KU LEUVEN

Regularity of minimal schemes apolar to a given form

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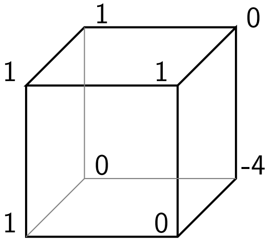
joint work with A. Bernardi & A. Oneto.

arxiv.org/abs/2309.12961

The problem

Motivation

Understanding "good" decompositions of a given symmetric tensor.



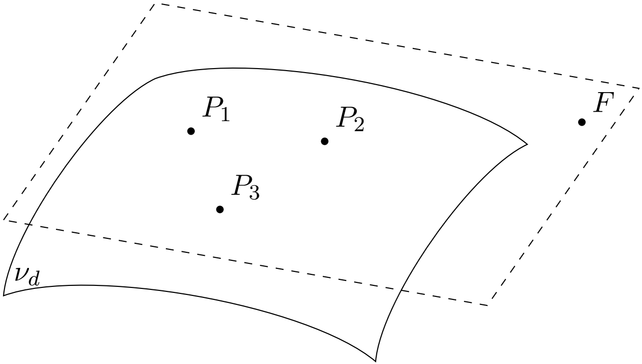
$$\longleftrightarrow \qquad x^3 + 3x^2y - 4y^3 = (x+2y)^2(x-y)$$

Notation: $\mathcal{S} = \Bbbk[X_0, \ldots, X_n] = \oplus_{d \geqslant 0} \mathcal{S}_d$ be a graded polynomial ring.

Geometrical intuition

Waring decomposition

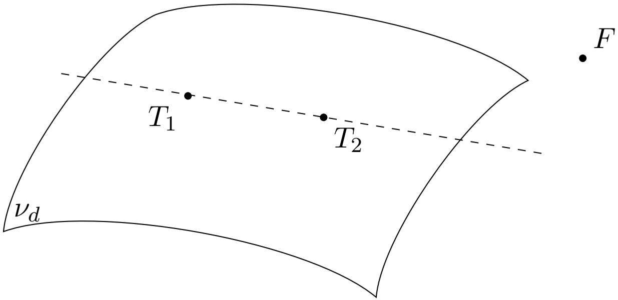
Span a given form $F \in \mathcal{S}_d$ with a minimal set of simple points on the Veronese variety.



Geometrical intuition

Tangential decomposition

Span a given form $F \in \mathcal{S}_d$ with a minimal set of points (counted with multiplicity) on the tangential variety to the Veronese variety.



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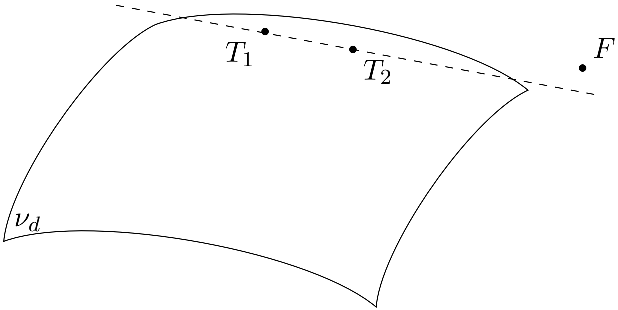

Geometrical intuition

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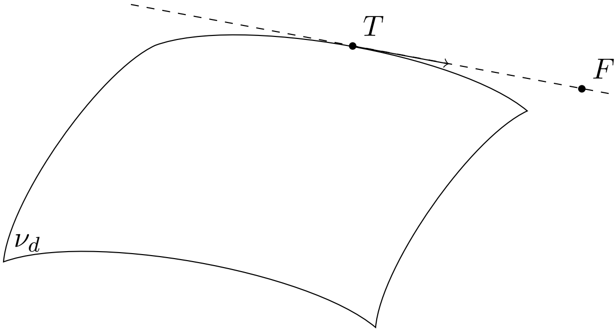
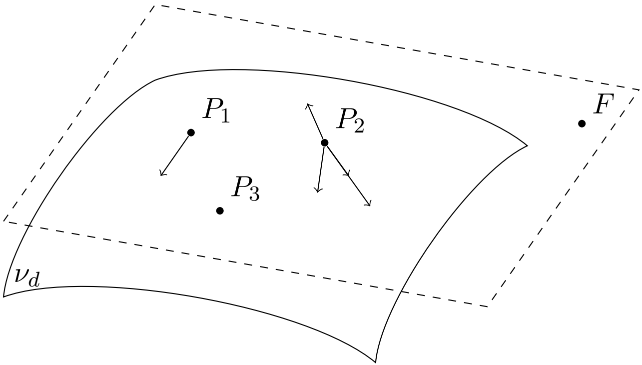
Span a given form $F \in \mathcal{S}_d$ with a minimal set of points (counted with multiplicity) on the tangential variety to the Veronese variety.



Geometrical intuition

Cactus decomposition

Span a given form $F \in \mathcal{S}_d$ with a minimal set of points (counted with multiplicity) on the osculating varieties to the Veronese variety.



But... We like polynomials

Notation: $\mathcal{R} = \Bbbk[Y_0, \ldots, Y_n] = \oplus_{d\geqslant 0}\mathcal{R}_d$ is the standard dual graded dual polynomial ring, acting on $\mathcal{S}$ *by derivation*, i.e. by extending

$$Y_i \circ F = \partial/\partial_{X_i} F.$$

Definition (Apolarity)

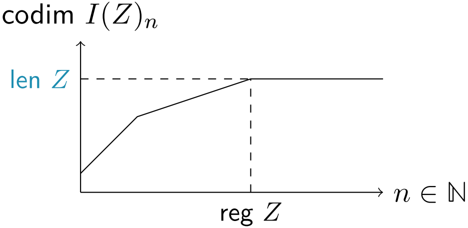
A 0-dimensional scheme $Z \subset \mathbb{P}^n$ is apolar to $F \in \mathcal{S}_d$ if its defining ideal ideal $I(Z)$ is contained in

$$\text{Ann}(F) = \{G \in \mathcal{R} \ : \ G \circ F = 0\}.$$

Problem formulation

(Big, open) Problem

Finding minimal 0-dimensional schemes $Z \subset \mathbb{P}^n$ apolar to $F \in \mathcal{S}_d$.



Problem formulation

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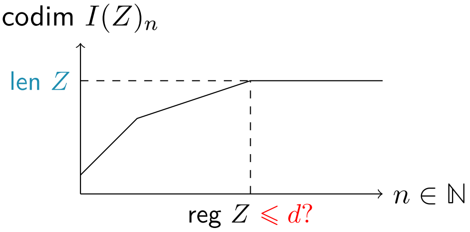


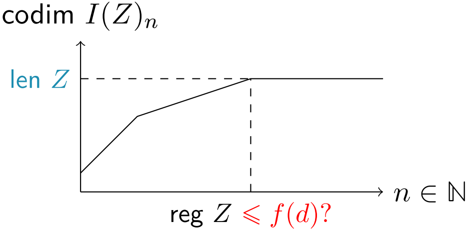
Our work

Searching for tight bounds on the regularity reg(Z) of such minimal Z.

Problem (re)formulation

(Big, open) **Problem**

Finding minimal 0-dimensional schemes $Z \subset \mathbb{P}^n$ apolar to $F \in \mathcal{S}_d$.



Question

Can the regularity of minimal schemes apolar to degree-d forms in $\Bbbk[X_1, X_2, X_3, \dots]$ be bounded only in terms of d?

Computing minimal schemes

Theoretically

- We have an algorithm. 🖹

In practice

- Challenging! It requires the solution of several polynomial systems.

🖹 A. Bernardi, D. Taufer, *Waring, tangential and cactus decompositions*, J. Math. Pures Appl. 143 (2020), pp. 1–30.

Generalized additive decompositions (GADs)

Definition (GAD)

Let $F \in \mathcal{S}_d$ and $L_1, \dots, L_s \in \mathcal{S}_1$ be pairwise non-proportional linear forms. A *generalized additive decomposition* (GAD) of F *supported at* $\{L_1, \dots, L_s\}$, is an expression

$$F = \sum_{i=1}^{s} L_i^{d-k_i} G_i, \quad \text{where } 0 \leqslant k_i \leqslant d, \text{ for all } i \in \{1, \dots, s\},$$

such that $L_i \nmid G_i$, for each $i \in \{1, \dots, s\}$.

Generalized additive decompositions (GADs)

$$F = 4x^3+12x^2y+6x^2z+18xy^2+6xyz+6xz^2+10y^3+3y^2z+3yz^2+2z^3.$$

Examples

$$= (x+y)^3 + (x+z)^3 + (x+y+z)^3 + (x+2y)^3, \qquad \leftarrow \text{"Waring"}$$

$$= (2x+3y+z)(x^2+3xy+xz+3y^2+z^2)$$
$$+(2x+y+z)(x^2+xy+xz+y^2-yz+z^2),$$

$$= (2x+2y+z)(2x^2+4xy+2xz+5y^2-yz+2z^2). \leftarrow \text{"Local"}$$

Generalized additive decompositions (GADs)

$$F = 4x^3+12x^2y+6x^2z+18xy^2+6xyz+6xz^2+10y^3+3y^2z+3yz^2+2z^3.$$

Non-examples

$$= (4x+4y+2z)(x^2+2xy+xz)+(2x+2y+z)(5y^2-yz+2z^2),$$

$$= (x+y)(x^2+2xy+y^2)+(x+z)^3+(x+y+z)^3+(x+2y)^3,$$

$$= z^0F+(x+y)^2(x-y)^2-(x^2-y^2)^2.$$

Natural apolar schemes

Definition (Natural apolar scheme) 🗎

Let $F = L^{d-k}G \in \mathcal{S}_d$ a local GAD.

i. Compute the projection $f \in \underline{\mathcal{S}}$ of F in the local chart $L = 1$.
ii. Compute the ideal $\operatorname{Ann}^{\lrcorner}(f) \subset \underline{\mathcal{R}}$.
iii. Return its global homogenization in $\mathcal{R}$ (w.r.t. L).

The scheme defined by the resulting ideal is called the *natural apolar scheme* to F w.r.t. L.

Boring, but efficient!

https://github.com/DTaufer/SchemesEvincedByGADs

🗎 A. Bernardi, J. Jelisiejew, P. M. Marques, K. Ranestad, *On polynomials with given Hilbert function and applications*, Collect. Math. 69 (2018), pp. 39–64.

Scheme evinced by GADs

Definition (Scheme evinced by a GAD)

Let

$$F = \sum_{i=1}^{s} L_i^{d-k_i} G_i$$

be a GAD of F, and let $Z(J_i)$ be the natural apolar scheme to $L_i^{d-k_i} G_i$ w.r.t. L_i. The scheme *evinced* by this GAD is

$$Z = \bigcup_{i=1}^{s} Z(J_i) = Z\left(\bigcap_{i=1}^{s} J_i\right).$$

Scheme evinced by GADs

Properties

- This construction is canonical, i.e. $Z \subset \mathbb{P}^n$ is a well-defined global gemetrical object.
- Z is apolar to F.
- $I(Z)$ is homogeneous, (locally) Gorenstein and saturated.
- Z is 0(-affine)-dimensional, i.e. it is a scheme of points.

Why GADs

Theorem 🗎

Let Z be a zero-dimensional scheme apolar to $F \in \mathcal{S}_d$.
Then Z contains a scheme evinced by a GAD of an extension $F^{\text{ext}} \in \mathcal{S}_D$, for some $D \geqslant d$.

Proposition

Let Z be a 0-dimensional scheme apolar and irredundant to $F \in \mathcal{S}_d$.
Then Z is evinced by a GAD of F if and only if the connected components of Z are contained in $(d+1)$-fat points.

🗎 W. Buczyńska, J. Buczyński, *Secant varieties to high degree Veronese reembeddings, catalecticant matrices and smoothable Gorenstein schemes*, J. Algebraic Geom. 23 (2014), pp. 63–90.
🗎 A. Bernardi, J. Brachat, B. Mourrain, *A comparison of different notions of ranks of symmetric tensors*, Linear Algebra Its Appl. 460 (2014), pp. 205–230.

Back to our example

Same F as before

$$\begin{aligned} L_1^3 + L_2^3 + L_3^3 + L_4^3 &\rightarrow [1,3,4,4,4,\dots], \quad \star \\ L_5Q_5 + L_6Q_6 &\rightarrow [1,3,5,6,6,\dots], \\ L_7Q_7 &\rightarrow [1,3,4,4,4,\dots]. \quad \star \end{aligned}$$

Observation

All the above GADs are regular in degree $3 = \deg F$. The "best" GADs are even regular in degree 2.

Question

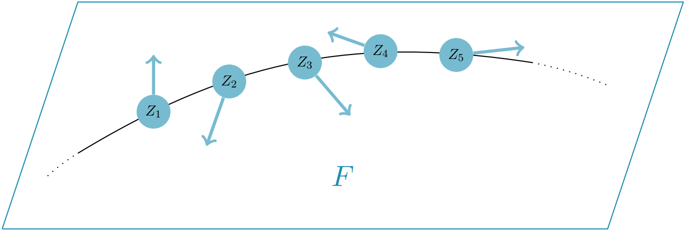
Is it always the case?

If "minimal" by inclusion: NO

Let Z be the scheme evinced by the GAD of $F \in \Bbbk[x, y, z_1, z_2, z_3]_3$:

$$x^2 z_1 + y^2 z_2 + (x+y)^2 z_3 + (x-y)^2 (z_1 - 3z_2 - 2z_3) + (x+2y)^2 (z_1 + z_2 + z_3).$$

Z is the union of five 2-jets $Z_1, \ldots, Z_5$.



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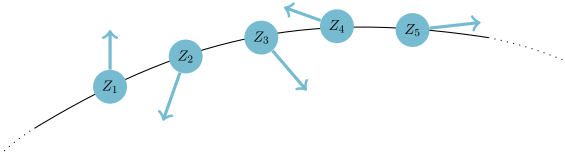


One can directly check that Z is irredundant for F, i.e.

$$Z' \subsetneq Z \implies Z' \not\subseteq \mathrm{Ann}(F),$$

but its Hilbert function is $[1, 5, 8, 9, 10, 10, \dots]$.

Conclusion: schemes irredundant to $F \in \mathcal{S}_d$ may be d-irregular.

Remark: Z is minimal by inclusion, but not by length.

Tangential decompositions

Proposition

Let $Z = Z_1 \cup \cdots \cup Z_s$ apolar to $F \in \mathcal{S}_d$ and such that $\mathrm{len}(Z_i) \leqslant 2$ for every $i \in \{1, \ldots, s\}$. If Z is of minimal length among such schemes, then Z is d-regular.

Proof idea: Z is evinced by a GAD

$$F = \sum_{i=1}^{s} L_i^{d-1} G_i \in \mathcal{S}_d.$$

If Z is d-irregular, then we have a relation among local inverse systems, involving either

- some $L_i^{d-1} G_i \rightsquigarrow$ new GAD of F evincing a proper subscheme, or
- only L_i^d's $\rightsquigarrow$ new GAD of F evincing a shorter scheme.

Remark: These new schemes are supported on the previous points!

Applied to our example

From the relation

$$(x-y)^2 = 2x^2 + 2y^2 - (x+y)^2$$

we produce a better GAD:

$$\begin{aligned} F = {} & x^2(3z_1 - 6z_2 - 4z_3) + y^2(2z_1 - 5z_2 - 4z_3) \\ & + (x+y)^2(-z_1 + 3z_2 + 3z_3) + (x+2y)^2(z_1 + z_2 + z_3). \end{aligned}$$

From the other relation

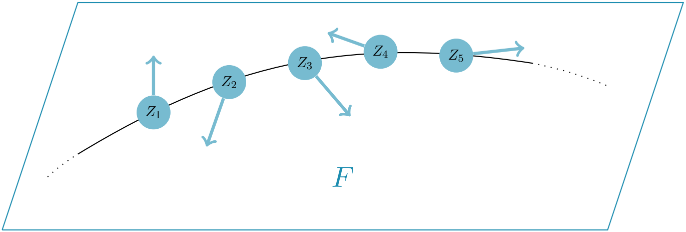
$$(x+2y)^2 = 2(x+y)^2 - x^2 + 2y^2$$

we produce an even better GAD of F:

$$x^2(2z_1 - 7z_2 - 5z_3) + y^2(4z_1 - 3z_2 - 2z_3) + (x+y)^2(z_1 + 5z_2 + 5z_3).$$

The scheme $Z' \subsetneq Z$ evincing the latter GAD is much shorter, and its Hilbert series is $[1, 5, 6, 6, \ldots]$.

Applied to our example



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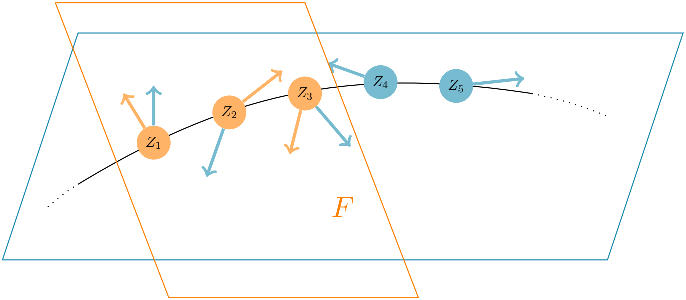

U. Perazzo, *Sulle varietá cubiche la cui hessiana svanisce identicamente*, Giornale di Matematiche (Battaglini), 38 (1900), pp. 337–354.

GADs with a few independent supports

Proposition

Let Z be the scheme evinced by the GAD

$$F = \sum_{i=1}^{s} L_i^{d-k_i} G_i \in \mathcal{S}_d,$$

such that $L_1, \ldots, L_s \in \mathcal{S}_1$ are $\Bbbk$-linearly independent. If either:

- $d > \max_{i \neq j}\{k_i + k_j\}$, or
- $d > \max_{i \neq j}\{k_i + k_j - 2\}$ and Z is irredundant,

then Z is d-regular.

Corollary: Minimal Z as above are d-regular whenever $\forall i \ : \ k_i < \frac{d}{2}$.

Short schemes

Proposition

Let $Z \subset \mathbb{P}^n$ be a 0-dimensional scheme apolar and irredundant to $F \in \mathcal{S}_d$. If

$$\operatorname{len}(Z) \leqslant 2d + 1,$$

then Z is d-regular.

Searching for possible counterexamples?

Identikit

- Need some long (len > 2) local component (no Waring/Tangential).
- Need either many supports $(s > n)$, or $\Bbbk$-dependent supports, or high-degree non-linear forms $(k_i \geqslant \frac{d}{2})$.
- Globally long schemes (len $> 2d + 1$).

Problem

Need to show that such a monster is minimal.

Open questions

- Are all the minimal apolar schemes d-regular?
- Is there a minimal apolar scheme that is d-regular?
- Are apolar schemes *generically* d-regular?
- What properties of the given tensor are needed for ensuring d-regularity?
- How to construct minimal apolar schemes with high regularity?
- Is d even tight? $d - 1$?

Thanks for your attention!

arxiv.org/abs/2309.12961