Some simple conditions for entire functions to have only real zeros

joint work with Anna Vishnyakova

Thu Hien Nguyen

Julius-Maximilians-Universität Würzburg

CIRM Workshop “Symmetry, Stability, and Interactions with Computation”

Luminy, Marseille, 15 November 2023

The $\mathcal{LP}$ class: a natural extension of univariate hyperbolic polynomials

The Laguerre–Pólya class

- A univariate real polynomial $P \in \mathbb{R}[x]$ is hyperbolic ($P \in \mathcal{HP}$) if all its zeros are real.

The Laguerre–Pólya class

- A univariate real polynomial $P \in \mathbb{R}[x]$ is hyperbolic ($P \in \mathcal{HP}$) if all its zeros are real.

Edmond Laguerre (1834 – 1886)

George Pólya (1887 – 1985)

The Laguerre–Pólya class

- A univariate real polynomial $P \in \mathbb{R}[x]$ is hyperbolic ($P \in \mathcal{HP}$) if all its zeros are real.

Edmond Laguerre (1834 – 1886)

George Pólya (1887 – 1985)

- The Laguerre–Pólya ($\mathcal{LP}$) is a class of real entire functions that are the uniform limits on compact sets of sequences of univariate hyperbolic polynomials.

Why do we care?

The Laguerre–Pólya class appears in:

- analysis e.g. asymptotics, connection to the Riemann hypothesis
- special functions e.g. Ramanujan-type q-series & hypergeometric functions
- stability problems & control theory
- combinatorics e.g. graph polys, discrete convexity
- probability e.g. interlacing polys
- operator theory e.g. hyperbolicity preserving operators
- total positivity e.g. Pólya frequency sequences, bi-infinite Toeplitz matrices
- analytic number theory
- statistical physics
- theory of mock modular forms

The $\mathcal{LP}$ class and totally positive sequences

- A sequence $(a_k)_{k=0}^{\infty}$, $a_k \geq 0$, is called the totally positive sequence (PF_∞), if all minors of the infinite matrix

$$\begin{Vmatrix} a_0 & a_1 & a_2 & a_3 & \dots \\ 0 & a_0 & a_1 & a_2 & \dots \\ 0 & 0 & a_0 & a_1 & \dots \\ 0 & 0 & 0 & a_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{Vmatrix}$$

are non-negative.

The $\mathcal{LP}$ class and totally positive sequences

- A sequence $(a_k)_{k=0}^{\infty}$, $a_k \geq 0$, is called the totally positive sequence (PF_∞), if all minors of the infinite matrix

$$\left\|\begin{matrix} a_0 & a_1 & a_2 & a_3 & \dots \\ 0 & a_0 & a_1 & a_2 & \dots \\ 0 & 0 & a_0 & a_1 & \dots \\ 0 & 0 & 0 & a_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}\right\|$$

  are non-negative.

M.Aissen, I.J.Schoenberg, A.M.Whitney and A.Edrei, 1952:

The $\mathcal{LP}$ class and totally positive sequences

- A sequence $(a_k)_{k=0}^{\infty}$, $a_k \geq 0$, is called the totally positive sequence (PF_∞), if all minors of the infinite matrix

$$\begin{Vmatrix} a_0 & a_1 & a_2 & a_3 & \dots \\ 0 & a_0 & a_1 & a_2 & \dots \\ 0 & 0 & a_0 & a_1 & \dots \\ 0 & 0 & 0 & a_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{Vmatrix}$$

  are non-negative.

M.Aissen, I.J.Schoenberg, A.M.Whitney and A.Edrei, 1952:
- the generating **polynomial** $P(z) = \sum_{k=0}^{n} a_k z^k \in \mathcal{HP}$ IFF $(a_0, a_1, \dots, a_n, 0, 0, \dots) \in PF_\infty$.

The $\mathcal{LP}$ class and totally positive sequences

- A sequence $(a_k)_{k=0}^{\infty}$, $a_k \geq 0$, is called the totally positive sequence (PF_∞), if all minors of the infinite matrix

$$\left\| \begin{array}{ccccc} a_0 & a_1 & a_2 & a_3 & \dots \\ 0 & a_0 & a_1 & a_2 & \dots \\ 0 & 0 & a_0 & a_1 & \dots \\ 0 & 0 & 0 & a_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \right\|$$

  are non-negative.

M.Aissen, I.J.Schoenberg, A.M.Whitney and A.Edrei, 1952:
- the generating **polynomial** $P(z) = \sum_{k=0}^{n} a_k z^k \in \mathcal{HP}$ IFF $(a_0, a_1, \dots, a_n, 0, 0, \dots) \in PF_\infty$.
- the generating **entire function** $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{HP}$ (f of order less than 1) IFF $(a_k)_{k=0}^{\infty} \in PF_\infty$.

Hadamard's canonical factorization

Theorem 1 (E. Laguerre and G. Pólya)

$$f \in \mathcal{LP} \Longleftrightarrow f(x) = cx^n e^{-\alpha x^2 + \beta x} \prod_{k=1}^{\infty} \left(1 - \frac{x}{x_k}\right) e^{\frac{x}{x_k}},$$

Hadamard's canonical factorization

Theorem 1 (E. Laguerre and G. Pólya)

$$f \in \mathcal{LP} \iff f(x) = cx^n e^{-\alpha x^2 + \beta x} \prod_{k=1}^{\infty} \left(1 - \frac{x}{x_k}\right) e^{\frac{x}{x_k}},$$

where

Hadamard's canonical factorization

Theorem 1 (E. Laguerre and G. Pólya)

$$f \in \mathcal{LP} \iff f(x) = cx^n e^{-\alpha x^2 + \beta x} \prod_{k=1}^{\infty} \left(1 - \frac{x}{x_k}\right) e^{\frac{x}{x_k}},$$

where

- $c, \alpha, \beta \in \mathbb{R}$, $\alpha \geq 0$, $n \in \mathbb{N} \cup \{0\}$,

Hadamard's canonical factorization

Theorem 1 (E. Laguerre and G. Pólya)

$$f \in \mathcal{LP} \iff f(x) = cx^n e^{-\alpha x^2 + \beta x} \prod_{k=1}^{\infty} \left(1 - \frac{x}{x_k}\right) e^{\frac{x}{x_k}},$$

where

- $c, \alpha, \beta \in \mathbb{R},\ \alpha \geq 0,\ n \in \mathbb{N} \cup \{0\}$,
- $x_k \in \mathbb{R}, x_k \neq 0,\ \sum_{k=1}^{\infty} \frac{1}{x_k^2} < \infty$.

Hadamard's canonical factorization

Theorem 1 (E. Laguerre and G. Pólya)

$$f \in \mathcal{LP} \iff f(x) = cx^n e^{-\alpha x^2 + \beta x} \prod_{k=1}^{\infty} \left(1 - \frac{x}{x_k}\right) e^{\frac{x}{x_k}},$$

where

- $c, \alpha, \beta \in \mathbb{R}$, $\alpha \geq 0$, $n \in \mathbb{N} \cup \{0\}$,
- $x_k \in \mathbb{R}, x_k \neq 0$, $\sum_{k=1}^{\infty} \frac{1}{x_k^2} < \infty$.

Note: the product on the right-hand side can be finite or empty.

Analytic closure of sets of hyperbolic polynomials

Theorem 2 (E. Laguerre and G. Pólya)

$(P_n)_{n=1}^{\infty}$, $P_n(0) = 1$, *is a sequence of univariate hyperbolic polynomials,*

Analytic closure of sets of hyperbolic polynomials

Theorem 2 (E. Laguerre and G. Pólya)

$(P_n)_{n=1}^{\infty}$, $P_n(0) = 1$, *is a sequence of univariate hyperbolic polynomials,*
$(P_n)_{n=1}^{\infty}$ *converges uniformly in the disk* $\{|z| \leq A\}$ *for some* $A > 0$.

Analytic closure of sets of hyperbolic polynomials

Theorem 2 (E. Laguerre and G. Pólya)

$(P_n)_{n=1}^{\infty}$, $P_n(0) = 1$, *is a sequence of univariate hyperbolic polynomials,* $(P_n)_{n=1}^{\infty}$ *converges uniformly in the disk* $\{|z| \leq A\}$ *for some* $A > 0$. *Then* $(P_n)_{n=1}^{\infty}$ *converges uniformly on compact subsets to* $f \in \mathcal{LP}$.

Examples

For a real entire function f of order $\rho < 2$:
f has only real zeros $\Longleftrightarrow f \in \mathcal{LP}$

Examples

For a real entire function f of order $\rho < 2$:
f has only real zeros $\Longleftrightarrow f \in \mathcal{LP}$

Example 3

$f(x) = e^{-x^2}$, $\rho(f) = 2$, $f \in \mathcal{LP}$.

Examples

For a real entire function f of order $\rho < 2$:
f has only real zeros $\Longleftrightarrow f \in \mathcal{LP}$

Example 3

$f(x) = e^{-x^2}$, $\rho(f) = 2$, $f \in \mathcal{LP}$.

Example 4 (a non-example)

$g(x) = e^{x^2}$, $\rho(g) = 2$, $g \notin \mathcal{LP}$.

Some previous results

Second quotients of Taylor coefficients

We will only consider $f(z) = \sum_{k=0}^{\infty} a_k z^k$ s.t. $a_k > 0 \; \forall k$.

Second quotients of Taylor coefficients

We will only consider $f(z) = \sum_{k=0}^{\infty} a_k z^k$ s.t. $a_k > 0 \ \forall k$.

$q_n = q_n(f) := \frac{a_{n-1}^2}{a_{n-2} a_n}, \ n \geq 2.$

Second quotients of Taylor coefficients

We will only consider $f(z) = \sum_{k=0}^{\infty} a_k z^k$ s.t. $a_k > 0 \ \forall k$.

$q_n = q_n(f) := \frac{a_{n-1}^2}{a_{n-2}a_n}, \ n \geq 2.$

Note: $a_n = \frac{a_1}{q_2^{n-1} q_3^{n-2} \ldots q_{n-1}^2 q_n} \left(\frac{a_1}{a_0}\right)^{n-1}, \ n \geq 2.$

Newton inequalities - a necessary condition

Polynomials(ultra log-concavity):
Let $P(z) = \sum_{k=0}^{n} a_k z^k$, $a_k > 0$. If P is hyperbolic then

$$\frac{a_k^2}{\binom{n}{k}^2} \geq \frac{a_{k-1}}{\binom{n}{k-1}} \frac{a_{k+1}}{\binom{n}{k+1}}, \qquad \text{or} \qquad q_k \geq \frac{k(n-k+2)}{(k-1)(n-k+1)} > 1$$

Newton inequalities - a necessary condition

Polynomials(ultra log-concavity):
Let $P(z) = \sum_{k=0}^{n} a_k z^k$, $a_k > 0$. If P is hyperbolic then

$$\frac{a_k^2}{\binom{n}{k}^2} \geq \frac{a_{k-1}}{\binom{n}{k-1}} \frac{a_{k+1}}{\binom{n}{k+1}}, \qquad \text{or} \qquad q_k \geq \frac{k(n-k+2)}{(k-1)(n-k+1)} > 1$$

Entire functions:

- If $f \in \mathcal{LP}$, then $q_k(f) \geq \frac{k}{k-1}$, $\forall k \geq 2$.
- If $\exists\ m = 2, 3, \ldots,$ s.t. $q_m(f) = \frac{m}{m-1}$, then $f(z) = ce^{\alpha z}, c > 0, \alpha > 0$ (folklore).

Hutchinson's constant

Theorem 5 (J.I. Hutchinson, 1923)

Inequalities $q_n \geq 4, \quad \forall n \geq 2$, are valid IFF:

Theorem 5 (J.I. Hutchinson, 1923)

Inequalities $q_n \geq 4$, $\forall n \geq 2$, are valid IFF:

1. the zeros of f are all real, simple, and negative ($f \in \mathcal{LP}$) and

Hutchinson's constant

Theorem 5 (J.I. Hutchinson, 1923)

Inequalities $q_n \geq 4, \ \forall n \geq 2$, are valid IFF:

1. the zeros of f are all real, simple, and negative ($f \in \mathcal{LP}$) and
2. the zeros of any polynomial $P(z, f) = \sum_{k=m}^{n} a_k z^k$, $m < n$, , are all real and non-positive.

The partial theta function

- The entire function $g_a(z) = \sum_{j=0}^{\infty} \frac{z^j}{a^{j^2}}$, $a > 1$, is called the partial theta function.

The partial theta function

- The entire function $g_a(z) = \sum_{j=0}^{\infty} \frac{z^j}{a^{j^2}}$, $a > 1$, is called the partial theta function.

Note: $q_n(g_a) = a^2, \quad \forall n \geq 2.$

The partial theta function

- The entire function $g_a(z) = \sum_{j=0}^{\infty} \frac{z^j}{a^{j^2}}$, $a > 1$, is called the partial theta function.

Note: $q_n(g_a) = a^2, \ \ \forall n \geq 2.$

S.O. Warnaar, Partial theta functions. I. Beyond the lost notebook, *Proc. London Math. Soc.*, **87**, No. 3 (2003), 363 – 395.

The partial theta function

- The entire function $g_a(z) = \sum_{j=0}^{\infty} \frac{z^j}{a^{j^2}}$, $a > 1$, is called the partial theta function.

Note: $q_n(g_a) = a^2, \quad \forall n \geq 2.$

S.O. Warnaar, Partial theta functions. I. Beyond the lost notebook, *Proc. London Math. Soc.*, **87**, No. 3 (2003), 363 – 395.

Theorem 6 (O. Katkova, T. Lobova-Eisner, A. Vishnyakova, 2003)

$\exists\ q_\infty \approx 3.23363666$ *s.t.:*

The partial theta function

- The entire function $g_a(z) = \sum_{j=0}^{\infty} \frac{z^j}{a^{j^2}}$, $a > 1$, is called the partial theta function.

Note: $q_n(g_a) = a^2, \quad \forall n \geq 2.$

S.O. Warnaar, Partial theta functions. I. Beyond the lost notebook, *Proc. London Math. Soc.*, **87**, No. 3 (2003), 363 – 395.

Theorem 6 (O. Katkova, T. Lobova-Eisner, A. Vishnyakova, 2003)

$\exists \ q_\infty \approx 3.23363666$ *s.t.:*

1. $g_a(z) \in \mathcal{LP} \iff a^2 \geq q_\infty$

The partial theta function

- The entire function $g_a(z) = \sum_{j=0}^{\infty} \frac{z^j}{a^{j^2}}$, $a > 1$, is called the partial theta function.

Note: $q_n(g_a) = a^2, \ \forall n \geq 2.$

S.O. Warnaar, Partial theta functions. I. Beyond the lost notebook, *Proc. London Math. Soc.*, **87**, No. 3 (2003), 363 – 395.

Theorem 6 (O. Katkova, T. Lobova-Eisner, A. Vishnyakova, 2003)

$\exists \ q_\infty \approx 3.23363666$ *s.t.:*

1. $g_a(z) \in \mathcal{LP} \iff a^2 \geq q_\infty$
2. $\forall \ n \geq 2 \ \ \exists \ c_n > 1$ *s.t.* $S_n(z, g_a) \in \mathcal{LP} \iff a^2 \geq c_n$

The partial theta function

- The entire function $g_a(z) = \sum_{j=0}^{\infty} \frac{z^j}{a^{j^2}}$, $a > 1$, is called the partial theta function.

Note: $q_n(g_a) = a^2, \ \forall n \geq 2.$

S.O. Warnaar, Partial theta functions. I. Beyond the lost notebook, *Proc. London Math. Soc.*, **87**, No. 3 (2003), 363 – 395.

Theorem 6 (O. Katkova, T. Lobova-Eisner, A. Vishnyakova, 2003)

$\exists \ q_\infty \approx 3.23363666$ *s.t.:*

1. $g_a(z) \in \mathcal{LP} \iff a^2 \geq q_\infty$
2. $\forall \ n \geq 2 \ \ \exists \ c_n > 1$ s.t. $S_n(z, g_a) \in \mathcal{LP} \iff a^2 \geq c_n$
3. $4 = c_2 > c_4 > c_6 > \ldots$ *and* $\lim_{n\to\infty} c_{2n} = q_\infty$
4. $3 = c_3 < c_5 < c_7 < \ldots$ *and* $\lim_{n\to\infty} c_{2n+1} = q_\infty$

Conditions in terms of Taylor coefficients

Applications of q_∞

Theorem 7 (V.Kostov, B.Shapiro, 2013)

If $\exists\, N \in \mathbb{N}$ *s.t.* $\forall n \geq N$ $S_n(z,f) = \sum_{k=0}^{n} a_k z^k \in \mathcal{LP}$. *Then*

$$\liminf_{n\to\infty} q_n(f) \geq q_\infty.$$

Applications of q_∞

Theorem 7 (V.Kostov, B.Shapiro, 2013)

If $\exists\ N \in \mathbb{N}$ *s.t.* $\forall n \geq N\ S_n(z, f) = \sum_{k=0}^{n} a_k z^k \in \mathcal{LP}$*. Then*

$$\liminf_{n\to\infty} q_n(f) \geq q_\infty.$$

Theorem 8 (A. Vishnyakova and Ng., 2018)

If q_n *are decreasing in* n*, and* $\lim_{n\to\infty} q_n = b \geq q_\infty$*. Then all the zeros of* f *are real and negative* $(f \in \mathcal{LP})$.

Theorem 7 (V.Kostov, B.Shapiro, 2013)

If $\exists\ N \in \mathbb{N}$ *s.t.* $\forall n \geq N$ $S_n(z,f) = \sum_{k=0}^{n} a_k z^k \in \mathcal{LP}$. *Then*

$$\liminf_{n\to\infty} q_n(f) \geq q_\infty.$$

Theorem 8 (A. Vishnyakova and Ng., 2018)

If q_n *are decreasing in* n*, and* $\lim_{n\to\infty} q_n = b \geq q_\infty$*. Then all the zeros of* f *are real and negative* $(f \in \mathcal{LP})$*.*

Theorem 9 (A. Vishnyakova and Ng., 2019)

If q_n *are increasing in* n*, and* $\lim_{n\to\infty} q_n = c < q_\infty$*. Then* $f \notin \mathcal{LP}$*.*

Entire functions with decreasing q_n

1. $f(z) = \sum_{k=0}^{\infty} \frac{z^k}{(k!)a^{k^2}}$: $q_k \to a^2$; $f \in \mathcal{LP}$ for $a > 1$

Entire functions with decreasing q_n

1. $f(z) = \sum_{k=0}^{\infty} \frac{z^k}{(k!) a^{k^2}}$: $q_k \to a^2$; $f \in \mathcal{LP}$ for $a > 1$

2. $h_a(z) = 1 + \sum_{k=1}^{\infty} \frac{z^k}{(a^k - 1)(a^{k-1} - 1) \cdots (a - 1)} = \prod_{k=1}^{\infty} \left(1 + \frac{z}{a^k}\right)$; has only real negative zeros for $a > 1$

Entire functions with decreasing q_n

1. $f(z) = \sum_{k=0}^{\infty} \frac{z^k}{(k!)a^{k^2}}$: $q_k \to a^2$; $f \in \mathcal{LP}$ for $a > 1$

2. $h_a(z) = 1 + \sum_{k=1}^{\infty} \frac{z^k}{(a^k-1)(a^{k-1}-1)\cdots(a-1)} = \prod_{k=1}^{\infty} \left(1 + \frac{z}{a^k}\right)$; has only real negative zeros for $a > 1$

3. $E_\rho(z, \mu) = \sum_{k=0}^{\infty} \frac{z_k}{\Gamma(\mu+k/\rho)}, \rho > 0, \mu \in \mathbb{R}$; $E_\rho \in \mathcal{LP}$ for $\rho \in (0, 1/2]$ – a generalisation of the classical Mittag-Leffler function (I.V. Ostrovskii and I. N. Peresyolkova, 1997)

Entire functions with increasing q_n

1. $\varphi_{m,a}(z) = \sum_{k=0}^{\infty} \frac{z^k}{a^{k^2}}(k!)^m$, $a > 1$, $m \geq 0$; conditions for which $m \geq 0 : \ \varphi_{m,a} \in \mathcal{LP}$ were studied by A. Bohdanov and A. Vishnyakova, 2016

Entire functions with increasing q_n

1. $\varphi_{m,a}(z) = \sum_{k=0}^{\infty} \frac{z^k}{a^{k^2}}(k!)^m$, $a > 1$, $m \geq 0$; conditions for which $m \geq 0: \ \varphi_{m,a} \in \mathcal{LP}$ were studied by A. Bohdanov and A. Vishnyakova, 2016

2. $f_a(z) = \sum_{k=0}^{\infty} \frac{z^k}{(a+1)(a^2+1)\cdots(a^k+1)}$, $a > 1$ – known as the q-Kummer function ${}_1\phi_1(q; -q; q, -z)$

Conditions for a q-Kummer function [1]

Theorem 10 (Ng., 2022)

$f_a(z) = \sum_{k=0}^{\infty} \frac{z^k}{(a+1)(a^2+1)\cdots(a^k+1)} \in \mathcal{LP}$ *for* $a > 1$ *IFF there exists* $z_0 \in (-(a^2+1), -(a+1))$ *s.t.* $f_a(z_0) \leq 0$.

[1] Partially supported by the Akhiezer Foundation.

Conditions for a q-Kummer function [1]

Theorem 10 (Ng., 2022)

$f_a(z) = \sum_{k=0}^{\infty} \frac{z^k}{(a+1)(a^2+1)\cdots(a^k+1)} \in \mathcal{LP}$ *for* $a > 1$ *IFF there exists* $z_0 \in (-(a^2+1), -(a+1))$ *s.t.* $f_a(z_0) \leq 0$.

Theorem 11 (Ng., 2022)

1. If $f_a \in \mathcal{LP}$, then $a \geq 3.90155$;
2. If $a \geq 3.91719$, then $f_a \in \mathcal{LP}$.

[1] Partially supported by the Akhiezer Foundation.

Conjecture about a q-Kummer function

Conjecture 1 (Ng., 2022)

$\exists\ a_0 \in [3.90155, 3.91719]$ s.t. $f_a \in \mathcal{LP}$ IFF $a \geq a_0$.

Conjecture about a q-Kummer function

Conjecture 1 (Ng., 2022)

$\exists$ $a_0 \in [3.90155, 3.91719]$ s.t. $f_a \in \mathcal{LP}$ IFF $a \geq a_0$.

Idea: $\forall n\ \exists a_n$ s.t. $F_{a_n} \in \mathcal{LP}$ IFF $a \geq a_n$.

- $a_2 \geq a_4 \geq a_6 \geq \cdots$
- $a_3 \leq a_5 \leq a_7 \leq \cdots$
- moreover, $a_{2m} \geq a_{2m+1}$

Then, $\exists \lim_{m\to 0} a_{2m} = b$, and $\exists \lim_{m\to\infty} a_{2m+1} = c$, s.t. $b = c = a_0$.

A criterion with additional conditions[2]

Theorem 12 (A. Vishnyakova and Ng., 2021)

Suppose that:

[2] The research was supported by the National Research Foundation of Ukraine.

A criterion with additional conditions[2]

Theorem 12 (A. Vishnyakova and Ng., 2021)

Suppose that:

1. $3 < q_2 \leq q_3 \leq q_4 \leq \dots;$

[2] The research was supported by the National Research Foundation of Ukraine.

A criterion with additional conditions[2]

Theorem 12 (A. Vishnyakova and Ng., 2021)

Suppose that:

1. $3 < q_2 \leq q_3 \leq q_4 \leq \ldots;$
2. If there exists $j_0 \geq 2$ s.t. $q_{j_0} < 4$, $q_{j_0+1} \geq 4$, and
   (i) $q_{j_0-1}/q_{j_0+1} \geq 0.525$; or
   (ii) $q_{j_0} \geq 3.4303$.

[2] The research was supported by the National Research Foundation of Ukraine.

A criterion with additional conditions[2]

Theorem 12 (A. Vishnyakova and Ng., 2021)

Suppose that:

1. $3 < q_2 \leq q_3 \leq q_4 \leq \ldots;$
2. If there exists $j_0 \geq 2$ s.t. $q_{j_0} < 4$, $q_{j_0+1} \geq 4$, and
   (i) $q_{j_0-1}/q_{j_0+1} \geq 0.525$; or
   (ii) $q_{j_0} \geq 3.4303$.

Then $f \in \mathcal{LP}$ IFF there exists $z_0 \in [\frac{-a_1}{a_2}, 0]$ s.t. $f(z_0) \leq 0$.

[2] The research was supported by the National Research Foundation of Ukraine.

Hutchinson's intervals[3]

J.I. Hutchinson: $q_n(f) \in [4, +\infty) \iff f \in \mathcal{LP}$

[3] This research was supported by the Mathematisches Forschungsinstitut Oberwolfach within the program Oberwolfach Leibniz Fellows.

Hutchinson's intervals[3]

J.I. Hutchinson: $q_n(f) \in [4, +\infty) \iff f \in \mathcal{LP}$

Theorem 13 (A. Vishnyakova and Ng., 2023+)

Suppose $q_n(f) \in [\alpha, \beta(\alpha)]$, *where* $1 + \sqrt{5} \leq \alpha < 4$, *and* $\beta(\alpha) = \frac{8}{\alpha(4-\alpha)}$.

[3]This research was supported by the Mathematisches Forschungsinstitut Oberwolfach within the program Oberwolfach Leibniz Fellows.

Hutchinson's intervals[3]

J.I. Hutchinson: $q_n(f) \in [4, +\infty) \iff f \in \mathcal{LP}$

Theorem 13 (A. Vishnyakova and Ng., 2023+)

Suppose $q_n(f) \in [\alpha, \beta(\alpha)]$, *where* $1 + \sqrt{5} \leq \alpha < 4$, *and* $\beta(\alpha) = \frac{8}{\alpha(4-\alpha)}$. *Then* $q_n(f) \in [\alpha, \beta(\alpha)]$ *for all* $n \geq 2$ *implies that* $f \in \mathcal{LP}$.

[3] This research was supported by the Mathematisches Forschungsinstitut Oberwolfach within the program Oberwolfach Leibniz Fellows.

Hutchinson's intervals[3]

J.I. Hutchinson: $q_n(f) \in [4, +\infty) \iff f \in \mathcal{LP}$

Theorem 13 (A. Vishnyakova and Ng., 2023+)

Suppose $q_n(f) \in [\alpha, \beta(\alpha)]$, *where* $1 + \sqrt{5} \leq \alpha < 4$, *and* $\beta(\alpha) = \frac{8}{\alpha(4-\alpha)}$. *Then* $q_n(f) \in [\alpha, \beta(\alpha)]$ *for all* $n \geq 2$ *implies that* $f \in \mathcal{LP}$.

[3]This research was supported by the Mathematisches Forschungsinstitut Oberwolfach within the program Oberwolfach Leibniz Fellows.

Hutchinson's intervals[3]

J.I. Hutchinson: $q_n(f) \in [4, +\infty) \iff f \in \mathcal{LP}$

Theorem 13 (A. Vishnyakova and Ng., 2023+)

Suppose $q_n(f) \in [\alpha, \beta(\alpha)]$, *where* $1+\sqrt{5} \leq \alpha < 4$, *and* $\beta(\alpha) = \frac{8}{\alpha(4-\alpha)}$. *Then* $q_n(f) \in [\alpha, \beta(\alpha)]$ *for all* $n \geq 2$ *implies that* $f \in \mathcal{LP}$.

arXiv:2008.04754

arXiv:2212.05692

[3]This research was supported by the Mathematisches Forschungsinstitut Oberwolfach within the program Oberwolfach Leibniz Fellows.

Hutchinson's intervals[3]

J.I. Hutchinson: $q_n(f) \in [4, +\infty) \iff f \in \mathcal{LP}$

Theorem 13 (A. Vishnyakova and Ng., 2023+)

Suppose $q_n(f) \in [\alpha, \beta(\alpha)]$, *where* $1 + \sqrt{5} \leq \alpha < 4$, *and* $\beta(\alpha) = \frac{8}{\alpha(4-\alpha)}$. *Then* $q_n(f) \in [\alpha, \beta(\alpha)]$ *for all* $n \geq 2$ *implies that* $f \in \mathcal{LP}$.

arXiv:2008.04754

arXiv:2212.05692

Many thanks

for attention!

[3] This research was supported by the Mathematisches Forschungsinstitut Oberwolfach within the program Oberwolfach Leibniz Fellows.

Appendix: Multiplier sequences as hyperbolicity preserving operators

Problem (C.Hermite): describe linear transformations that map polynomials with all roots in a given area into the set of polynomials with all roots in another given area.

$\mathcal{LP}$ and operator theory

Problem (C.Hermite): describe linear transformations that map polynomials with all roots in a given area into the set of polynomials with all roots in another given area.

Our setting: $\mathbb{R} \to \mathbb{R}$.

Hyperbolicity preserving operators

- A linear operator $T : \mathbb{R}[x] \to \mathbb{R}[x]$ is called a hyperbolicity preserver (HPO) if it preserves $\mathcal{HP}$.

Hyperbolicity preserving operators

- A linear operator $T : \mathbb{R}[x] \to \mathbb{R}[x]$ is called a hyperbolicity preserver (HPO) if it preserves $\mathcal{HP}$.

Example 1. Hermite-Poulain theorem: a finite order linear differential operator with constant coefficients $T = a_0 + a_1 \frac{d}{dx} + \cdots + a_k \frac{d^k}{dx^k}$ is an HPO IFF its symbol polynomial $Q(t) = a_0 + a_1 t + \cdots + a_k t^k \in \mathcal{HP}$.

Hyperbolicity preserving operators

- A linear operator $T : \mathbb{R}[x] \to \mathbb{R}[x]$ is called a hyperbolicity preserver (HPO) if it preserves $\mathcal{HP}$.

Example 1. Hermite-Poulain theorem: a finite order linear differential operator with constant coefficients $T = a_0 + a_1 \frac{d}{dx} + \cdots + a_k \frac{d^k}{dx^k}$ is an HPO IFF its symbol polynomial $Q(t) = a_0 + a_1 t + \cdots + a_k t^k \in \mathcal{HP}$.

Example 2. G.Pólya & J. Schur:

- A real sequence $(\gamma_k)_{k=0}^{\infty}$ is a multiplier sequence ($\mathcal{MS}$) if:

$$P(x) = \sum_{k=0}^{n} a_k z^k \in \mathcal{HP} \Longrightarrow Q(x) = \sum_{k=0}^{n} \gamma_k a_k z^k \in \mathcal{HP}.$$

Hyperbolicity preserving operators

- A linear operator $T:\mathbb{R}[x]\to\mathbb{R}[x]$ is called a hyperbolicity preserver (HPO) if it preserves $\mathcal{HP}$.

Example 1. Hermite-Poulain theorem: a finite order linear differential operator with constant coefficients $T=a_0+a_1\frac{d}{dx}+\cdots+a_k\frac{d^k}{dx^k}$ is an HPO IFF its symbol polynomial $Q(t)=a_0+a_1t+\cdots+a_kt^k\in\mathcal{HP}$.

Example 2. G.Pólya & J. Schur:

- A real sequence $(\gamma_k)_{k=0}^{\infty}$ is a multiplier sequence ($\mathcal{MS}$) if:

$$P(x)=\sum_{k=0}^{n}a_kz^k\in\mathcal{HP}\Longrightarrow Q(x)=\sum_{k=0}^{n}\gamma_k a_kz^k\in\mathcal{HP}.$$

Note: $\mathcal{MS}$ is a linear operator acting diagonally in the monomial basis of $\mathbb{C}[x]$: $T_{\mathcal{A}}(x^i)=\alpha_i x^i$.

$\mathcal{LP}$ and multiplier sequences

Theorem 14 (G. Pólya and J. Schur, 1914)

Let $(\gamma_k)_{k=0}^{\infty}$ be a real sequence. TFAE:

$\mathcal{LP}$ and multiplier sequences

Theorem 14 (G. Pólya and J. Schur, 1914)

Let $(\gamma_k)_{k=0}^{\infty}$ be a real sequence. TFAE:

1. $(\gamma_k)_{k=0}^{\infty} \in \mathcal{MS}$

$\mathcal{LP}$ and multiplier sequences

Theorem 14 (G. Pólya and J. Schur, 1914)

Let $(\gamma_k)_{k=0}^{\infty}$ be a real sequence. TFAE:

1. $(\gamma_k)_{k=0}^{\infty} \in \mathcal{MS}$
2. $\forall\ n \in \mathbb{N}: \quad P_n(z) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k z^k \in \mathcal{HP}$

Theorem 14 (G. Pólya and J. Schur, 1914)

Let $(\gamma_k)_{k=0}^{\infty}$ be a real sequence. TFAE:

1. $(\gamma_k)_{k=0}^{\infty} \in \mathcal{MS}$
2. $\forall\ n \in \mathbb{N}:\ P_n(z) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k z^k \in \mathcal{HP}$
3. $\Phi(z) := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} z^k$ converges absolutely in $\mathbb{C}$, and $\Phi \in \mathcal{LP}$

Theorem 14 (G. Pólya and J. Schur, 1914)

Let $(\gamma_k)_{k=0}^{\infty}$ be a real sequence. TFAE:

1. $(\gamma_k)_{k=0}^{\infty} \in \mathcal{MS}$
2. $\forall\ n \in \mathbb{N}:\ \ P_n(z) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k z^k \in \mathcal{HP}$
3. $\Phi(z) := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} z^k$ converges absolutely in $\mathbb{C}$, and $\Phi \in \mathcal{LP}$

Every new entire function from $\mathcal{LP}$ generates a new multiplier sequence.

$\mathcal{LP}$ and multiplier sequences

Theorem 14 (G. Pólya and J. Schur, 1914)

Let $(\gamma_k)_{k=0}^{\infty}$ be a real sequence. TFAE:

1. $(\gamma_k)_{k=0}^{\infty} \in \mathcal{MS}$
2. $\forall\ n \in \mathbb{N}:\ P_n(z) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k z^k \in \mathcal{HP}$
3. $\Phi(z) := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} z^k$ converges absolutely in $\mathbb{C}$, and $\Phi \in \mathcal{LP}$

Every new entire function from $\mathcal{LP}$ generates a new multiplier sequence.

Corollary 15 (A. Vishnyakova and Ng., 2019)

Let $(a_k)_{k=0}^{\infty}$ be a real positive sequence s.t. its $(q_k)_{k=2}^{\infty}$ is increasing in k. Then

$$(k!a_k)_{k=0}^{\infty} \in \mathcal{MS} \Longrightarrow \lim_{k\to\infty} q_k \geq q_\infty.$$