Reach of the Segre variety

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Joint work with Paul Breiding

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Segre–Veronese variety

Definition (Veronese variety)

The (spherical) Veronese variety is powers of linear forms on the sphere:

$$\mathbb{V}_{n,d} = \{\pm \ell^d \mid \ell \text{ is a linear form in } x_0, \ldots, x_n\} \cap \mathbb{S}.$$

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Let $\mathbf{n} = (n_1, \ldots, n_r)$, $\mathbf{d} = (d_1, \ldots, d_r)$.

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The (spherical) Segre–Veronese variety is the set of partially symmetric rank-one tensors, i.e.

$$\mathbb{X}_{\mathbf{n},\mathbf{d}} := \left\{\pm \ell_1^{d_1} \otimes \cdots \otimes \ell_r^{d_r} \mid \ell_i \text{ is a linear form in } x_0, \ldots, x_{n_i}\right\} \cap \mathbb{S}.$$

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For $d_1 = \cdots = d_r = 1$, we obtain the *(spherical) Segre variety*, or the set of rank-one tensors.

Reach

Definition (Medial axis)

Given a set $S \subset \mathbb{S}$, the medial axis $\mathrm{Med}(S)$ is the set of all points $\mathbf{G} \in \mathbb{S}$ such that there exist at least two points $\mathbf{F}_1, \mathbf{F}_2 \in S$ with $d_{\mathbb{S}}(\mathbf{G}, S) = d_{\mathbb{S}}(\mathbf{F}_i, \mathbf{G})$, $i = 1, 2$.

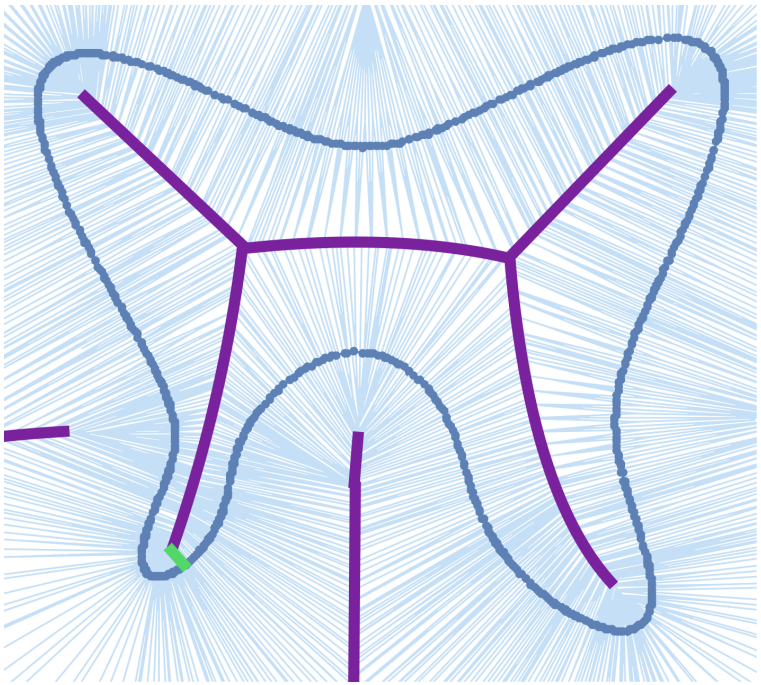


Figure: The medial axis of a variety (purple). Figure courtesy of Madeline Brandt and Madeleine Weinstein.

Reach

Definition (Reach)

The reach of S is defined as

$$\tau(S) := \inf_{\mathbf{F} \in S} d_{\mathbb{S}}(\mathbf{F}, \mathrm{Med}(S)).$$

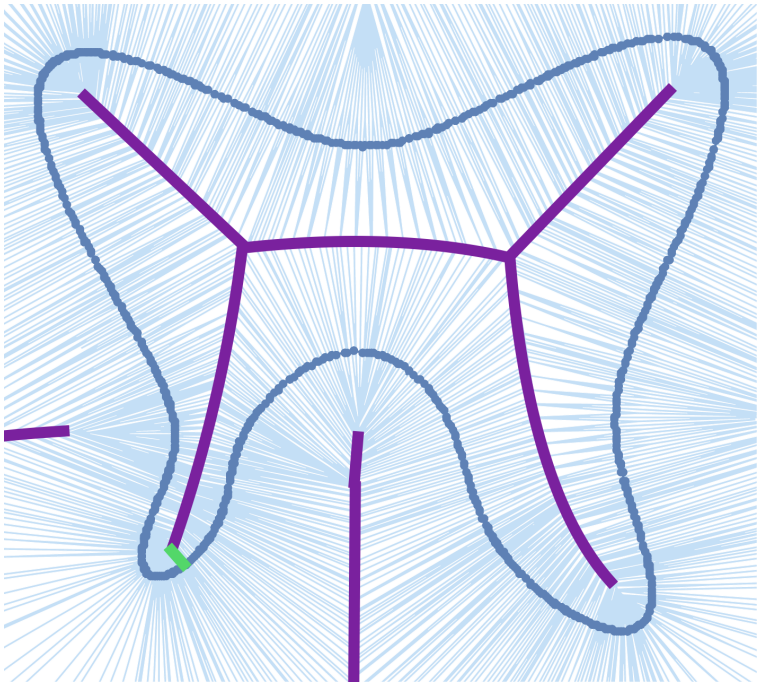
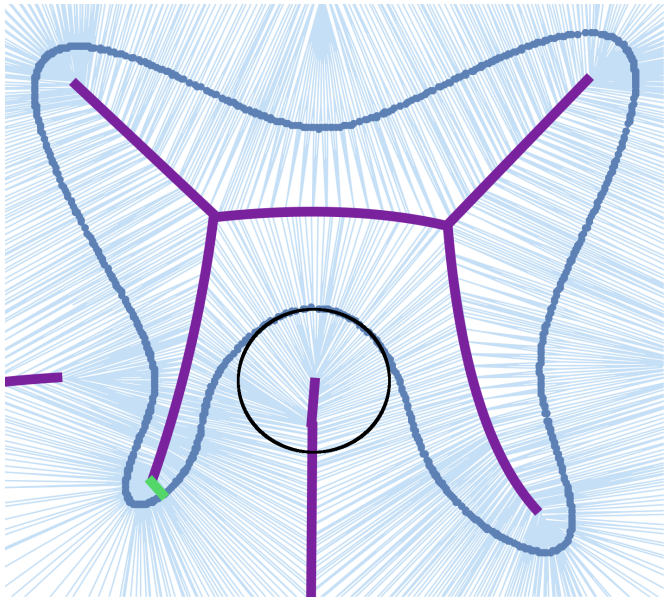


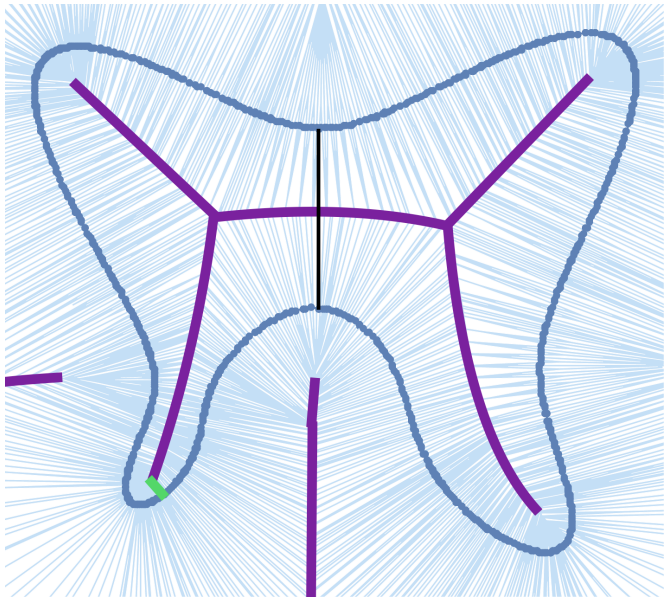
Figure: Reach = length of green line.

Reach

The reach is determined by points of maximal curvature and bottlenecks.



(a) Maximal curvature (ρ_1)



(b) Bottlenecks (ρ_2)

The reach of the spherical Segre–Veronese manifold is the minimum of the two functions

$$\tau(\mathbb{X}_{\mathbf{n},\mathbf{d}}) = \min\{\rho_1, \rho_2\}.$$

Maximal curvature

$$\rho_1 = \frac{1}{\sup\{\|P_{\mathbf{E}}(\gamma''(0))\| \mid \gamma \text{ is a geodesic in } \mathbb{X}_{\mathbf{n},\mathbf{d}} \text{ parametrized by arc length}\}},$$

where $P_{\mathbf{E}}$ is the orthogonal projection onto $N_{\mathbf{E}}\mathbb{X}_{\mathbf{n},\mathbf{d}}$.

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Lemma (Kostlan 1993, 2002)

The Bombieri-Weyl inner product on $\mathbb{X}_{\mathbf{n},\mathbf{d}}$ is invariant under orthogonal change of variables; i.e., for orthogonal matrices $Q_1 \in O(n_1+1), \ldots, Q_r \in O(n_r+1)$ we have

$$\begin{aligned}\langle(\mathbf{f}_1 \circ Q_1) \otimes \cdots \otimes (\mathbf{f}_r \circ Q_r), (\mathbf{g}_1 \circ Q_1) \otimes \cdots \otimes (\mathbf{g}_r \circ Q_r)\rangle \\ = \langle \mathbf{f}_1 \otimes \cdots \otimes \mathbf{f}_r, \mathbf{g}_1 \otimes \cdots \otimes \mathbf{g}_r \rangle.\end{aligned}$$

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Thus, we can take

$$\mathbf{E} := x_0^{d_1} \otimes \cdots \otimes x_0^{d_r} \in \mathbb{X}_{\mathbf{n},\mathbf{d}}.$$

Maximal curvature

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where, after rotating as necessary,

$$\gamma_i(t) = \left(\cos\left(d_i^{-1/2}\, a_i t\right)\, x_0 + \sin\left(d_i^{-1/2}\, a_i t\right)\, x_1\right)^{d_i}.$$

$\gamma(t)$ parametrized by arc length $\Rightarrow \sum_{i=1}^{r} a_i^2 = 1$.

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Maximize $\|P_{\mathbf{E}}(\gamma''(0))\|$ subject to $\sum_{i=1}^r a_i^2 = 1$:

$$a_{i\max} = \sqrt{\frac{d_i}{d}} \quad \Rightarrow \quad \sup\left\{\|P_{\mathbf{E}}(\gamma''(0))\|\right\} = \sqrt{\frac{2(d-1)}{d}}.$$

Bottlenecks

$$\rho_2 = \min\left\{ \tfrac{1}{2}\, d_{\mathbb{S}}(\mathbf{F}, \mathbf{E}) \;\middle|\; \mathbf{F} \in \mathbb{X}_{\mathbf{n},\mathbf{d}},\ \mathbf{F} \neq \mathbf{E} \ \text{ and } \ \mathbf{F} - \mathbf{E} \in N_{\mathbf{E}}\mathbb{X}_{\mathbf{n},\mathbf{d}} \oplus \mathbb{R} \cdot \mathbf{E} \right\}$$

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Therefore, the bottlenecks of $\mathbb{X}_{\mathbf{n},\mathbf{d}}$ all have width $\arccos 0 = \frac{\pi}{2}$, so $\rho_2 = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}$.

Reach of $\mathbb{X}_{\mathbf{n},\mathbf{d}}$

Theorem (Breiding-E.)

Let $\mathbf{d} = (d_1, \ldots, d_r)$ *and* $\mathbf{n} = (n_1, \ldots, n_r)$ *be* r*-tuples of positive integers, and let* $d := d_1 + \cdots + d_r \geq 2$ *be the total degree. The reach of the (spherical) Segre–Veronese manifold is*

$$\tau(\mathbb{X}_{\mathbf{n},\mathbf{d}}) = \begin{cases} \frac{\pi}{4}, & d \leq 5 \\ \sqrt{\frac{d}{2(d-1)}}, & d > 5. \end{cases}$$

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In particular, the reach only depends on the total degree d *and not on the dimensions of the Veronese varieties* $\mathbb{V}_{n_i,d_i}$*.*

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Then

$$\mathrm{vol}(U(\varepsilon)) = \sum_{0 \le 2i \le n} \kappa_i \cdot J_i(\varepsilon),$$

with

$$\kappa_i = \mathrm{vol}(\mathbb{X}_{\mathbf{n},\mathbf{d}}) \int_{\mathbf{F} \in N_{\mathbf{E}}\mathbb{X}_{\mathbf{n},\mathbf{d}} \,:\, \|\mathbf{F}\|=1} m_{2i}(L_{\mathbf{F}}) \, \mathrm{d}\mathbf{F},$$

where $m_{2i}(L_{\mathbf{F}})$ denotes the sum of the $2i$-principal minors of the Weingarten map $L_{\mathbf{F}}$ in normal direction $\mathbf{F}$.

Volume of the tubular neighborhood

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Lemma

If $\mathbf{F}$ *is Gaussian,* $L_{\mathbf{F}}$ *is a random symmetric matrix with independent blocks*

$$L_{\mathbf{F}} \sim \begin{bmatrix} L_1 & \cdots & L_{1,r} \\ & \ddots & \\ (L_{1,r})^T & \cdots & L_r \end{bmatrix}, \qquad \begin{aligned} L_k &\sim \sqrt{\tfrac{d_k(d_k-1)}{2}}\,\mathrm{GOE}(n_i), \\ L_{i,j} &\sim N(0, I_{n_i} \otimes I_{n_j}) \end{aligned} .$$

Example

For the Segre manifold with $n_1 = n_2 = 2$, $n_3 = n_4 = 1$ and degrees $\mathbf{1} = (1, 1, 1, 1)$, the Weingarten map is given by

$$L_{\mathbf{1}}(2, 2, 1, 1) = \left[\begin{array}{cc|cc|cc} 0 & 0 & \mathbf{F}_{1100} & \mathbf{F}_{1200} & \mathbf{F}_{1010} & \mathbf{F}_{1001} \\ 0 & 0 & \mathbf{F}_{2100} & \mathbf{F}_{2200} & \mathbf{F}_{0201} & \mathbf{F}_{2001} \\ \hline \mathbf{F}_{1100} & \mathbf{F}_{2100} & 0 & 0 & \mathbf{F}_{0110} & \mathbf{F}_{0101} \\ \mathbf{F}_{1200} & \mathbf{F}_{2200} & 0 & 0 & \mathbf{F}_{0210} & \mathbf{F}_{0201} \\ \hline \mathbf{F}_{1010} & \mathbf{F}_{2010} & \mathbf{F}_{0110} & \mathbf{F}_{0210} & 0 & \mathbf{F}_{0011} \\ \hline \mathbf{F}_{1001} & \mathbf{F}_{2001} & \mathbf{F}_{0101} & \mathbf{F}_{0201} & \mathbf{F}_{0011} & 0 \end{array}\right],$$

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where the entries are all i.i.d. standard Gaussian.
Expectation is linear, so

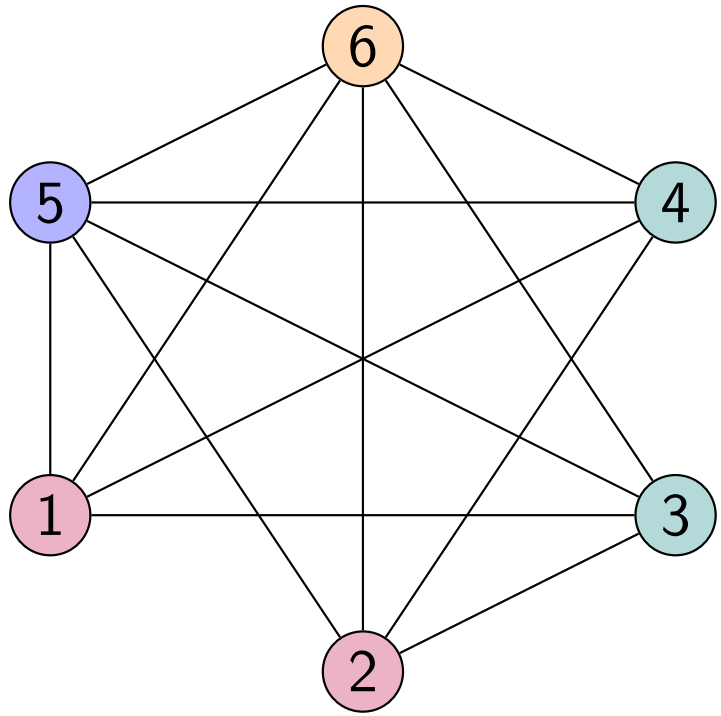
$$\mathbb{E}\,\det L_{\mathbf{1}}(2,2,1,1) = \sum_{\sigma \in \mathfrak{S}_6} \operatorname{sgn}(\sigma)\, \mathbb{E} \prod_{i=1}^{6} \ell_{i,\sigma(i)},$$

where $\mathfrak{S}_m$ is the symmetric group on m elements.

Perfect matchings in graphs and random determinants

Thus, we can compute $\mathbb{E}\det L_{\mathbf{1}}(2,2,1,1)$ by counting perfect matchings in the following graph with $n_1+n_2+n_3+n_4=6$ vertices and four groups $\mathcal{I}_1=\{1,2\},\mathcal{I}_2=\{3,4\},\mathcal{I}_3=\{5\},\mathcal{I}_4=\{6\}$:

0	0	$\ell_{1,3}$	$\ell_{1,4}$	$\ell_{1,5}$	$\ell_{1,6}$
0	0	$\ell_{2,3}$	$\ell_{2,4}$	$\ell_{2,5}$	$\ell_{2,6}$
$\ell_{3,1}$	$\ell_{3,2}$	0	0	$\ell_{3,5}$	$\ell_{3,6}$
$\ell_{4,1}$	$\ell_{4,2}$	0	0	$\ell_{4,5}$	$\ell_{4,6}$
$\ell_{5,1}$	$\ell_{5,2}$	$\ell_{5,3}$	$\ell_{5,4}$	0	$\ell_{5,6}$
$\ell_{6,1}$	$\ell_{6,2}$	$\ell_{6,3}$	$\ell_{6,4}$	$\ell_{6,5}$	0



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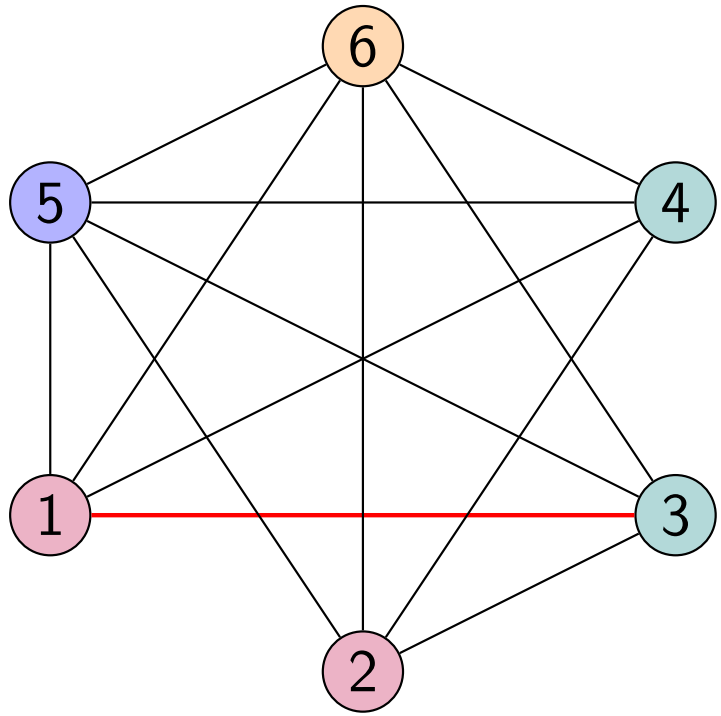

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$\ell_{3,1}$	$\ell_{3,2}$	0	0	$\ell_{3,5}$	$\ell_{3,6}$
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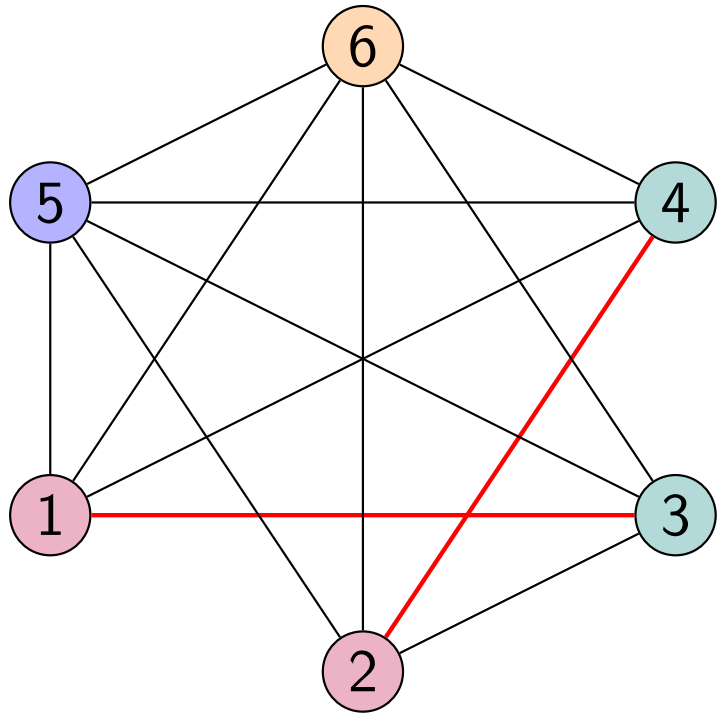
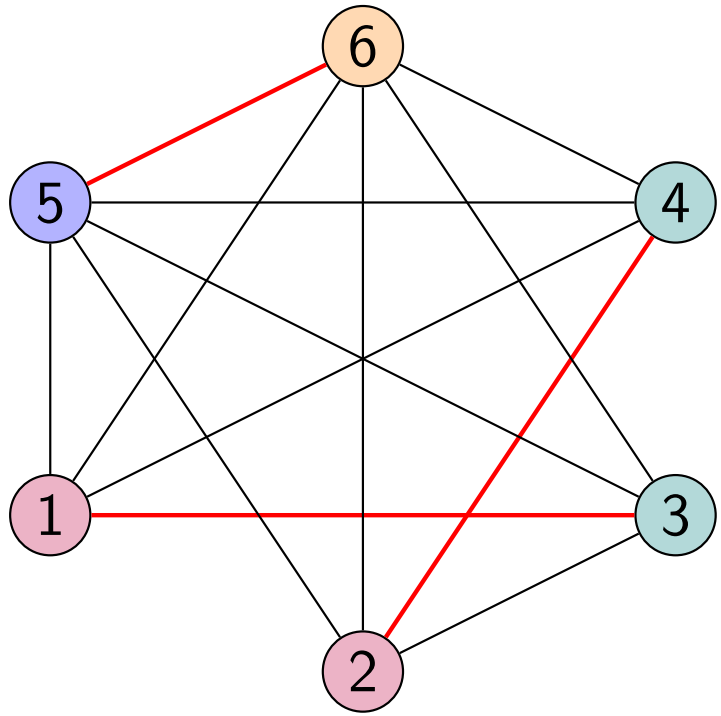
$\ell_{5,1}$	$\ell_{5,2}$	$\ell_{5,3}$	$\ell_{5,4}$	0	$\ell_{5,6}$
$\ell_{6,1}$	$\ell_{6,2}$	$\ell_{6,3}$	$\ell_{6,4}$	$\ell_{6,5}$	0



Perfect matchings in graphs and random determinants

Thus, we can compute $\mathbb{E}\det L_{\mathbf{1}}(2,2,1,1)$ by counting perfect matchings in the following graph with $n_1+n_2+n_3+n_4=6$ vertices and four groups $\mathcal{I}_1=\{1,2\},\mathcal{I}_2=\{3,4\},\mathcal{I}_3=\{5\},\mathcal{I}_4=\{6\}$:

0	0	$\ell_{1,3}$	$\ell_{1,4}$	$\ell_{1,5}$	$\ell_{1,6}$
0	0	$\ell_{2,3}$	$\ell_{2,4}$	$\ell_{2,5}$	$\ell_{2,6}$
$\ell_{3,1}$	$\ell_{3,2}$	0	0	$\ell_{3,5}$	$\ell_{3,6}$
$\ell_{4,1}$	$\ell_{4,2}$	0	0	$\ell_{4,5}$	$\ell_{4,6}$
$\ell_{5,1}$	$\ell_{5,2}$	$\ell_{5,3}$	$\ell_{5,4}$	0	$\ell_{5,6}$
$\ell_{6,1}$	$\ell_{6,2}$	$\ell_{6,3}$	$\ell_{6,4}$	$\ell_{6,5}$	0



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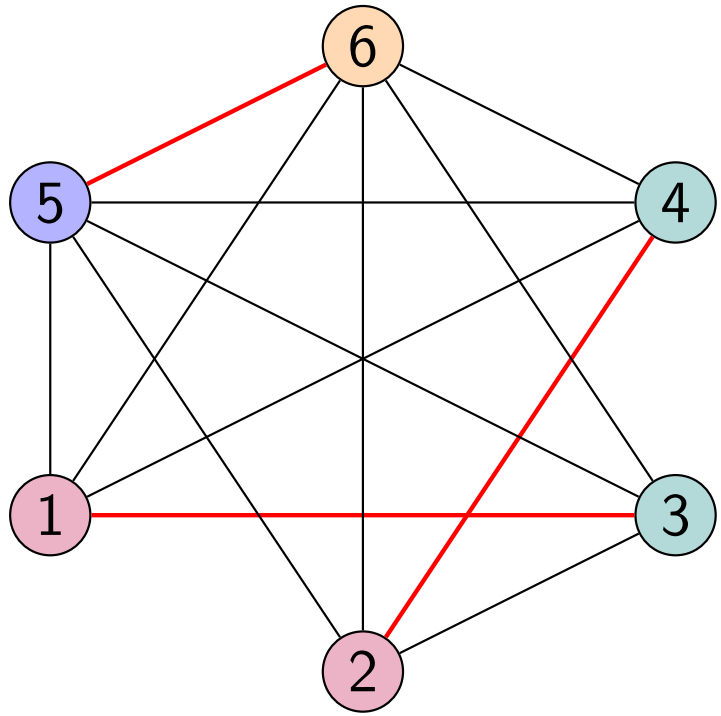
0	0	$\ell_{2,3}$	$\ell_{2,4}$	$\ell_{2,5}$	$\ell_{2,6}$
$\ell_{3,1}$	$\ell_{3,2}$	0	0	$\ell_{3,5}$	$\ell_{3,6}$
$\ell_{4,1}$	$\ell_{4,2}$	0	0	$\ell_{4,5}$	$\ell_{4,6}$
$\ell_{5,1}$	$\ell_{5,2}$	$\ell_{5,3}$	$\ell_{5,4}$	0	$\ell_{5,6}$
$\ell_{6,1}$	$\ell_{6,2}$	$\ell_{6,3}$	$\ell_{6,4}$	$\ell_{6,5}$	0



In total these are 10 matches, which shows that $D_{\mathbf{1}}(2,2,1,1)=-10$.

Perfect matchings in graphs and random determinants

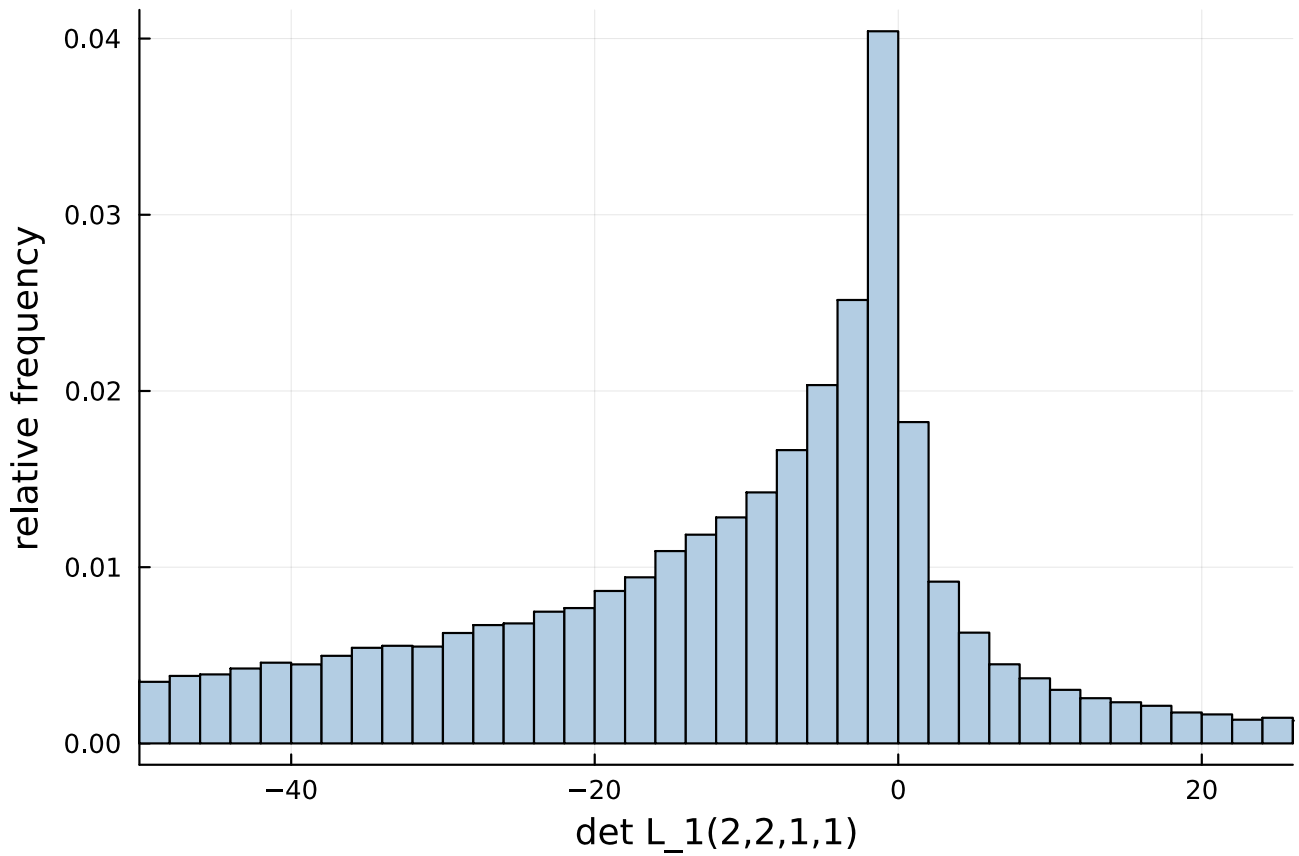


Figure: The empirical distribution of $\det L_{\mathbf{1}}(2,2,1,1)$ for 10^5 sample points. The empirical mean of this sample is -9.9995. We showed that the actual mean value is -10.

Thanks!

arxiv.org/abs/2307.04224

Bottlenecks

$$\rho_2 = \min\left\{ \tfrac{1}{2}\, d_{\mathbb{S}}(\mathbf{F}, \mathbf{E}) \;\middle|\; \mathbf{F} \in \mathbb{X}_{\mathbf{n},\mathbf{d}},\ \mathbf{F} \neq \mathbf{E} \ \text{ and } \ \mathbf{F} - \mathbf{E} \in N_{\mathbf{E}}\mathbb{X}_{\mathbf{n},\mathbf{d}} \oplus \mathbb{R} \cdot \mathbf{E} \right\}$$

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where

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$$\begin{aligned}\langle \mathbf{F}-\mathbf{E}, x_0^{d_1-1}x_k \otimes x_0^{d_2} \otimes \cdots \otimes x_0^{d_r}\rangle &= \langle \mathbf{F}, x_0^{d_1-1}x_k \otimes x_0^{d_2} \otimes \cdots \otimes x_0^{d_r}\rangle \\ &= \langle \ell_1^{d_1}, x_0^{d_1-1}x_k\rangle \prod_{i=2}^{r}\langle \ell_i^{d_i}, x_0^{d_i}\rangle = \langle \ell_1, x_0\rangle^{d_1-1}\langle \ell_1, x_k\rangle \prod_{i=2}^{r}\langle \ell_i, x_0\rangle^{d_i} \\ &= 0 \ \forall k \in \{1,\dots,n_1\} \quad \text{if } \ell_1 = x_0 \text{ or } \langle \ell_1, x_0\rangle = 0.\end{aligned}$$

Bottlenecks

Iterating, we have

$$\langle \mathbf{F} - \mathbf{E}, \mathbf{G} \rangle = 0 \quad \Leftrightarrow \quad \begin{cases} \text{either } \ell_i = x_0 \text{ for all } i \Leftrightarrow \mathbf{F} = \mathbf{E} \\ \text{or } \langle \ell_i, x_0 \rangle = 0 \text{ for at least one } i \Leftrightarrow \langle \mathbf{F}, \mathbf{E} \rangle = 0. \end{cases}$$

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Therefore, the bottlenecks of $\mathbb{X}_{\mathbf{n},\mathbf{d}}$ all have width $\arccos 0 = \frac{\pi}{2}$,
so $\rho_2 = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}$.