Probabilistic approaches to rational points on algebraic surfaces

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The problem

smooth, projective, geometrically integral

$X/\mathbb{Q}$: nice geometrically rational surface.

$\overline{X} := X \times_{\mathbb{Q}} \overline{\mathbb{Q}} \simeq \mathbb{P}^2_{\overline{\mathbb{Q}}}$

Example: $-K_X$ ample (del Pezzo surfaces); $d := K_X^2$

degree

$d = 4 : \quad \{Q_1 = Q_2 = 0\} \subset \mathbb{P}^4_{\mathbb{Q}}$

$d = 3 : \quad \{C(x,y,z,w) = 0\} \subset \mathbb{P}^3_{\mathbb{Q}} \longleftarrow$ cubic surfaces

$d = 2 : \quad \{w^2 = f_4(x,y,z)\} \subset \mathbb{P}_{\mathbb{Q}}(1,1,1,2)$

$d = 1 : \quad \{w^2 = z^3 + a_4(x,y)z + b_6(x,y)\} \subset \mathbb{P}_{\mathbb{Q}}(1,1,2,3)$

The problem

$X/\mathbb{Q}$: nice geometrically rational surface.

Determine if $X(\mathbb{Q}) \neq \emptyset$.

Necessary condition: $X(\mathbb{Q}_p) \neq \emptyset$ for all primes $p \leq \infty$.

X is projective $\implies$ $X(\mathbb{A}_{\mathbb{Q}}) = \prod_{p \leq \infty} X(\mathbb{Q}_p)$.

The problem

$X/\mathbb{Q}$: nice geometrically rational surface.

Determine if $X(\mathbb{Q}) \neq \emptyset$.

Necessary condition: $X(\mathbb{A}_{\mathbb{Q}}) \neq \emptyset$.

Not sufficient (Cassels—Guy, 1966):

$$X : \quad 5x^3 + 9y^3 + 10z^3 + 12w^3 = 0.$$

Obstruction Sets

$$X(\mathbb{Q}) \subseteq \underline{X(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}}} \subseteq X(\mathbb{A}_{\mathbb{Q}})$$

Brauer—Manin set

Brauer—Manin obstruction:

$$X(\mathbb{A}_{\mathbb{Q}}) \neq \varnothing \text{ but } X(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}} = \varnothing \implies X(\mathbb{Q}) = \varnothing.$$

Conjecture (Colliot-Thélène—Sansuc, 1979):

$$X(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}} \neq \varnothing \implies X(\mathbb{Q}) \neq \varnothing\,.$$

Obstruction Sets

$X(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}}$ is defined using $\mathrm{Br}\, X \,/\, \mathrm{Br}_0\, X$

Brauer group:
$H^2_{\mathrm{et}}(X, \mathbb{G}_m)_{\mathrm{tors}}$

Constant Algebras:
$\mathrm{im}\,(\mathrm{Br}\, \mathbb{Q} \to \mathrm{Br}\, X)$
from $X \to \mathrm{Spec}(\mathbb{Q})$

For nice geometrically rational surfaces over $\mathbb{Q}$:

$$\mathrm{Br}\, X \,/\, \mathrm{Br}_0\, X \xrightarrow{\simeq} H^1\Big(\mathrm{Gal}\,\big(\overline{\mathbb{Q}}/\mathbb{Q}\big), \mathrm{Pic}\, \overline{X}\Big)$$

(Hochschild—Serre spectral sequence).

Case of Cubic surfaces

$$\overline{X} \simeq \mathrm{Bl}_{\{P_1,\ldots,P_6\}} \mathbb{P}^2_{\overline{\mathbb{Q}}} \implies \text{Pic } \overline{X} \simeq \mathbb{Z}^7$$

in general position

Corrected post-talk

$K/\mathbb{Q}$: Galois closure of field where the 27 lines are defined. This is a finite extension.

If $X(\mathbb{A}_{\mathbb{Q}}) \neq \emptyset$ then $\text{Pic } X_K \xrightarrow{\simeq} \left(\text{Pic } \overline{X}\right)^{\text{Gal}(\overline{\mathbb{Q}}/K)}$ as $\text{Gal}(K/\mathbb{Q})$-modules, and inflation gives an isomorphism

$$H^1\left(\text{Gal}\,(K/\mathbb{Q}), \text{Pic } X_K\right) \xrightarrow{\simeq} H^1\left(\text{Gal}\,\left(\overline{\mathbb{Q}}/\mathbb{Q}\right), \text{Pic } \overline{X}\right)$$

Case of Cubic surfaces

So: to understand Br $X/\text{Br}_0\, X$, we must understand

$$\text{Gal}(K/\mathbb{Q}) \curvearrowright \text{Pic}\, X_K.$$

Pic X_K has an intersection form (curves on a surface!)

$\text{Gal}(K/\mathbb{Q}) \curvearrowright \text{Pic}\, X_K$ respects this form and fixes K_X.

Get $\rho_X\colon \text{Gal}(K/\mathbb{Q}) \to O\left(K_X^{\perp}\right)$.

Finite $\implies$ Finite image!

Case of Cubic surfaces

So: to understand Br $X/\mathrm{Br}_0\, X$, we must understand

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Pic X_K has an intersection form (curves on a surface!)

$\mathrm{Gal}(K/\mathbb{Q}) \curvearrowright \mathrm{Pic}\, X_K$ respects this form and fixes K_X.

Get ρ_X: $\mathrm{Gal}(K/\mathbb{Q}) \to O\left(K_X^{\perp}\right)$

Manin: $K_X^{\perp}$ carries a root system of type E_6

$$\implies \mathrm{im}(\rho_X) \leq W(E_6)$$

Weyl group;
51,840 elements!

Case of Cubic surfaces

Using these ideas, Swinnerton—Dyer (1993) showed that $\mathrm{Br}\, X/\mathrm{Br}_0\, X$ is isomorphic to one of:

$$0,\ \mathbb{Z}/2\mathbb{Z},\ \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z},\ \mathbb{Z}/3\mathbb{Z},\ \mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$$

Goal: Given a smooth cubic surface $X/\mathbb{Q}$, efficiently compute $\mathrm{Br}\, X/\mathrm{Br}_0\, X$.

This can already be done in magma.
So why bother?

(1) Cubic surfaces are a “proof of concept” case for us.

(2) Existing methods help benchmark new ideas.

Case of Cubic surfaces

Using these ideas, Swinnerton—Dyer (1993) showed that $\mathrm{Br}\, X/\mathrm{Br}_0\, X$ is isomorphic to one of:

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Goal: Given a smooth cubic surface $X/\mathbb{Q}$,
efficiently compute $\mathrm{Br}\, X/\mathrm{Br}_0\, X$.

Do not do the naive thing: compute $K/\mathbb{Q}$.
This can be a number field of very large degree!

Reduction modulo p

$X/\mathbb{Q}$: smooth cubic surface;
$K/\mathbb{Q}$: a splitting field for X;
p : a prime of good reduction for X;
$\mathfrak{p}$: a prime ideal in $\mathcal{O}_K$ above p.

Have repn's

$$\rho_X : \mathrm{Gal}(K/\mathbb{Q}) \to O\left(K_X^{\perp}\right)$$

$$\rho_{X_p} : \mathrm{Gal}(\mathbb{F}_{\mathfrak{p}}/\mathbb{F}_p) \to O\left(K_{X_p}^{\perp}\right)$$

Frob $\in \mathrm{Gal}(\mathbb{F}_{\mathfrak{p}}/\mathbb{F}_p)$ gives conj. class $\left[\mathrm{Frob}_{\mathfrak{p}}\right] \subseteq \mathrm{Gal}(K/\mathbb{Q})$

$\rho_{X_p}(\mathrm{Frob}) \in O\left(K_{X_p}^{\perp}\right)$ gives conj. class in $\mathrm{im}(\rho_X)$.

Strategy

Given a smooth cubic surface $X/\mathbb{Q}$:

(1) Reduce X over lots of good primes;

(2) Use Frob to compute conj. classes of $\mathrm{im}(\rho_X)$;

(3) Decide the most likely $\mathrm{im}(\rho_X) \leq W(E_6)$;

(4) Compute $H^1\big(\mathrm{Gal}\,(K/\mathbb{Q}), \mathrm{Pic}\, X_K\big) \simeq \mathrm{Br}\, X/\mathrm{Br}_0\, X$.

The group $W(E_6)$

- Order: $51{,}840 = 2^7 \cdot 3^4 \cdot 5^1$
- Orders of elements: $\{1, 2, 3, 4, 5, 6, 8, 9, 10, 12\}$
- 25 Conjugacy Classes
- 350 Conjugacy classes of subgroups
- Orders of subgroups:
  {1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 16, 18, 20, 24, 27, 32, 36, 40, 48, 54, 60, 64, 72, 80, 81, 96, 108, 120, 128, 144, 160, 162, 192, 216, 240, 288, 320, 324, 360,384, 432, 576, 648, 720, 960, 1152, 1296, 1440, 1920, 25920, 51840}

Spectral Profile of a subgroup

Let $H \leq W(E_6)$ be a subgroup.
For each $h \in H$, compute its $W(E_6)$-conjugacy class.
This is one of 25 possibilities $C_1, \ldots, C_{25}$
Tabulate and normalize.

Example: a particular H of order 48

C	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
#	1	7	2	0	6	2	4	2	0	0	0	0	0	0	0	5	9	0	2	0	4	0	0	4	0

Spectral profile of H:

$$\left[\frac{1}{48}, \frac{7}{48}, \frac{1}{24}, 0, \frac{1}{8}, \frac{1}{24}, \frac{1}{12}, \frac{1}{24}, 0, 0, 0, 0, 0, 0, 0, \frac{5}{48}, \frac{3}{16}, 0, \frac{1}{24}, 0, \frac{1}{12}, 0, 0, \frac{1}{12}, 0\right]$$

Spectral Profile of a subgroup

The 350 conjugacy classes of subgroups of $W(E_6)$ give rise to 339 distinct spectral profiles.

Among the 11 collisions (Gassmann equivalent subgroups) between 22 subgroups, 6 pairs lead to the same Brauer group for a cubic surface.

So spectral profiles can be used to compute the Brauer group among $340 = 350 - 22 + 12$ conjugacy classes of subgroups of $W(E_6)$.

An Example

$$X: \quad -x^2w + 2xyz + 4xyw - 5xzw - xw^2 + y^2z$$
$$-3y^2w + 2yz^2 - 4yzw + z^3 - z^2w - 3w^3 = 0$$

Compute Frobenius conjugacy classes for 40 primes:
$\{p \leq 197\} \backslash \{2, 3, 5, 7, 19\}$.

Observed conjugacy classes:

C	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
#	0	4	1	0	0	5	0	8	0	0	0	0	0	0	0	1	7	0	0	0	14	0	0	0	0

What is the most likely spectral profile?

Detour: Bayesian Basics

Suppose we are give a coin and told that the probability of it landing heads is either $p = 0.3$, 0.4, or 0.5.

We toss the coin 1000 times. It falls heads 482 times. What is the most likely value for p?

Scenarios for $[p, 1-p]$:

A	B	C
$[0.3, 0.7]$	$[0.4, 0.6]$	$[0.5, 0.5]$

A priori: $P(A) = P(B) = P(C) = \frac{1}{3}$.

Detour: Bayesian Basics

A	B	C
[0.3,0.7]	[0.4,0.6]	[0.5,0.5]

Let E be the event:
482 heads are observed after 1000 tosses.

$$L(E \mid A) = \frac{1000!}{482!518!} \cdot (0.3)^{482} \cdot (0.7)^{518} \approx 7.65 \times 10^{-34}$$

$$L(E \mid B) = \frac{1000!}{482!518!} \cdot (0.4)^{482} \cdot (0.6)^{518}, \approx 2.67 \times 10^{-8}$$

$$L(E \mid C) = \frac{1000!}{482!518!} \cdot (0.5)^{482} \cdot (0.5)^{518} \approx 0.0132$$

Using binomial distribution

Detour: Bayesian Basics

A	B	C
$[0.3, 0.7]$	$[0.4, 0.6]$	$[0.5, 0.5]$

Let E be the event:
482 heads are observed after 1000 tosses.

$$\text{Posterior}(A \mid E) = \frac{P(A) \cdot L(A \mid E)}{P(A) \cdot L(A \mid E) + P(B) \cdot L(B \mid E) + P(C) \cdot L(C \mid E)}$$

$$= \frac{(0.3)^{482}(0.7)^{518}}{(0.5)^{482}(0.5)^{518} + (0.4)^{482}(0.6)^{518} + (0.3)^{482}(0.7)^{518}}$$

$$\approx 5.79728017 \times 10^{-32}$$

Detour: Bayesian Basics

A	B	C
$[0.3,0.7]$	$[0.4,0.6]$	$[0.5,0.5]$

Let E be the event:
482 heads are observed after 1000 tosses.

$$\text{Posterior}(B \mid E) = \frac{P(B) \cdot L(B \mid E)}{P(A) \cdot L(A \mid E) + P(B) \cdot L(B \mid E) + P(C) \cdot L(C \mid E)}$$

$$= \frac{(0.4)^{482}(0.6)^{518}}{(0.5)^{482}(0.5)^{518} + (0.4)^{482}(0.6)^{518} + (0.3)^{482}(0.7)^{518}}$$

$$\approx 2.01956718 \times 10^{-6}$$

Detour: Bayesian Basics

A — $[0.3, 0.7]$

B — $[0.4, 0.6]$

C — $[0.5, 0.5]$

Let E be the event:
482 heads are observed after 1000 tosses.

$$\text{Posterior}(C \mid E) = \frac{P(C) \cdot L(C \mid E)}{P(A) \cdot L(A \mid E) + P(B) \cdot L(B \mid E) + P(C) \cdot L(C \mid E)}$$

$$= \frac{(0.5)^{482}(0.5)^{518}}{(0.5)^{482}(0.5)^{518} + (0.4)^{482}(0.6)^{518} + (0.3)^{482}(0.7)^{518}}$$

$$\approx 0.99999798$$

Back to our Example

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$$-3y^2w + 2yz^2 - 4yzw + z^3 - z^2w - 3w^3 = 0$$

Observed conjugacy classes:

C	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
#	0	4	1	0	0	5	0	8	0	0	0	0	0	0	0	1	7	0	0	0	14	0	0	0	0

What is the most likely spectral profile?

We can apply the same principle! Use a multinomial distribution to replace the binomial distribution

Back to our Example

There are 339 spectral profiles $SP_1, \ldots, SP_{339}$.
Think of SP_i as a vector of probabilities of observing a particular conjugacy class:

$$SP_i = [p_{i,1}, p_{i,2}, \ldots, p_{i,25}]; \qquad \sum_j p_{i,j} = 1$$

All SP_i start out equally likely to be right: $P(SP_i) = \dfrac{1}{339}$.

We have collected a vector of observations

$$\alpha = [\alpha_1, \ldots, \alpha_{25}]$$

Back to our Example

Likelihood of observing α given that SP_i is correct:

$$L(\alpha \mid SP_i) = \frac{(\alpha_1 + \cdots + \alpha_{25})\,!}{\alpha_1\,!\cdots\alpha_{25}\,!} \prod_{j=1}^{25} p_{ij}^{\alpha_j}\,.$$

Posterior probabilities:

$$\text{Posterior}(SP_i \,|\, \alpha) = \frac{P(SP_i) \cdot L(\alpha \mid SP_i)}{\sum_{j=1}^{339} P(SP_j) \cdot L(\alpha \mid SP_j)}\,.$$

This idea works

Theorem (James—V.A. '23): Let $0 < \epsilon < 1$. Let $X/\mathbb{Q}$ denote a smooth cubic surface. Let SP be the spectral profile of $\mathrm{im}(\rho_X) \leq W(E_6)$.

There is an $x = x(\epsilon)$ such that for all sets of observations $\alpha = [\alpha_1, \ldots, \alpha_{25}]$ with $\alpha_1 + \cdots + \alpha_{25} > x$ we have

$$\text{Posterior}(SP \mid \alpha) > 1 - \epsilon\,.$$

Elsenhans—Jahnel Database

Elsenhans & Jahnel have a publicly available list of 350 smooth cubic surfaces, one for each conjugacy class of subgroups of $W(E_6)$.

The above ideas by themselves can only probabilistically determine the Brauer group of 340 of these surfaces.

They do! In 184 seconds.
(2.3 GHz Quad-Core Intel Core i7)

Magma internals do too! In 1002 seconds.