Algorithms for fundamental invariants and equivariants of a finite group

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Joint work with **Erick Rodriguez Bazan**

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Dedicated to the memory of Karin Gatermann

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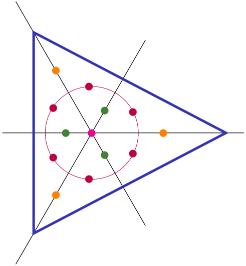
The group of order 6 with generators s_1 and s_2 and relationships

$$s_1^2 = s_2^2 = (s_1 s_2)^3 = 1.$$



$\mathfrak{D}_3$: the group of symmetry of the triangle

$$\tau(s_1) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \tau(s_2) = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

$\mathfrak{S}_3$: the group of coordinate permutations in $\mathbb{R}^3$

$$\rho(s_1) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \rho(s_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

- The ρ-**invariants** of the representation of $\mathfrak{S}_3$ by permutation of coordinates are the polynomials $h \in \mathbb{R}[x,y,z]$ s.t. $h \circ \rho(s) = h$.

$$h_1 = x + y + z, \quad h_2 = yz + zx + xy, \quad h_3 = xyz$$

generate the **invariant ring** $\mathbb{R}[\mathrm{x}]^{\mathfrak{S}_3} = \mathbb{R}[h_1, h_2, h_3]$

- The $(\rho:\tau)$-**equivariants** are the row vectors of polynomials

$$\mathrm{q} = \begin{bmatrix} q_1 & q_2 \end{bmatrix} \text{ s.t. } \mathrm{q} \circ \rho(s) = \mathrm{q}\,\tau(s)$$

They are *generated* by $\mathrm{q}_1 = \begin{bmatrix} \sqrt{3}\,(x+y-2z) & 3(y-x) \end{bmatrix}$,

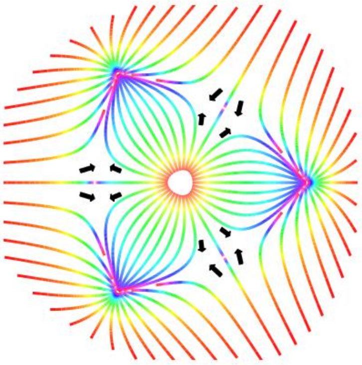
$$\mathrm{q}_2 = \begin{bmatrix} \sqrt{3}\,(2\,z(z-x-y)-y^2-x^2x+4xy) & 3\,(x-y)\,(x+y-2z) \end{bmatrix}$$

to form the $\mathbb{R}[\mathrm{x}]^{\mathfrak{S}_3}$-**module** $\mathbb{R}[\mathrm{x}]^{\mathfrak{S}_3}_{\tau}$

$$\mathbb{R}[\mathrm{x}]^{\mathfrak{S}_3}_{\tau} = \mathbb{R}[\mathrm{x}]^{\mathfrak{S}_3}\,\mathrm{q}_1 \oplus \mathbb{R}[\mathrm{x}]^{\mathfrak{S}_3}\,\mathrm{q}_2$$

Dynamical systems with symmetry

[Chossat Lauterbach 00, Gatermann 01]



Noonburg's Neural Network model

$$\begin{aligned}\dot{x}_1 &= 1 - x_1(c + x_2^2 + x_3^2)\\ \dot{x}_2 &= 1 - x_2(c + x_3^2 + x_1^2)\\ \dot{x}_3 &= 1 - x_3(c + x_1^2 + x_2^2)\end{aligned}$$

[Gatermann, ISSAC 90]

$$\begin{bmatrix} \frac{dx_1}{dt} & \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} p_1(x_1, x_2) & p_2(x_1, x_2) \end{bmatrix}$$

$$\mathrm{p}\left(\mathrm{x}\cdot\tau(g)\right) = \mathrm{p}(\mathrm{x})\cdot\tau(g)$$

Central configurations of N-bodies

$$\sum_{j\neq i} \frac{\mathrm{r}_i - \mathrm{r}_j}{|\mathrm{r}_i - \mathrm{r}_j|} = \lambda\,\mathrm{r}_i,\ 1 \le i \le N$$

2D case [Faugeres Svarz, ISSAC 12]

Symmetry adapted bases

Sum of squares [Gatermann & Parillo 06]

$$x^2 + y^2 + z^2 - (yz + zx + xy) = \frac{3}{4}\,(y - z)^2 + \frac{1}{4}\,(2x - y - z)^2$$

In a **symmetry adapted basis** the Gram/moment matrix is block diagonal, with some blocks repeated.

Global optimization [Gatermann Parillo], [Riener *et al.*] Approximation theory [Rodriguez H.], [Collowald H.] Combinatorics [Stanley], Cryptography, . . . as well as Physics, chemistry [Fässler Stiefels], [Muggli], [Cassam Chennai *et al.*], . . .

Computing higher degree **symmetry adapted bases** is a major motivation for the results presented today.

Fundamental invariants and equivariants

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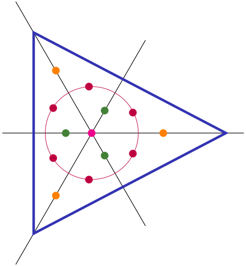
irreducible



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$\mathbb{R}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and its orthogonal complement are invariant: reducible

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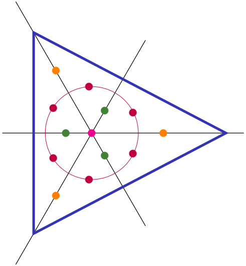

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$\mathbb{R}\begin{bmatrix}1\\1\\1\end{bmatrix}$ and its orthogonal complement are invariant: reducible

A finite group has finitely many irreducible representations.

Fundamental invariants and equivariants

Consider a finite group $\mathfrak{G}$ and a representation $\varrho : \mathfrak{G} \rightarrow \mathrm{GL}_n(\mathbb{R})$

- $\mathfrak{r}^{(1)}, \ldots, \mathfrak{r}^{(\mathfrak{t})}$ the irreducible representations of $\mathfrak{G}$:

$$\mathfrak{r}^{(1)}(g) = [1] \qquad \mathfrak{r}^{(\ell)} : \mathfrak{G} \rightarrow \mathrm{GL}_{\mathfrak{n}_\ell}(\mathbb{C})$$

- A $\mathfrak{r}^{(\ell)}$-equivariant is a row vector $\mathrm{q} = \begin{bmatrix} q_1 & \ldots & q_{\mathfrak{n}_\ell} \end{bmatrix} \in \mathbb{R}[\mathrm{x}]^{\mathfrak{n}_\ell}$ s.t.

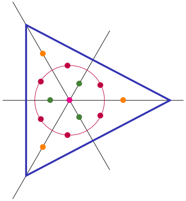
$$\mathrm{q}\left(\varrho(g)x\right) = \mathrm{q}(x)\, \mathfrak{r}^{(\ell)}(g) \quad \text{for all } g \in \mathfrak{G}$$

$\mathfrak{r}^{(\ell)}$-equivariants form the module $\mathbb{C}[\mathrm{x}]^{\mathfrak{G}}_{\mathfrak{r}^{(\ell)}}$

The fundamental invariants and equivariants consist of

- the generators of the ring $\mathbb{C}[\mathrm{x}]^{\mathfrak{G}}$
- the generators of the $\mathbb{C}[\mathrm{x}]^{\mathfrak{G}}$-modules $\mathbb{C}[\mathrm{x}]^{\mathfrak{G}}_{\mathfrak{r}^{(\ell)}}$, $2 \leq \ell \leq \mathfrak{t}$

$\varrho : \mathfrak{S}_3 \to \mathrm{GL}_3(\mathbb{R})$ s.t. $\rho(s_1) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \rho(s_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$



	s_1	s_2
$\mathfrak{r}^{(1)}$	$[1]$	$[1]$
$\mathfrak{r}^{(2)}$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$
$\mathfrak{r}^{(3)}$	$[-1]$	$[-1]$

Fundamental invariants and equivariants:

- $\mathbb{C}[\mathrm{x}]^{\mathfrak{S}_3} = \mathbb{C}[x+y+z,\ yz+zx+xy,\ xyz]$
- $\mathbb{C}[\mathrm{x}]^{\mathfrak{S}_3}_{\mathfrak{r}^{(2)}} = \mathbb{C}[\mathrm{x}]^{\mathfrak{S}_3}\,\mathrm{q}_1 \oplus \mathbb{C}[\mathrm{x}]^{\mathfrak{S}_3}\,\mathrm{q}_2$ where
  $\mathrm{q}_1 = \left[\sqrt{3}\,(x+y-2z) \quad 3(y-x)\right],$
  and $\mathrm{q}_2 = \left[\sqrt{3}\,(2\,z(z-x-y)-y^2-x^2x+4xy) \quad 3\,(x-y)\,(x+y-2z)\right]$
- $\mathbb{C}[\mathrm{x}]^{\mathfrak{S}_3}_{\mathfrak{r}^{(3)}} = \mathbb{C}[\mathrm{x}]^{\mathfrak{S}_3}\,(y-z)(z-x)(x-y)$

Symmetry adapted bases

$\varrho : \mathfrak{G} \to \mathrm{GL}_n(\mathbb{R})$ $\mathfrak{r}^{(1)}, \ldots, \mathfrak{r}^{(\mathfrak{t})}$ the irreducible representations of $\mathfrak{G}$

$$\varrho(g) = Q \begin{bmatrix} \mathrm{I}_{m_1} \otimes \mathfrak{r}^{(1)}(g) & & \\ & \ddots & \\ & & \mathrm{I}_{m_{\mathfrak{t}}} \otimes \mathfrak{r}^{(\mathfrak{t})}(g) \end{bmatrix} Q^{-1}$$

$$\mathcal{Q} = \bigcup_{\ell=1}^{\mathfrak{t}} \mathcal{Q}^{(\ell)} \text{ with } \mathcal{Q}^{(\ell)} = \left\{ \left[q_{i1}^{(\ell)}, \ldots, q_{i\mathfrak{n}_\ell}^{(\ell)} \right] \mid 1 \leq i \leq m_\ell \right\}$$

Symmetry adapted bases

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$$\varrho(g) = Q \begin{bmatrix} \mathrm{I}_{m_1} \otimes \mathfrak{r}^{(1)}(g) & & \\ & \ddots & \\ & & \mathrm{I}_{m_\mathfrak{t}} \otimes \mathfrak{r}^{(\mathfrak{t})}(g) \end{bmatrix} Q^{-1} = P \begin{bmatrix} \mathfrak{r}^{(1)}(g) \otimes \mathrm{I}_{m_1} & & \\ & \ddots & \\ & & \mathfrak{r}^{(\mathfrak{t})}(g) \otimes \mathrm{I}_{m_\mathfrak{t}} \end{bmatrix} P^{-1}$$

$$\mathcal{Q} = \bigcup_{\ell=1}^{\mathfrak{t}} \mathcal{Q}^{(\ell)} \text{ with } \mathcal{Q}^{(\ell)} = \left\{ \left[q_{i1}^{(\ell)}, \ldots, q_{i\mathfrak{n}_\ell}^{(\ell)} \right] \mid 1 \le i \le m_\ell \right\}$$

P provides a symmetry adapted basis.

It can be computed thanks to
$$\pi_{ij}^{(\ell)} = \sum_{g \in \mathfrak{G}} \mathfrak{r}_{ij}^{(\ell)}(g^{-1})\ \varrho(g)$$

- $q_{11}^{(\ell)}, \ldots, q_{m_\ell 1}^{(\ell)}$ a basis of $\pi_{11}^{(\ell)}(\mathbb{R}^n)$
- $q_{ij}^{(\ell)} = \pi_{j1}^{(\ell)}\left(q_{i1}^{(\ell)}\right)$

Block diagonalisation of equivariant linear maps

Equivariant linear map

$$\Phi : U \rightarrow V \qquad \begin{array}{rcl} \mu : \mathfrak{G} & \rightarrow & \mathrm{GL}(U) \\ \nu : \mathfrak{G} & \rightarrow & \mathrm{GL}(V) \end{array} \qquad \Phi(\mu(g)\, u) = \nu(g)\, \Phi(u)$$

In s.a.b. of U and V the matrix of Φ is

$$\mathrm{diag}\left(\mathrm{I}_{\mathfrak{n}_\ell} \otimes \Phi^{(\ell)} \,|\, 1 \leq \ell \leq \mathfrak{t}\right) = \begin{bmatrix} \Phi^{(1)} & & & & & \\ & \ddots & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & \Phi^{(\mathfrak{t})} & \\ & & & & & \ddots & \\ & & & & & & \Phi^{(\mathfrak{t})} \end{bmatrix}$$

Symmetry adapted bases of $\mathbb{C}[\mathrm{x}]$ and basic equivariants

$\varrho : \mathfrak{G} \to \mathrm{GL}_n(\mathbb{C})$ $\qquad$ $\mathbb{C}[\mathrm{x}] = \mathbb{C}[x_1, \ldots, x_n]$

$$\rho : \mathfrak{G} \to \mathrm{GL}_n(\mathbb{C}[\mathrm{x}]_d) \qquad \rho(g)(f) = f \circ \varrho(g^{-1})$$

$$\mathbb{C}[\mathrm{x}]_d = \mathbb{C}[\mathrm{x}]_d^{(1)} \oplus \ldots \oplus \mathbb{C}[\mathrm{x}]_d^{(\mathrm{t})}$$

$\mathbb{C}[\mathrm{x}]_d^{(\ell)}$ spanned by the components of $\mathrm{q}_1^{(\ell)}, \ldots, \mathrm{q}_{m_\ell}^{(\ell)}$ where $\mathrm{q}_k^{(\ell)} \in \mathbb{C}[\mathrm{x}]_{\mathfrak{r}(\ell)}^{\mathfrak{G}}$

$\mathbb{C}[\mathrm{x}]^{\mathfrak{G}}$-combination of generators of $\mathbb{C}[\mathrm{x}]_{\mathfrak{r}(2)}^{\mathfrak{G}}, \ldots, \mathbb{C}[\mathrm{x}]_{\mathfrak{r}(\mathrm{t})}^{\mathfrak{G}}$ provide **s.a.b.** of $\mathbb{C}[\mathrm{x}]$

Our contributions: Fundamental invariants and equivariants

From a s.a.b of $\mathbb{C}[\mathrm{x}]_{\leq d}$ compute minimal generators for $\mathbb{C}[\mathrm{x}]^{\mathfrak{G}}$ and $\mathbb{C}[\mathrm{x}]_{\mathfrak{r}(\ell)}^{\mathfrak{G}}$

and these provide generators of $\mathbb{C}[\mathrm{x}]_{\tau}^{\mathfrak{G}}$ for any representation τ.

Fundamental invariants and equivariants

Previously for finite, or compact, groups

Computing invariants: with the Reynolds operator: $\pi^{(1)} : \mathbb{C}[\mathrm{x}] \to \mathbb{C}[\mathrm{x}]^{\mathfrak{G}}$

$$\pi^{(1)}(f) = \frac{1}{|\mathfrak{G}|} \sum_{g \in \mathfrak{G}} f \circ \rho(g)$$

- Using the Hilbert series of the invariant ring, known from Molien's formula, for termination [Sturmfels 93], [Gatermann 00]
- Computing a homogeneous system of parameters and then using the Hilbert series of the invariant ring [Sturmfels 93], [Derksen & Kemper 02]
- Using the Hilbert ideal [Derksen 99], [King 13]

Computing equivariants: the module of $(\rho : \tau)$-equivariants can be identified to a submodule of the ring of $(\rho \oplus \tau)$-invariants [Gatermann 00]

Algorithms for fundamental invariants and equivariants

finite groups

- Reflection groups : ideal interpolation along an orbit

  The invariants are read on a H-basis of the ideal J of a generic orbit.
  The equivariants on the s.a.b. of the least interpolation space (a.k.a. the orthogonal complement of J^0) [Rodriguez Bazan & H, JSC 23]

- Free module generators over primary invariants

  from the s.a.b. of an invariant complement of the ideal generated by the primary invariants [Rodriguez Bazan & H , JSC 21]

- Minimal set of generating invariants and equivariants

  Computing invariants and equivariants degree by degree.
  Constructing
  - the H-basis of the Hilbert ideal N
  - a s.a.b. of its orthogonal complement.

Hilbert ideal and covariant algebra : the key concepts

$\varrho : \mathfrak{G} \to \mathrm{GL}_n(\mathbb{C})$ $\rho : \mathfrak{G} \to \mathrm{GL}_n(\mathbb{C})$

Hilbert ideal: $N = \langle h \mid h \in \mathbb{C}[\mathrm{x}]^{\mathfrak{G}} \setminus \mathbb{C} \rangle$

[Hilbert 1890] $h_1, \ldots, h_k$ generate $\mathbb{C}[\mathrm{x}]^{\mathfrak{G}}$ as a ring iff $N = \langle h_1, \ldots, h_k \rangle$

Covariant algebra: $\mathbb{C}[\mathrm{x}]/N$ a $\mathbb{C}$-vector space

$\mathfrak{G}$ finite $\Rightarrow$ $\mathbb{C}[\mathrm{x}]/N$ finite dimensional $\mathbb{C}$-vector space

If $\mathbb{C}[\mathrm{x}] = N \oplus Q$ then $Q \cong \mathbb{C}[\mathrm{x}]/N$.

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If $\mathbb{C}[\mathrm{x}] = N \oplus Q$ then $Q \cong \mathbb{C}[\mathrm{x}]/N$. Choose Q invariant

$\mathcal{Q} = \bigcup_{\ell=1}^{\mathfrak{t}} \mathcal{Q}^{(\ell)}$ with $\mathcal{Q}^{(\ell)} = \left\{ \left[q_{i1}^{(\ell)}, \ldots, q_{i\mathfrak{n}_\ell}^{(\ell)} \right] \mid 1 \leq i \leq m_\ell \right\}$ s.a.b. of Q

[Nakayama] and consequence

$$\mathbb{C}[\mathrm{x}] = \mathbb{C}[\mathrm{x}]^{\mathfrak{G}} \oplus \bigoplus_{\ell=2}^{\mathfrak{t}} \bigoplus_{j=1}^{\mathfrak{n}_\ell} \sum_{i=1}^{m_\ell} \mathbb{C}[\mathrm{x}]^{\mathfrak{G}} q_{ij}^{(\ell)}$$

$\Rightarrow$ $\mathcal{Q}^{(\ell)}$ is a basis of $\mathbb{C}[\mathrm{x}]^{\mathfrak{G}}_{\mathfrak{r}^{(\ell)}}$ as a $\mathbb{C}[\mathrm{x}]^{\mathfrak{G}}$-module.

Basic ideas of the algorithm

Compute degree by degree

- an orthogonal H-basis H of the Hilbert ideal $N = \langle\, h \,|\, h \in \mathbb{C}[\mathrm{x}]^{\mathfrak{G}} \setminus \mathbb{C} \rangle$
- a s.a.b. $Q = \bigcup_\ell Q^{(\ell)}$ of the orthogonal complement of N in $\mathbb{C}[x]$

Then

- $H = \{h_1, \ldots, h_k\}$ is a minimal generating set of invariants
- $Q^{(\ell)}$ is a basis of $\mathbb{C}[\mathrm{x}]^{\mathfrak{G}}_{\mathfrak{r}^{(\ell)}}$ as a $\mathbb{C}[\mathrm{x}]^{\mathfrak{G}}$-module .

Basically

- $\mathbb{C}[\mathrm{x}]_d = \Psi_d(H_{d-1}) \overset{\perp}{\oplus} \langle K_d\rangle_{\mathbb{C}} \overset{\perp}{\oplus} \langle R_d\rangle_{\mathbb{C}}$
  $$\Psi_d(H) = \sum_{h\in H} \langle\, p\, h \mid \deg(p) + \deg(h) = d \,\rangle$$
- $H_d \leftarrow H_{d-1} \cup K_d;\qquad Q_d \leftarrow Q_{d-1} \cup R_d.$

taking into account the $\rho - \tau_d$ equivariance of Ψ

Fundamental invariants and equivariants

Algorithm [H. & Rodriguez Bazan]

$d := 0;\ R_0^{(1)} = \{1\};$

do $d \leftarrow d + 1$

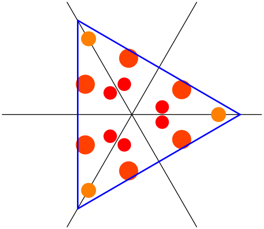
- $\mathbb{C}[\mathrm{x}]_d^{(1)} = \psi_d^{(1)}(H_{d-1}) \overset{\perp}{\oplus} \langle K_d \rangle_{\mathbb{C}}$
- $\mathbb{C}[\mathrm{x}]_d^{(\ell,1)} = \psi_d^{(\ell,1)}(H_{d-1}) \overset{\perp}{\oplus} \langle R_d^{(\ell,1)} \rangle_{\mathbb{C}} \overset{\text{lemma}}{=} \psi_d^{(1)}\left(Q_{d-1}^{(\ell,1)}\right) \overset{\perp}{\oplus} \langle R_d^{(\ell,1)} \rangle_{\mathbb{C}}$
- $H_d \leftarrow H_{d-1} \cup K_d, \quad Q_d^{(\ell)} \leftarrow Q_{d-1}^{(\ell)} \cup R_d^{(\ell)}$

until $\bigcup_{\ell=1}^{\mathfrak{t}} R_d^{(\ell)} = \emptyset$ i.e. $\langle H_d \rangle \cap \mathbb{C}[\mathrm{x}]_d = \mathbb{C}[\mathrm{x}]_d$

Output:

- $H = \{h_1, \ldots, h_k\}$ is a minimal generating set of invariants
- $Q^{(\ell)}$ is a minimal basis of $\mathbb{C}[\mathrm{x}]_{\mathfrak{r}^{(\ell)}}^{\mathfrak{G}}$ as a $\mathbb{C}[\mathrm{x}]^{\mathfrak{G}}$-module

Reflection symmetry $\mathfrak{D}_3$



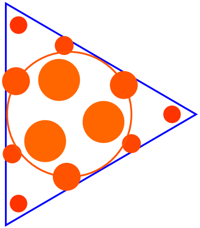
	s_1	s_2
$\mathfrak{r}^{(1)}$	$\begin{bmatrix}1\end{bmatrix}$	$\begin{bmatrix}1\end{bmatrix}$
$\mathfrak{r}^{(2)}$	$\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}1 & \sqrt{3}\\ \sqrt{3} & 1\end{bmatrix}$
$\mathfrak{r}^{(3)}$	$\begin{bmatrix}-1\end{bmatrix}$	$\begin{bmatrix}-1\end{bmatrix}$

$$H = \left\{x^2+y^2,\ x(x^2-3y^2)\right\}$$

$$Q^{(1)} = \{1\},\ Q^{(2)} = \left\{[x,y]\,,\left[y^2-x^2, 2xy\right]\right\},\ Q^{(3)} = \left\{y(y^2-3x^2)\right\}$$

$$\mathbb{R}[x,y]^{\mathfrak{D}_3} = \mathbb{R}[x^2+y^2, x(x^2-3y^2)], \qquad \mathbb{R}[x,y]^{\mathfrak{D}_3}_{\mathfrak{r}^{(3)}} = y(y^2-3x^2)\,\mathbb{R}[x,y]^{\mathfrak{D}_3}$$

$$\mathbb{R}[x,y]^{\mathfrak{D}_3}_{\mathfrak{r}^{(2)}} = [x,y]\,\mathbb{R}[x,y]^{\mathfrak{D}_3} \ \oplus\ \left[y^2-x^2, 2xy\right]\mathbb{R}[x,y]^{\mathfrak{D}_3}$$



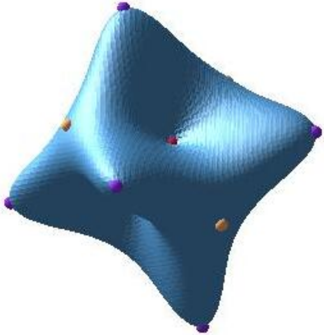
	g
$\mathfrak{r}^{(1)}$	$\left[1\right]$
$\mathfrak{r}^{(2)}$	$\left[e^{\mathrm{i}\frac{2\pi}{3}}\right]$
$\mathfrak{r}^{(3)}$	$\left[e^{-\mathrm{i}\frac{2\pi}{3}}\right]$
$\mathfrak{r}^{(2)} \oplus \mathfrak{r}^{(3)}$	$\frac{1}{2}\begin{bmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$

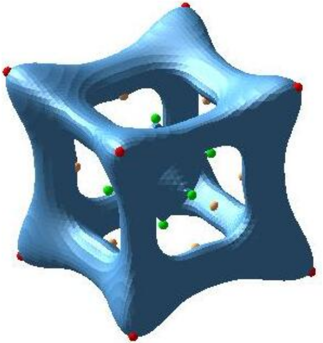
$$H = \left\{x^2+y^2,\ x(x^2-3y^2),\ y(y^2-3x^2)\right\}$$

$$Q^{(1)} = \{1\}, \qquad Q^{(2+3)} = \left\{[x,y]\,,\left[y^2-x^2, 2xy\right]\right\}$$

$$\mathbb{R}[x,y]^{\mathfrak{C}_3} = \mathbb{R}[x^2+y^2, x(x^2-3y^2), y(y^2-3x^2)],$$
$$\mathbb{R}[x,y]^{\mathfrak{C}_3}_{\mathfrak{r}^{(2+3)}} = [x,y]\,\mathbb{R}[x,y]^{\mathfrak{C}_3} + \left[y^2-x^2, 2xy\right]\mathbb{R}[x,y]^{\mathfrak{C}_3}$$

Thanks





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