H^{j''}_{st}\left(\mathcal{P}''(\Omega)\right).$
2. $H^j_{st}\left((F^p \circ \mathcal{P})(\Omega)\right) = H^j_{st}\left((\mathcal{P} \circ F^p)(\Omega)\right) = H^j_{st}\left(\mathcal{P}(\Omega)\right).$

Relationship with arithmetic complexes

$\mathrm{Sym}^{d+1}\,\Omega$

$H_i(\tilde{C}_\bullet(d)) \simeq H_{st}^{d+1-i}\left(\mathrm{Sym}^{d+1}\,\Omega\right)$

each p-adic expression contributes one copy of **k** to cohomology

Arithmetic Koszul complex $\tilde{C}_\bullet(d)$

p-adic expressions
$d + 1 = \sum \alpha_i \cdot p^i$
$\alpha_i \equiv 0, 1 \bmod p$

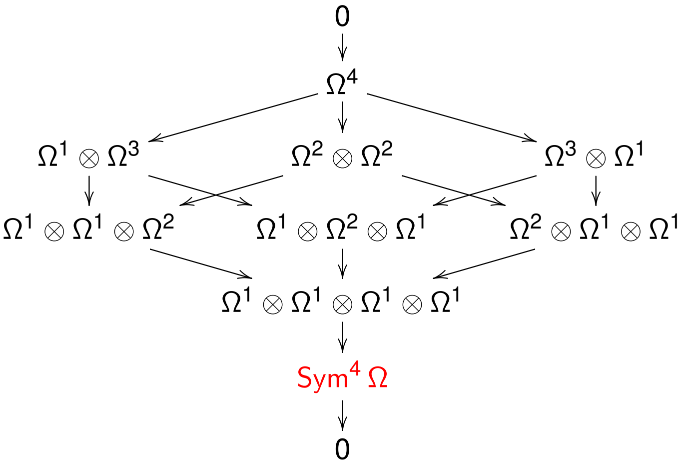
Examples:

$d = 0$: $H^1_{st}(\Omega) = \mathbf{k}$.

$d = 1, p = 2$: $H^1_{st}(\mathrm{Sym}^2\,\Omega) = H^2_{st}(\mathrm{Sym}^2\,\Omega) = \mathbf{k}$.

$d = 8, p = 3$: $H^i_{st}(\mathrm{Sym}^9\,\Omega) = \mathbf{k}$ for $i = 1, 2, 5, 6$.

Resolution by exterior powers (approx. from above)



Akin,
Buchsbaum,
Weyman,
Rota,
Totaro,
Santana,
Yudin,
. . . .
resolutions of
GL-modules

- $H_{st}^{i_1+\cdots+i_k}\left(\Omega^{i_1}\otimes\cdots\otimes\Omega^{i_k}\right) = \mathbf{k}$, and $H^j = 0$ for $j \neq i_1+\cdots+i_k$.
- $\Omega^{i_1}\otimes\Omega^{i_2} \longrightarrow \Omega^{i_1+i_2}$ induces an isomorphism in cohomology.
- $\Omega^{i_1+i_2} \longrightarrow \Omega^{i_1}\otimes\Omega^{i_2}$ is multiplication by $\binom{i_1+i_2}{i_1}$ in cohomology.

Truncated symmetric powers (approx. from below)

For $V = \mathbf{k}^n$ we let

$$S = \mathrm{Sym}(V) \simeq \mathbf{k}[x_1, \cdots, x_n], \text{ the symmetric algebra of } V.$$

If $\mathrm{char}(\mathbf{k}) = p > 0$, we have the (p–)truncated symmetric algebra

$$T_p S = \mathrm{Sym}(V)/\langle F^p V\rangle \simeq \mathbf{k}[x_1, \cdots, x_n]/\langle x_1^p, \cdots, x_n^p\rangle.$$

whose degree d part is the truncated symmetric power $T_p \mathrm{Sym}^d V$.

Example: In characteristic 2, we have $T_2 \mathrm{Sym}^d = \bigwedge^d$ for all $d \geq 0$.

For $\alpha = (\alpha_0, \cdots, \alpha_k)$, we have a simple polynomial functor F^α given by

$$F^\alpha V = T_p \mathrm{Sym}^{\alpha_0} V \otimes F^p(T_p \mathrm{Sym}^{\alpha_1} V) \otimes \cdots \otimes F^{p^k}(T_p \mathrm{Sym}^{\alpha_k} V).$$

Doty: the composition factors of Sym^d are F^α, with $\sum \alpha_i \cdot p^i = d$.

Theorem (R–VandeBogert)

The only non-zero cohomology for $F^\alpha\Omega$ is $H^{|\alpha|_p}(\mathbf{P}, F^\alpha\Omega) = \mathbf{k}$, and it occurs if and only if $\alpha_i \equiv 0, 1 \mod p$ for all i.

Hooks and two-column partitions

Recall the general identification of stable cohomology

$$H^j_{st}(\mathbb{S}_\lambda \Omega) = H^j_{st}(-|\lambda|, \lambda_1, \lambda_2, \cdots).$$

Theorem (R–VandeBogert)

For $m \geq d \geq 0$ we have

$$H^j_{st}(-m-d, 2^d, 1^{m-d}) = H^{2m+1-j}_{st}(-m-1, d+1, 1^{m-d}).$$

Equivalently, if $\lambda = (2^d, 1^{m-d}) = (m, d)'$ is the partition conjugate to (m, d), and $\mu = (d+1, 1^{m-d})$ is a related hook partition, then

$$H^j_{st}(\mathbb{S}_\lambda \Omega) = H^{2m+1-j}_{st}(\mathbb{S}_\mu \Omega).$$

Remark(able): there is no reference to $\mathrm{char}(\mathbf{k})$ in the above theorem! If $\mathrm{char}(\mathbf{k}) = p > 0$ then the cohomology groups above are the same as

$$H_{j-m}(C_\bullet(\underline{w})), \text{ where } \underline{w} = (p^k - m - d - 1, 1^d) \text{ for } p^k \gg 0.$$

Thank You!