The geometry of stable lattices in Bruhat-Tits buildings

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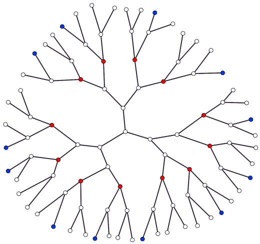
PRESENTING JOINT WORK WITH **Y. El Maazouz, M. A. Hahn, G. Nebe, B. Sturmfels**



Arithmetic, Geometry, Cryptography and Coding Theory

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Let K be a field with a surjective valuation map

$$\mathrm{val} : K \to \mathbb{Z} \cup \{\infty\}.$$

Denote

- $\mathcal{O}_K = \{x \in K : \mathrm{val}(x) \geq 0\}$ is the valuation ring of K,
- $\mathfrak{m}_K = \{x \in K : \mathrm{val}(x) > 0\} \lhd \mathcal{O}_K$ unique maximal,
- $\pi \in K$ such that $\mathrm{val}(\pi) = 1$ is a uniformizer and $\mathfrak{m}_K = \mathcal{O}_K \pi$.

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The valuation val can be extended to K^d or $K^{d\times d}$ coordinate-wise:

$$\mathrm{val}_3(2, 15, -1/36) = (0, 1, -2), \text{ for } K = \mathbb{Q} \text{ or } \mathbb{Q}_3$$

$$\mathrm{val}_t \begin{pmatrix} 0 & t^{-5} + t^{-1} \\ -1/3 & 87t^7 - t^{11} \end{pmatrix} = \begin{pmatrix} \infty & -5 \\ 0 & 7 \end{pmatrix} \text{ for } K = \mathbb{Q}((t))$$

Definition.

- A ($\mathcal{O}_K$-)lattice in K^d is a free $\mathcal{O}_K$-submodule L of rank d.
- An order (in $K^{d\times d}$) is a lattice Λ that is also a ring: it is called maximal if it is not properly contained in any other order.
- A Λ-lattice is a lattice L with $\Lambda L \subseteq L$, i.e. L is Λ-stable.

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Example.

- $\mathcal{O}_K^d$ is the standard lattice in K^d,
- $\mathcal{O}_K^{d\times d}$ is a maximal order in $K^{d\times d}$ and its stable lattices are

$$[\mathcal{O}_K^d] = \{\pi^n \mathcal{O}_K^d : n \in \mathbb{Z}\}.$$

Definition. Lattices with $L' = \pi^n L$ are called homothetic. The homothety class of L is denoted $[L]$.

Let $\Lambda \subseteq K^{d\times d}$ be an order and $L \subseteq K^d$ a lattice. The endomorphism ring of L is $\mathrm{End}_{\mathcal{O}_K}(L) = \{X \in K^{d\times d} : XL \subseteq L\}$.

If L, L' are Λ-lattices then the following hold:

- $\pi^n L$ is a Λ-lattice and $\mathrm{End}_{\mathcal{O}_K}(\pi^n L) = \mathrm{End}_{\mathcal{O}_K}(L)$.
- $L \cap L'$ and $L + L'$ are Λ-lattices.

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Notation. $Q(\Lambda) = \{\, [L] : L \text{ is } \Lambda\text{-stable} \,\}$

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- $\Lambda_1 \subseteq \Lambda_2 \;\Rightarrow\; Q(\Lambda_2) \subseteq Q(\Lambda_1)$

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Definition.

- The order Λ is closed if $\Lambda = \bigcap_{[L]\in Q(\Lambda)} \mathrm{End}_{\mathcal{O}_K}(L)$.

Remark. The closed orders are precisely those that are determined by the collection of their stable lattices. For these orders:

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Example. Let $\mathcal{O}_K^{2\times 2} \supset \Lambda_1 \supset \Lambda_2$ be defined by

$$\Lambda_1 = \{X \in \mathcal{O}_K^{2\times 2} : X_{21} \equiv 0 \bmod \pi\} = \begin{pmatrix} \mathcal{O}_K & \mathcal{O}_K \\ \pi\mathcal{O}_K & \mathcal{O}_K \end{pmatrix},$$

$$\Lambda_2 = \{X \in \Lambda_1 : X_{11} \equiv X_{22} \bmod \pi\}.$$

Then $Q(\Lambda_1) = Q(\Lambda_2) = \{[\mathcal{O}_K \oplus \pi\mathcal{O}_K], [\mathcal{O}_K^2]\}$ and Λ_1 is closed, while Λ_2 is not.

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Main Goal. Study the interplay between Λ and $Q(\Lambda)$ in the language of Bruhat-Tits buildings.

Let $\mathcal{E} = (e_1, \ldots, e_d)$ be a basis of K^d and $u = (u_1, \ldots, u_d) \in \mathbb{Z}^d$. Then

$$L_u = \pi^{u_1}\mathcal{O}_K e_1 \oplus \ldots \oplus \pi^{u_d}\mathcal{O}_K e_d$$

is a diagonal lattice (wrt $\mathcal{E}$). Two lattices have compatible bases if they are diagonal with respect to the same basis.

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Example. Let $\mathcal{E}$ be the standard basis. Then:

- $L_{\mathbf{0}} = \mathcal{O}_K^d$.
- Given two lattices there always exists a basis of K^d such that the lattices are diagonal with respect to that basis.
- In the last example the stable lattices were $L_{(0,1)}$ and $L_{(0,0)}$ and the order Λ_1 could have been described as

$$\Lambda_1 = \left\{ X \in K^{2\times 2} : \mathrm{val}(X) \geq \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}.$$

Let $M = (m_{ij}) \in \mathbb{Z}^{d\times d}$. Then the set

$$\Lambda(M) = \{X \in K^{d\times d} : \mathrm{val}(X) \geq M\} \subset K^{d\times d}$$

is naturally a free $\mathcal{O}_K$-submodule of rank d^2.

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Example.

- If $M = 0$, then $\Lambda(M) = \mathcal{O}_K^{d\times d}$.
- For $K = \mathbb{Q}$, $d = 3$, and $\mathrm{val} = \mathrm{val}_p$:

$$\mathrm{val}_p \underbrace{\begin{pmatrix} 1 & 1 & p \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{M} \text{ but } \begin{pmatrix} \star & \star & 0 \\ \star & \star & \star \\ \star & \star & \star \end{pmatrix} = \mathrm{val} \underbrace{\begin{pmatrix} 2+p & 2+p & 1+2p \\ 3 & 3 & 2+p \\ 3 & 3 & 2+p \end{pmatrix}}_{X^2}$$

so $\Lambda(M)$ is not a ring.

Orders of the form $\Lambda(M)$ are called graduated, monomial, tiled or split. Their study was pioneered by Plesken and Zassenhaus.

Proposition (Plesken). $\Lambda(M)$ is an order if and only if

$$m_{ii} = 0, \; m_{ij} + m_{jk} \geq m_{ik} \text{ for } 1 \leq i, j, k \leq d.$$

Write $M \in \mathbb{Z}_0^{d \times d}$ if $m_{ii} = 0$.

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Example. In the previous examples we saw:

$$M = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \text{ in which case } \Lambda(M) \text{ is a ring,}$$

$$M = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ which satisfies } m_{12} + m_{23} = 0 < 1 = m_{13}.$$

Remark. We will soon see that graduated orders are closed.

Theorem (Plesken). A lattice L in K^d is stable under $\Lambda(M)$ if and only if there exists $u \in \mathbb{Z}^d$ with

$$u_i - u_j \leq m_{ij} \quad \text{for} \quad 1 \leq i, j \leq d,$$

such that $L = L_u = \pi^{u_1}\mathcal{O}_K e_1 \oplus \ldots \oplus \pi^{u_d}\mathcal{O}_K e_d$. Moreover, two $\Lambda(M)$-lattices L_u and L_v are isomorphic if and only if $[L_u] = [L_v]$.

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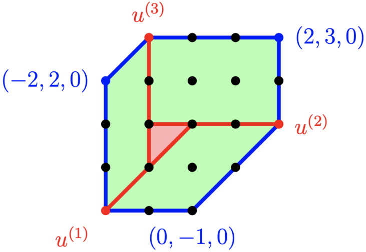

Here

$$M = \begin{pmatrix} 0 & 1 & 2 \\ 4 & 0 & 3 \\ 2 & 1 & 0 \end{pmatrix}$$

and dots represent hom. classes of $\Lambda(M)$-lattices.

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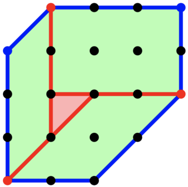

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Definition. $Q_M = \{[u] \in \mathbb{R}^d/\mathbb{R}\mathbf{1} : u_i - u_j \leq m_{ij}\}$ is a polytrope.

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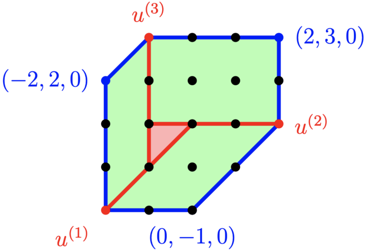

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Theorem. The following is a well-defined bijection:

$$Q_M \cap (\mathbb{Z}^d/\mathbb{Z}\mathbf{1}) \longrightarrow Q(\Lambda(M)) = \{[L] : L \text{ is } \Lambda(M)\text{-stable}\}$$
$$[u] \longmapsto [L_u]$$

The min-plus and max-plus algebras $(\mathbb{R}, \underline{\oplus}, \odot)$ and $(\mathbb{R}, \overline{\oplus}, \odot)$ are defined by the operations

$$a \underline{\oplus} b = \min\{a, b\}, \quad a \overline{\oplus} b = \max\{a, b\}, \quad a \odot b = a + b.$$

Example. $L_u \cap L_v = L_{u \overline{\oplus} v}$ and $L_u + L_v = L_{u \underline{\oplus} v}$

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Example. If $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $N = \begin{pmatrix} -2 & 0 \\ 3 & 1 \end{pmatrix}$, and $u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ then

$$M \overline{\odot} N = \begin{pmatrix} 4 & 2 \\ 3 & 1 \end{pmatrix}, \qquad M \overline{\odot} u = \begin{pmatrix} 2 \\ 2 \end{pmatrix},$$

$$M \underline{\odot} N = \begin{pmatrix} -2 & 0 \\ -1 & 1 \end{pmatrix}, \qquad M \underline{\odot} u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

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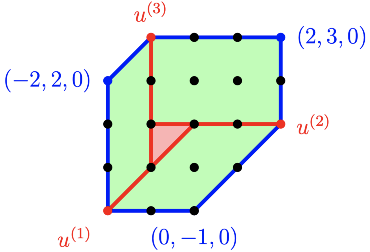
Consequences.

- $\Lambda(M)$ is an order if and only if $M \,\underline{\odot}\, M = M$
- L_u is a stable lattice if and only if $M \,\underline{\odot}\, u^\top \geq u^\top$
- L_u is $\Lambda(M)$-stable iff $[u] \in Q_M \cap (\mathbb{Z}^d/\mathbb{Z}\mathbf{1})$
- $\Lambda(M) = \bigcap_{[u] \in Q_M \cap (\mathbb{Z}^d/\mathbb{Z}\mathbf{1})} \mathrm{End}_{\mathcal{O}_K(L_u)}$

Theorem. Let $M \in \mathbb{Z}_0^{d\times d}$ satisfy $M \odot M = M$. Then Q_M is both a min-plus and a max-plus simplex. The min-plus vertices u are the columns of M and represent L_u's that are projective $\Lambda(M)$-modules. The max-plus vertices v are the rows of $-M$, and they represent the injective $\Lambda(M)$-modules L_v.



Recall that

$$M = \begin{pmatrix} 0 & 1 & 2 \\ 4 & 0 & 3 \\ 2 & 1 & 0 \end{pmatrix}.$$

Consequence.

$$\Lambda(M) = \cap_{[u]\in Q_M\cap(\mathbb{Z}^d/\mathbb{Z}\mathbf{1})} \mathrm{End}_{\mathcal{O}_K}(L_u) = \cap_{[u]\in Q_M\cap(\mathbb{Z}^d/\mathbb{Z}\mathbf{1})} \mathrm{End}_{\mathcal{O}_K}(L_u)$$

Definition. The Bruhat-Tits building $\mathcal{B}_d(K)$ is a simplicial complex where:

- the vertices are equivalence classes of lattices in K^d,
- $([L_1], \ldots, [L_s])$ is a simplex if $L_1 \supset L_2 \supset \ldots L_s \supset \pi L_1$ (up to reordering and picking representatives)

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Question. What's life like when you are not quarantined in one apartment?

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Remark. If Λ is an order, then $Q(\Lambda)$ is non-empty, convex, and bounded in $\mathcal{B}_d(K)$.

Question. What are the convex objects that arise as $Q(\Lambda)$ when Λ is not graduated?

An example for $d = 2$ and $d = 3$:

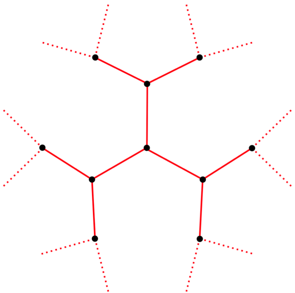
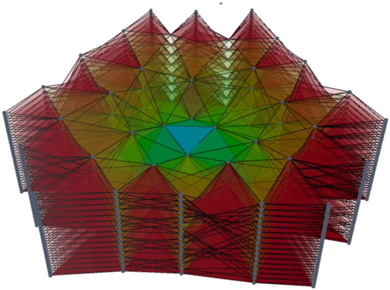


Figure 5. The building $\mathcal{B}(SL_2(\mathbb{Q}), \nu_2)$ is an infinite tree.



(b) The affine building $\mathcal{B}(SL_3(\mathbb{Q}), \nu_2)$ up to distance 5 from the chamber in the centre.

Bekker, Solleveld - The Buildings Gallery: visualising buildings (2021)

More in the online gallery: https://buildings.gallery

For L_1, L_2 lattices in K^d, define

$$\begin{aligned}\mathrm{dist}([L_1],[L_2]) &= \min\{s \mid \exists L_1' \in [L_1], L_2' \in [L_2],\ \pi^s L_1' \subseteq L_2' \subseteq L_1'\}\\ &= \min\{s \mid \exists m \text{ with } \pi^s L_1 \subseteq \pi^m L_2 \subseteq L_1\}\end{aligned}$$

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Then the following hold:

- dist agrees with the tropical distance in one apartment:

$$\mathrm{dist}(u,v) = \max_{1\leq i\leq d}(u_i - v_i) - \min_{1\leq j\leq d}(u_j - v_j).$$

- if $d = 2$, then dist is the same as the graph distance.

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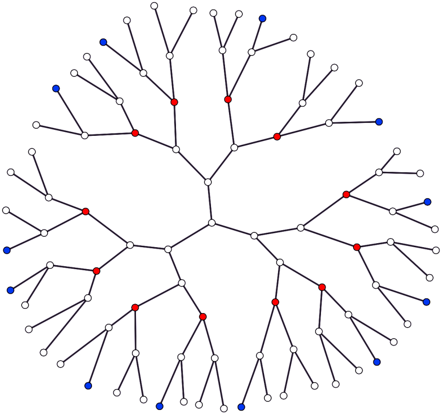
For each $r \geq 0$ (integer), define

$$\mathrm{B}_r = \{[L] \mid \operatorname{dist}([\mathcal{O}_K^d],[L]) \leq r\}$$

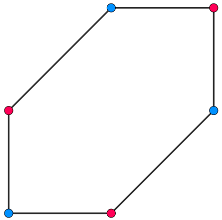
so, with the appropriate basis, $\mathrm{B}_r \cap \mathcal{A}$ is a tropical ball of radius r.

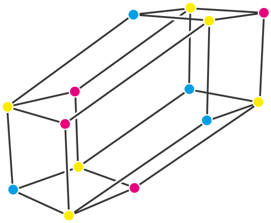
 A ball in $\mathcal{B}_2(\mathbb{Q}_2)$



This is B_5 inside of $\mathcal{B}_2(\mathbb{Q}_2)$.

The tropical ball of radius r around $[\mathcal{O}_K^d]$ is $Q_{rJ_d} = Q(\Lambda(rJ_d))$ where $rJ_d \in \mathbb{Z}_0^{d\times d}$ has all r's outside the main diagonal.





These spheres of radius 1 for $d=3$ and $d=4$ correspond to

$$J_3 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad J_4 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Assume that the residue field of K has q elements.

For $r = 1$, we have:

- $|Q_{J_d}| = |\,\mathrm{B}_1 \cap \mathcal{A}| = 2^d - 1$,
- $|\,\mathrm{B}_1\,| = 1 + \sum_{\ell=1}^{d-1} |\mathrm{Gr}(\ell, \mathbb{F}_q^d)|$, e.g.

$$|\,\mathrm{B}_1\,|_{d=2} = q + 2 \quad \text{and} \quad |\,\mathrm{B}_1\,|_{d=3} = 2q^2 + 2q + 3.$$

In general a precise count is given in terms of partitions of rd.

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Asymptotically ($q \to \infty$), we have:

$$|\mathrm{B}_r| \sim \begin{cases} q^{d^2 r/4} & \text{if } d \text{ even,} \\ (r+1)q^{(d^2-1)r/4} & \text{if } d \text{ odd,} \end{cases} \quad \text{and} \quad \frac{|\mathrm{B}_r \cap \mathcal{A}|}{|\mathrm{B}_r|} \to 0.$$

Lemma. The ball B_r is equal to $Q(\Lambda_r(0))$ where

$$\Lambda_r(0) = \{X \in \Lambda(rJ_d) : X_{11} \equiv X_{22} \equiv \ldots \equiv X_{dd} \bmod \pi^r\}\,.$$

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Remark. Every proper free $R_{K,r}$-submodule $[L]$ of $R_{K,r}^d$ defines an element of $\partial\,\mathrm{B}_r$ $(\mathrm{dist}([L], [\mathcal{O}_K^d]) = r)$ and is a vertex of the ball: there exists $\mathcal{A}$ such that $[L]$ is a vertex of $\mathrm{B}_r \cap \mathcal{A}$.

A bolytrope with center $Q_M \subseteq \mathcal{A}$ and radius $r \geq 0$ is

$$\mathrm{B}_r(M) = \{[L] \mid \mathrm{dist}([L], Q_M) \leq r\}.$$

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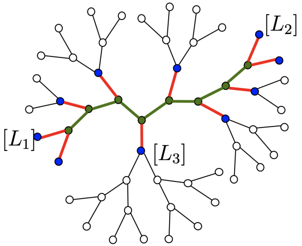
Theorem. $Q(\Lambda_r(M)) = \mathrm{B}_r(M)$.

Theorem. If $d = 2$, then all closed orders are bolytrope orders.

This extends Tu's result (2011) for finite residue fields.



Definition. The idealizer of a lattice $\Lambda \subset K^{d\times d}$ is

$$\mathrm{I}(\Lambda) = \{X \in K^{d\times d} : X\Lambda \subseteq \Lambda,\ \Lambda X \subseteq \Lambda\}.$$

The radical idealizer chain of an order Λ is $(\Omega_i(\Lambda))_{i\geq 0}$ where

$$\Omega_0(\Lambda) = \Lambda \quad \text{and} \quad \Omega_{i+1}(\Lambda) = \mathrm{I}(\mathrm{Jac}(\Omega_i(\Lambda))) :$$

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this is ascending, finite, and satisfies $Q(\Omega_1) \subseteq Q(\Lambda) \subseteq \mathrm{B}_1(Q(\Omega_1))$.
The corresponding chain of stable lattices is descending and finite.

Remark.

- $\Omega_1(\Lambda(M + J_d)) = \Lambda(M)$
- $\Omega_1(\Lambda_r(0)) = \Lambda_{r-1}(0)$
- $\Omega_1(\Lambda_r(M)) = \Lambda_{r-1}(M)$ (+ Ind.) $\rightsquigarrow Q(\Lambda_r(M)) = \mathrm{B}_r(M)$

Spherical codes in buildings...

...if time allows (?)

Let $r > 0$ be an integer and recall $\partial\,\mathrm{B}_r = \mathrm{B}_r \setminus \mathrm{B}_{r-1}$.

A spherical code in $\mathcal{B}_d(K)$ is a subset $\mathcal{C} \subseteq \partial\,\mathrm{B}_r$ with $|\mathcal{C}| \geq 2$.

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Spherical codes in the Euclidean setting can be defined from sphere packings and have various applications in the field of telecommunication.

In view of these applications, it is desirable to produce sizeable codes of large internal distance and small length. Optimal codes have the "best possible" coexistence constraints on these parameters.

The minimum distance of $\mathcal{C}$ is

$$\mathrm{dist}(\mathcal{C}) = \min\{\mathrm{dist}([L_1],[L_2]) \mid [L_1],[L_2] \in \mathcal{C}, [L_1] \neq [L_2]\}.$$

Define $V_r = R_{K,r}^d = \mathcal{O}_K^d / \pi^r \mathcal{O}_K^d$ and assume $q = |R_{K,1}|$ is finite.

Then the following hold:

- V_1 is a vector space over the residue field of K,
- identify $[L] \in \partial\, \mathrm{B}_r$ with $L \leq V_r$ with $\pi^{r-1} V_r \not\subseteq L \not\subset \pi V_r$,
- vertices of B_r represent free $R_{K,r}$-submodules of V_r.

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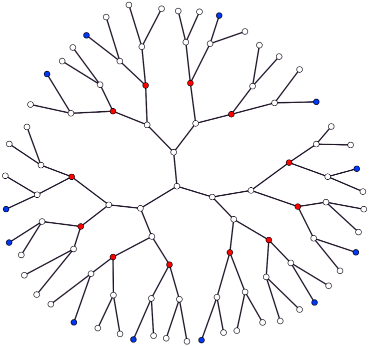
In particular, spherical codes in Bruhat-Tits buildings are instances of submodule codes over chain rings (new distance!).

Definition. $\mathrm{Gr}(n, V_r) = \{$free $R_{K,r}$-submods of V_r of rank $n\}$

Remark. If $r = 1$, then we recover the usual Grassmannian.

Definition. Let $1 \leq \alpha \leq r$. A Sperner code with parameters (d, r, α) is $\mathcal{C} \subseteq \mathrm{Gr}(\lceil d/2 \rceil, V_r)$ such that the following is a bijection:

$$\mathcal{C} \longrightarrow \mathrm{Gr}(\lceil d/2 \rceil, \pi^{\alpha-1} V_r), \quad L \longmapsto \pi^{\alpha-1} L.$$



The blue points form a Sperner code with parameters $(2, 5, 3)$.

Theorem. Let $1 \leq \alpha \leq r$ and let $\mathcal{C} \subseteq \mathrm{B}_r$ be a spherical code of maximal size with $\mathrm{dist}(\mathcal{C}) = 2\alpha$. Then

- $|\mathcal{C}| \geq |\,\mathrm{Gr}(\lceil d/2 \rceil, V_{r+1-\alpha})|$,
- if $d = 2$ or $\alpha = r$, then $|\mathcal{C}| = |\,\mathrm{Gr}(\lceil d/2 \rceil, V_{r+1-\alpha})|$.

In particular, Sperner codes are optimal if $d = 2$ or $\alpha = r$.

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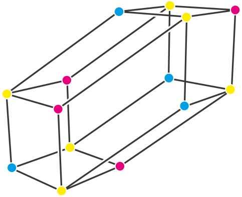
Remark. The last result works thanks to the fact that we can reduce to codes of free modules and results in extremal combinatorics (Sperner (1928), Stanley (1991), Wang (1998)).

Question. Are Sperner codes always optimal?

Codes in one apartment generalize permutation codes and, given the choice of an apartment $\mathcal{A}$, are defined as

$$\mathcal{C}_{\mathcal{A}}([u]) = \{[L_v] : v \in \mathrm{Sym}(d) \cdot u\}, \text{ with } u_1 = r \geq u_2 \geq \ldots u_d = 0.$$



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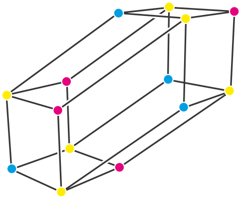

- Bounds relating minimum distance and cardinality: optimal codes in one apartment (far from general optimality).
- Easy encoding and decoding process.

Question. Other interesting codes in one apartment?

ArXiv: 2107.00503, 2111.11244, 2202.13370

And check out our Mathrepo page:
https://mathrepo.mis.mpg.de/OrdersPolytropes/index.html