# SDP bounds for distance-avoiding sets on compact spaces

Bram Bekker
TU Delft

Olga Kuryatnikova
Erasmus University Rotterdam

Juan C. Vera
Tilburg University

Fernando Mário de Oliveira
TU Delft

# Witsenhausen's problem

What is the measure $\alpha_n$ of the largest measurable subset of $S^{n-1}$ avoiding orthogonal points?

## Witsenhausen's problem

What is the measure $\alpha_n$ of the largest measurable subset of $S^{n-1}$ avoiding orthogonal points?

## Double cap conjecture (Kalai; 2015)

The set $\{x \in S^{n-1} : |e \cdot x| > \cos(\pi/4)\}$ achieves $\alpha_n$.

# An upper bound is given by

## $\vartheta$-number

$$\vartheta = \sup \int_{S^{n-1}} \int_{S^{n-1}} A(x,y)\, dx dy$$

s.t.

- $\int_{S^{n-1}} A(x,x) dx = 1$
- $A(x,y) = 0$, if $x \cdot y = 0$
- $A \in C_{\text{sym}}(S^{n-1})_{\succcurlyeq 0}$

# An upper bound is given by

## $\vartheta$-number

$$\vartheta = \sup \int_{S^{n-1}} \int_{S^{n-1}} A(x,y)\, dx dy$$

s.t.

- $\int_{S^{n-1}} A(x,x) dx = 1$
- $A(x,y) = 0$, if $x \cdot y = 0$
- $A \in C_{\text{sym}}(S^{n-1})_{\succcurlyeq 0}$

$A \succcurlyeq 0 \Leftrightarrow$ for all finite $U \subset S^{n-1}$

$$A[U] : U \times U \to \mathbb{R}$$

is positive semidefinite.

# Lasserre's hierarchy

Consider continuous functions

$$A: \mathrm{sub}_k(S^{n-1}) \times \mathrm{sub}_k(S^{n-1}) \to \mathbb{R}$$

such that:

# Lasserre's hierarchy

Consider continuous functions

$$A: \mathrm{sub}_k(S^{n-1}) \times \mathrm{sub}_k(S^{n-1}) \to \mathbb{R}$$

such that:

- $\int_{S^{n-1}} A(\{x\},\{x\})dx = 1$
- $A(S,T) = A(S',T') \qquad \forall S \cup T = S' \cup T'$
- $A(S,T) = 0 \text{ if } 0 \in \{x \cdot y : x, y \in S \cup T\}$

# Lasserre's hierarchy

Consider continuous functions

$$A: \mathrm{sub}_k(S^{n-1}) \times \mathrm{sub}_k(S^{n-1}) \to \mathbb{R}$$

such that:

- $\int_{S^{n-1}} A(\{x\},\{x\})dx = 1$
- $A(S,T) = A(S',T') \qquad \forall S \cup T = S' \cup T'$
- $A(S,T) = 0$ if $0 \in \{x \cdot y : x,y \in S \cup T\}$

This does not lead to a tractable upper bound for any $k \geq 2$

# Completely and copositive cone

$$\mathrm{COP}(\mathrm{S}^{\mathrm{n}-1}) = \{A \in L^2_{\mathrm{sym}}(S^{n-1}) : \langle A, f \otimes f \rangle \geq 0 \quad \forall f \in L^2(S^{n-1}), f \geq 0\}$$

$$\mathrm{CP}(\mathrm{S}^{\mathrm{n}-1}) = \mathrm{COP}(\mathrm{S}^{\mathrm{n}-1})^*$$

$$= \left\{\mathrm{A} \in L^2_{\mathrm{sym}}(S^{n-1}) : \langle A, Z \rangle \geq 0 \quad \forall Z \in \mathrm{COP}(S^{n-1})\right\}$$

# Completely and copositive cone

$$\mathrm{COP}(\mathrm{S}^{\mathrm{n}-1}) = \{A \in L^2_{\mathrm{sym}}(S^{n-1}) : \langle A, f \otimes f \rangle \geq 0 \quad \forall f \in L^2(S^{n-1}), f \geq 0\}$$

$$\mathrm{CP}(\mathrm{S}^{\mathrm{n}-1}) = \mathrm{COP}(\mathrm{S}^{\mathrm{n}-1})^*$$

$$= \left\{\mathrm{A} \in L^2_{\mathrm{sym}}(S^{n-1}) : \langle A, Z \rangle \geq 0 \quad \forall Z \in \mathrm{COP}(S^{n-1})\right\}$$

**Theorem** (DeCorte, Oliveira, Vallentin)

$$\vartheta_{\mathrm{CP}(S^{n-1})} = \alpha_n$$

# Completely and copositive cone

$$\mathrm{COP}(\mathrm{S}^{\mathrm{n}-1}) = \{A \in L^2_{\mathrm{sym}}(S^{n-1}) : \langle A, f \otimes f \rangle \geq 0 \quad \forall f \in L^2(S^{n-1}), f \geq 0\}$$

$$\mathrm{CP}(\mathrm{S}^{\mathrm{n}-1}) = \mathrm{COP}(\mathrm{S}^{\mathrm{n}-1})^*$$

$$= \left\{\mathrm{A} \in L^2_{\mathrm{sym}}(S^{n-1}) : \langle A, Z \rangle \geq 0 \quad \forall Z \in \mathrm{COP}(S^{n-1})\right\}$$

## Theorem (DeCorte, Oliveira, Vallentin)

$$\vartheta_{\mathrm{CP}(S^{n-1})} = \alpha_n$$

Optimizing over the completely positive cone is also not tractable

We approximate the copositive cone by cones

$$\mathcal{C}_r(S^{n-1}) = \{A \in L^2_{\mathrm{sym}}(S^{n-1}) \colon \mathcal{R}_{S_{r+2}}\left(A \otimes 1^{\otimes r}\right) \geq 0\}$$

We approximate the copositive cone by cones

$$\mathcal{C}_r(S^{n-1}) = \{A \in L^2_{\mathrm{sym}}(S^{n-1}) : \mathcal{R}_{S_{r+2}}\left(A \otimes 1^{\otimes r}\right) \geq 0\}$$

$$\mathcal{C}_1(S^{n-1}) \subseteq \mathcal{C}_2(S^{n-1}) \subseteq \cdots \subseteq \mathrm{COP}(\mathrm{S}^{\mathrm{n}-1})$$

We approximate the copositive cone by cones

$$\mathcal{C}_r(S^{n-1}) = \{A \in L^2_{\text{sym}}(S^{n-1}) \colon \mathcal{R}_{S_{r+2}}\left(A \otimes 1^{\otimes r}\right) \geq 0\}$$

$$\mathcal{C}_1(S^{n-1}) \subseteq \mathcal{C}_2(S^{n-1}) \subseteq \cdots \subseteq \mathrm{COP}(S^{n-1})$$

**Theorem** (Bekker, Kuryatnikova, Oliveira, Vera)

If $A \in L^2_{\text{sym}}(S^{n-1})$ is such that $\langle A, Z\rangle \geq 0$ for all $Z \in \bigcup_{r\geq 1} \mathcal{C}_r(S^{n-1})$, then $A \in \mathrm{CP}(S^{n-1})$.

## The copositive hierarchy

$$\vartheta_r = \sup \int_{S^{n-1}} \int_{S^{n-1}} A(x,y)\, dxdy$$

s.t.

- $\int_{S^{n-1}} A(x,x)dx = 1$
- $A(x,y) = 0$, if $x \cdot y = 0$
- $A \in \mathcal{C}_r(S^{n-1})^*,\ A \succcurlyeq 0$

## The copositive hierarchy

$$\vartheta_r = \sup \int_{S^{n-1}} \int_{S^{n-1}} A(x,y)\, dxdy$$

s.t.

- $\int_{S^{n-1}} A(x,x) dx = 1$
- $A(x,y) = 0$, if $x \cdot y = 0$
- $A \in \mathcal{C}_r(S^{n-1})^*,\ A \succcurlyeq 0$

## Theorem (Bekker, Kuryatnikova, Oliveira, Vera)

If $n \geq 3,\ \lim_r(\vartheta_r) = \alpha_n$.

For any $n,\ r \geq 1$: $\text{lass}_{r+2} \leq \vartheta_r$.

| Dimension | Lower bound | Previous best upper bound | New best upper bound | Percentage gap closed |
|---|---|---|---|---|
| 3 | 0.2928 | 0.3015 | 0.2977 | 43% |
| 4 | 0.1816 | 0.2168 | 0.1943 | 64% |
| 5 | 0.1161 | 0.1677 | 0.1346 | 64% |
| 6 | 0.0755 | 0.1338 | 0.0981 | 61% |
| 7 | 0.0498 | 0.1174 | 0.0758 | 62% |
| 8 | 0.0331 | 0.0998 | 0.0612 | 58% |