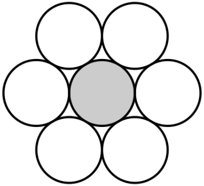
The second level of the Lasserre hierarchy for the kissing number

Nando Leijenhorst
Joint work with David de Laat and Willem de Muinck-Keizer

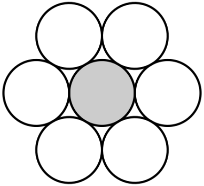
Delft University of Technology, The Netherlands

November 2023

The kissing number



The kissing number



$I \subset S^{n-1}$ is independent if $x \cdot y \leq \cos\theta$

SDP bounds

SDP bounds

- Delsarte-Goethals-Seidel LP-bound (1977)

SDP bounds

- Delsarte-Goethals-Seidel LP-bound (1977)
- Bachoc-Vallentin three-point bound (2007)

SDP bounds

- Delsarte-Goethals-Seidel LP-bound (1977)
- Bachoc-Vallentin three-point bound (2007)
- Lasserre hierarchy for packing problems (de Laat, Vallentin, 2015)

SDP bounds

- Delsarte-Goethals-Seidel LP-bound (1977)
- Bachoc-Vallentin three-point bound (2007)
- Lasserre hierarchy for packing problems (de Laat, Vallentin, 2015)
- Second step of Lasserre hierarchy for energy minimization in dimension 3 (de Laat, 2019)

SDP bounds

- Delsarte-Goethals-Seidel LP-bound (1977)
- Bachoc-Vallentin three-point bound (2007)
- Lasserre hierarchy for packing problems (de Laat, Vallentin, 2015)
- Second step of Lasserre hierarchy for energy minimization in dimension 3 (de Laat, 2019)
- Second and third step of Lasserre hierarchy for equiangular lines (de Laat, Machado, de Muinck-Keizer, 2023+)

SDP bounds

- Delsarte-Goethals-Seidel LP-bound (1977)
- Bachoc-Vallentin three-point bound (2007)
- Lasserre hierarchy for packing problems (de Laat, Vallentin, 2015)
- Second step of Lasserre hierarchy for energy minimization in dimension 3 (de Laat, 2019)
- Second and third step of Lasserre hierarchy for equiangular lines (de Laat, Machado, de Muinck-Keizer, 2023+)
- This talk: Second step of Lasserre hierarchy for kissing number for dimensions $n \geq 4$.

The Lasserre hierarchy for the kissing number

- $\mathcal{I}_k$: independent sets of size at most k

The Lasserre hierarchy for the kissing number

- $\mathcal{I}_k$: independent sets of size at most k
- $C(\mathcal{I}_t \times \mathcal{I}_t)_{\succeq 0}^{O(n)}$: the $O(n)$-invariant positive definite kernels on $\mathcal{I}_t$.

The Lasserre hierarchy for the kissing number

- $\mathcal{I}_k$: independent sets of size at most k
- $C(\mathcal{I}_t \times \mathcal{I}_t)_{\succeq 0}^{O(n)}$: the $O(n)$-invariant positive definite kernels on $\mathcal{I}_t$.
- $A_t : C(\mathcal{I}_t \times \mathcal{I}_t) \to C(\mathcal{I}_{2t})$ is the operator

$$A_t(K)(S) = \sum_{\substack{J, J' \in \mathcal{I}_t \\ J \cup J' = S}} K(J, J')$$

The Lasserre hierarchy for the kissing number

- $\mathcal{I}_k$: independent sets of size at most k
- $C(\mathcal{I}_t \times \mathcal{I}_t)_{\geq 0}^{O(n)}$: the $O(n)$-invariant positive definite kernels on $\mathcal{I}_t$.
- $A_t : C(\mathcal{I}_t \times \mathcal{I}_t) \to C(\mathcal{I}_{2t})$ is the operator

$$A_t(K)(S) = \sum_{\substack{J,J' \in \mathcal{I}_t \\ J \cup J' = S}} K(J, J')$$

Then the t-th step of the Lasserre hierarchy is given by:

$$las_t(n) = \min\{K(\emptyset, \emptyset) : A_t K(S) \leq -\chi_{\mathcal{I}_{=1}}(S) \quad S \in \mathcal{I}_{2t} \setminus \{\emptyset\}, \\ K \in C(\mathcal{I}_t \times \mathcal{I}_t)_{\geq 0}^{O(n)}\}$$

Proof that las_t gives an upper bound

Let $C \subset S^{n-1}$ be a independent set and K a feasible kernel. Then

$$\sum_{\substack{J, J' \subseteq C \\ |J|, |J'| \leq t}} K(J, J')$$

Proof that las_t gives an upper bound

Let $C \subset S^{n-1}$ be a independent set and K a feasible kernel. Then

$$0 \leq \sum_{\substack{J, J' \subseteq C \\ |J|, |J'| \leq t}} K(J, J')$$

Proof that las_t gives an upper bound

Let $C \subset S^{n-1}$ be a independent set and K a feasible kernel. Then

$$0 \leq \sum_{\substack{J,J' \subseteq C \\ |J|,|J'| \leq t}} K(J, J') = \sum_{\substack{S \subseteq C \\ |S| \leq 2t}} A_t K(S)$$

Proof that las_t gives an upper bound

Let $C \subset S^{n-1}$ be a independent set and K a feasible kernel. Then

$$0 \leq \sum_{\substack{J,J' \subseteq C \\ |J|,|J'| \leq t}} K(J, J') = \sum_{\substack{S \subseteq C \\ |S| \leq 2t}} A_t K(S) \leq K(\varnothing, \varnothing) - |C|$$

Proof that las_t gives an upper bound

Let $C \subset S^{n-1}$ be a independent set and K a feasible kernel. Then

$$0 \leq \sum_{\substack{J, J' \subseteq C \\ |J|, |J'| \leq t}} K(J, J') = \sum_{\substack{S \subseteq C \\ |S| \leq 2t}} A_t K(S) \leq K(\varnothing, \varnothing) - |C|$$

so $K(\varnothing, \varnothing) \geq |C|$.

The Lasserre hierarchy: computational problems

1. Represent kernels in $C(\mathcal{I}_t \times \mathcal{I}_t)_{\geq 0}^{O(n)}$ in terms of positive semidefinite matrices, as polynomials in the inner products of points in $S \in \mathcal{I}_{2t}$.

The Lasserre hierarchy: computational problems

1. Represent kernels in $C(\mathcal{I}_t \times \mathcal{I}_t)_{\succeq 0}^{O(n)}$ in terms of positive semidefinite matrices, as polynomials in the inner products of points in $S \in \mathcal{I}_{2t}$.
2. Reformulate as semidefinite program using sums-of-squares polynomials

The Lasserre hierarchy: computational problems

1. Represent kernels in $C(\mathcal{I}_t \times \mathcal{I}_t)_{\geq 0}^{O(n)}$ in terms of positive semidefinite matrices, as polynomials in the inner products of points in $S \in \mathcal{I}_{2t}$.
   - Mathematically: adapt the approach of de Laat, Machado and de Muinck-Keizer to get polynomials

2. Reformulate as semidefinite program using sums-of-squares polynomials

The Lasserre hierarchy: computational problems

1. Represent kernels in $C(\mathcal{I}_t \times \mathcal{I}_t)_{\geq 0}^{O(n)}$ in terms of positive semidefinite matrices, as polynomials in the inner products of points in $S \in \mathcal{I}_{2t}$.
   - Mathematically: adapt the approach of de Laat, Machado and de Muinck-Keizer to get polynomials
2. Reformulate as semidefinite program using sums-of-squares polynomials
3. Do everything for a high enough degree to get new bounds

The Lasserre hierarchy: computational problems

1. Represent kernels in $C(\mathcal{I}_t \times \mathcal{I}_t)_{\succeq 0}^{O(n)}$ in terms of positive semidefinite matrices, as polynomials in the inner products of points in $S \in \mathcal{I}_{2t}$.
   - Mathematically: adapt the approach of de Laat, Machado and de Muinck-Keizer to get polynomials
   - Use symmetry to simplify calculating the integral over $O(n)$
2. Reformulate as semidefinite program using sums-of-squares polynomials
3. Do everything for a high enough degree to get new bounds

Description of the kernels

Let (π, V_π) be an irreducible, unitary representation of $O(n)$.

Description of the kernels

Let (π, V_π) be an irreducible, unitary representation of $O(n)$. Let $\mathrm{Hom}_{O(n)}(\mathcal{I}_t, V_\pi)$ be the space of continuous, $O(n)$-equivariant maps from $\mathcal{I}_t$ to V_π. Let $\{\phi_{\pi,j}\}_j$ be a complete system for this space. Let K_π be a positive semidefinite matrix.

Description of the kernels

Let (π, V_π) be an irreducible, unitary representation of $O(n)$. Let $\mathrm{Hom}_{O(n)}(\mathcal{I}_t, V_\pi)$ be the space of continuous, $O(n)$-equivariant maps from $\mathcal{I}_t$ to V_π. Let $\{\phi_{\pi,j}\}_j$ be a complete system for this space. Let K_π be a positive semidefinite matrix.Then the kernel $K : \mathcal{I}_t \times \mathcal{I}_t \to \mathbb{R}$

$$K(x,y) = \sum_{\pi} \sum_{i,j} K_{\pi,i,j} \langle \phi_{\pi,i}(x), \phi_{\pi,j}(y) \rangle$$

is continuous, positive definite and $O(n)$-invariant.

From equiangular lines to the kissing number

Recall the zonal matrices for the three-point bound:

$$Y_k^n(u, v, t)_{ij} = u^i v^j \sqrt{(1-u^2)(1-v^2)}^k P_k^n\left(\frac{t-uv}{\sqrt{(1-u^2)(1-v^2)}}\right).$$

From equiangular lines to the kissing number

Recall the zonal matrices for the three-point bound:

$$Y_k^n(u,v,t)_{ij} = u^i v^j \sqrt{(1-u^2)(1-v^2)}^k P_k^n\left(\frac{t-uv}{\sqrt{(1-u^2)(1-v^2)}}\right).$$

For las_2, we get

$$Z(S(x,y))$$

From equiangular lines to the kissing number

Recall the zonal matrices for the three-point bound:

$$Y_k^n(u,v,t)_{ij} = u^i v^j \sqrt{(1-u^2)(1-v^2)}^k P_k^n\left(\frac{t-uv}{\sqrt{(1-u^2)(1-v^2)}}\right).$$

For las_2, we get

$$Q(x)Z(S(x,y))Q(y)$$

with $Q(x)$ a diagonal matrix with square roots of polynomials in the inner products

From equiangular lines to the kissing number

Recall the zonal matrices for the three-point bound:

$$Y_k^n(u,v,t)_{ij} = u^i v^j \sqrt{(1-u^2)(1-v^2)}^k P_k^n\left(\frac{t-uv}{\sqrt{(1-u^2)(1-v^2)}}\right).$$

For las_2, we get

$$w(x) \otimes Q(x) Z(S(x,y)) Q(y) \otimes w(y)^T$$

with $Q(x)$ a diagonal matrix with square roots of polynomials in the inner products, and $w(x)$ a basis for the space of polynomials in the inner products of the vectors in x.

Simplifying the computations

We have

$$\langle \phi_{\pi,i}(x), \phi_{\pi,j}(y) \rangle = \int_{O(n)} \langle \rho(\omega M \epsilon) e_i, \rho(\omega M S \epsilon) e_j \rangle dM$$

Simplifying the computations

We have

$$\begin{aligned}\langle \phi_{\pi,i}(x), \phi_{\pi,j}(y)\rangle &= \int_{O(n)} \langle \rho(\omega M\epsilon)e_i, \rho(\omega MS\epsilon)e_j\rangle dM \\ &= \sum_k \int_{O(n)} \langle \rho(\omega M\epsilon)e_i, e_k\rangle\langle e_k, \rho(\omega MS\epsilon)e_j\rangle dM\end{aligned}$$

Simplifying the computations

We have

$$\begin{aligned}\langle \phi_{\pi,i}(x), \phi_{\pi,j}(y)\rangle &= \int_{O(n)} \langle \rho(\omega M \epsilon) e_i, \rho(\omega M S \epsilon) e_j \rangle dM \\ &= \sum_k \int_{O(n)} \langle \rho(\omega M \epsilon) e_i, e_k \rangle \langle e_k, \rho(\omega M S \epsilon) e_j \rangle dM\end{aligned}$$

1. All terms are proportional to each other and we can calculate the constants.

Simplifying the computations

We have

$$\langle \phi_{\pi,i}(x), \phi_{\pi,j}(y) \rangle = \int_{O(n)} \langle \rho(\omega M \epsilon) e_i, \rho(\omega M S \epsilon) e_j \rangle dM$$
$$= \sum_k \int_{O(n)} \langle \rho(\omega M \epsilon) e_i, e_k \rangle \langle e_k, \rho(\omega M S \epsilon) e_j \rangle dM$$

1. All terms are proportional to each other and we can calculate the constants.
2. We can take the real part of the inner products before the product

Simplifying the computations

We have

$$\langle \phi_{\pi,i}(x), \phi_{\pi,j}(y) \rangle = \int_{O(n)} \langle \rho(\omega M \epsilon) e_i, \rho(\omega M S \epsilon) e_j \rangle dM$$
$$= \sum_k \int_{O(n)} \langle \rho(\omega M \epsilon) e_i, e_k \rangle \langle e_k, \rho(\omega M S \epsilon) e_j \rangle dM$$

1. All terms are proportional to each other and we can calculate the constants.
2. We can take the real part of the inner products before the product
3. We can reduce the polynomials in the $2t \times t$ block of S to polynomials in the $t \times t$ block of S.

Simplifying the computations

We have

$$\langle \phi_{\pi,i}(x), \phi_{\pi,j}(y) \rangle = \int_{O(n)} \langle \rho(\omega M \epsilon) e_i, \rho(\omega M S \epsilon) e_j \rangle dM$$
$$= \sum_k \int_{O(n)} \langle \rho(\omega M \epsilon) e_i, e_k \rangle \langle e_k, \rho(\omega M S \epsilon) e_j \rangle dM$$

1. All terms are proportional to each other and we can calculate the constants.
2. We can take the real part of the inner products before the product
3. We can reduce the polynomials in the $2t \times t$ block of S to polynomials in the $t \times t$ block of S.
4. For certain π, powers of the determinant of X appear in $\rho(X)$. Similar to (1) we get proportional terms, and we can calculate the constants.

Simplifying the computations

We have

$$\langle \phi_{\pi,i}(x), \phi_{\pi,j}(y) \rangle = \int_{O(n)} \langle \rho(\omega M \epsilon) e_i, \rho(\omega M S \epsilon) e_j \rangle dM$$
$$= \sum_k \int_{O(n)} \langle \rho(\omega M \epsilon) e_i, e_k \rangle \langle e_k, \rho(\omega M S \epsilon) e_j \rangle dM$$

1. All terms are proportional to each other and we can calculate the constants.
2. We can take the real part of the inner products before the product
3. We can reduce the polynomials in the $2t \times t$ block of S to polynomials in the $t \times t$ block of S.
4. For certain π, powers of the determinant of X appear in $\rho(X)$. Similar to (1) we get proportional terms, and we can calculate the constants.
5. In the product, we only need to take terms into account that don't integrate to 0.

Sum-of-squares representations

For $S \in \mathcal{I}_{2t}$, we have

$$A_t K(S) \leq -\chi_{\mathcal{I}_{=1}}(S).$$

This now gives polynomial constraints in $\binom{|S|}{2}$ variables.

Sum-of-squares representations

For $S \in \mathcal{I}_{2t}$, we have

$$A_t K(S) \leq -\chi_{\mathcal{I}_{=1}}(S).$$

This now gives polynomial constraints in $\binom{|S|}{2}$ variables.
We have: $S \subseteq S^{n-1}$ if and only if the Gram matrix of S is positive semidefinite if and only if all principal minors are nonnegative.

Sum-of-squares representations

For $S \in \mathcal{I}_{2t}$, we have

$$A_t K(S) \leq -\chi_{\mathcal{I}_{=1}}(S).$$

This now gives polynomial constraints in $\binom{|S|}{2}$ variables.
We have: $S \subseteq S^{n-1}$ if and only if the Gram matrix of S is positive semidefinite if and only if all principal minors are nonnegative.
Putinar's Positivstellensatz: If

$$p(x) > 0 \quad \text{on } \{x : g_j(x) \geq 0\}$$

then (under mild conditions on g_j)

$$p(x) = s_0(x) + \sum_j g_j(x) s_j(x)$$

where s_j are sum-of-squares polynomials.

Comparing computations with other bounds

n	lower bound	LP	3-point	Las_2
4	24	25.56	24.05	24.00
5	40	46.34	44.97	44.36
6	72	82.63	78.14	77.84

Table: The best values obtained with different bounds for dimension 4-6

Uniqueness of the 24-cell

Recall the proof that las_t gives an upper bound: Let C be a valid configuration of points and K a feasible kernel. Then

$$0 \leq \sum_{\substack{J,J' \subseteq C \\ |J|,|J'| \leq t}} K(J, J') = \sum_{S \subseteq C, |S| \leq 2t} A_t K(S) \leq K(\emptyset, \emptyset) - |C|$$

so $K(\emptyset, \emptyset) \geq |C|$.

Uniqueness of the 24-cell

Recall the proof that las_t gives an upper bound: Let C be a valid configuration of points and K a feasible kernel. If $K(\emptyset, \emptyset) = |C|$, then

$$0 = \sum_{\substack{J,J' \subseteq C \\ |J|,|J'| \leq t}} K(J, J') = \sum_{S \subseteq C, |S| \leq 2t} A_t K(S) = K(\emptyset, \emptyset) - |C|.$$

Uniqueness of the 24-cell

Recall the proof that las_t gives an upper bound: Let C be a valid configuration of points and K a feasible kernel. If $K(\emptyset, \emptyset) = |C|$, then

$$0 = \sum_{\substack{J, J' \subseteq C \\ |J|, |J'| \leq t}} K(J, J') = \sum_{S \subseteq C, |S| \leq 2t} A_t K(S) = K(\emptyset, \emptyset) - |C|.$$

→ A sharp bound gives information about the configuration that matches the bound.

Uniqueness of the 24-cell

Recall the proof that las_t gives an upper bound: Let C be a valid configuration of points and K a feasible kernel. If $K(\varnothing,\varnothing) = |C|$, then

$$0 = \sum_{\substack{J,J'\subseteq C \\ |J|,|J'|\leq t}} K(J,J') = \sum_{S\subseteq C, |S|\leq 2t} A_t K(S) = K(\varnothing,\varnothing) - |C|.$$

→ A sharp bound gives information about the configuration that matches the bound.

Uniqueness of the 24-cell

Recall the proof that las_t gives an upper bound: Let C be a valid configuration of points and K a feasible kernel. If $K(\emptyset, \emptyset) = |C|$, then

$$0 = \sum_{\substack{J, J' \subseteq C \\ |J|, |J'| \leq t}} K(J, J') = \sum_{S \subseteq C, |S| \leq 2t} A_t K(S) = K(\emptyset, \emptyset) - |C|.$$

→ A sharp bound gives information about the configuration that matches the bound.
The rounding procedure of Dostert, de Laat and Moustrou is too slow:

Uniqueness of the 24-cell

Recall the proof that las_t gives an upper bound: Let C be a valid configuration of points and K a feasible kernel. If $K(\emptyset, \emptyset) = |C|$, then

$$0 = \sum_{\substack{J, J' \subseteq C \\ |J|, |J'| \leq t}} K(J, J') = \sum_{S \subseteq C, |S| \leq 2t} A_t K(S) = K(\emptyset, \emptyset) - |C|.$$

→ A sharp bound gives information about the configuration that matches the bound.

The rounding procedure of Dostert, de Laat and Moustrou is too slow:

- Their largest example gave a system of equations of size $\approx 700 \times 23.000$, which took around 8 hours to round

Uniqueness of the 24-cell

Recall the proof that las_t gives an upper bound: Let C be a valid configuration of points and K a feasible kernel. If $K(\emptyset, \emptyset) = |C|$, then

$$0 = \sum_{\substack{J, J' \subseteq C \\ |J|, |J'| \leq t}} K(J, J') = \sum_{S \subseteq C, |S| \leq 2t} A_t K(S) = K(\emptyset, \emptyset) - |C|.$$

→ A sharp bound gives information about the configuration that matches the bound.

The rounding procedure of Dostert, de Laat and Moustrou is too slow:

- Their largest example gave a system of equations of size $\approx 700 \times 23.000$, which took around 8 hours to round
- We have a system of size $\approx 4000 \times 330.000$, which would take approximately 1000 times as long (a year)

Thank you!